Zero-cycles on varieties over finite fields

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Abstract

We prove that for a smooth projective variety X of dimension d defined over a finite field k, the structure map $\sigma : X \longrightarrow \text{Spec } k$ induces an isomorphism $\sigma_* : CH^{d+1}(X, 1) \cong CH^1(k, 1) = k^*$. We also prove that the higher Chow groups $CH^{d+s-1}(X, s)$ of one-dimensional cycles are torsion for $s \geq 2$.

1 Introduction

For any field k, Milnor [Mi] defined a sequence of groups $K_0^M(k), K_1^M(k), K_2^M(k), \ldots$ which later came to be known as *Milnor K-groups*. These were studied extensively by Bass and Tate [BT], Suslin [Su], Kato [Ka1], [Ka2] and others. In [Som], Somekawa investigates a generalization of this definition proposed by Kato: given semi-abelian varieties $G_1, \ldots, G_s$ over a field k, there is a group $K(k; G_1, \ldots, G_s)$ which is isomorphic to $K_s^M(k)$ in the case that $G_1 = \ldots = G_s = \mathbf{G}_m$, the multiplicative group scheme. Raskind and Spiess [RS] use a similar idea to define a Milnor-type group $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r))$ associated to a family $X_1, \ldots, X_r$ of smooth projective varieties over k and prove that this group is isomorphic to the Chow group $CH_0(X_1 \times_k \ldots \times_k X_r)$ of zero-cycles. These two definitions were amalgamated in [A2] to define "mixed" K-groups $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r); G_1, \ldots, G_s)$ where the G_i are as above and this time $X_1, \ldots, X_r$ are only assumed to be quasiprojective. In the case $G_1 = \ldots = G_s = \mathbf{G}_m$, we use the abbreviated notation $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r); \mathbf{G}_m)$ for this group.

The main result of [A2] is the following:

Theorem 1.1. *Let X be a smooth quasiprojective variety of dimension d over a field k. Then there are natural isomorphisms:*

$$K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \cong CH^{d+s}(X, s) \cong H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

The middle group above is Bloch's higher Chow group and the (Zariski) sheaf on the far right is the sheaf associated to the presheaf $U \mapsto K^M_s(\mathcal{O}_X(U))$. Since there is an explicit formula (see [A2] for details):

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \cong \frac{\bigoplus_{x \in X^d} K^M_s k(x)}{\bigoplus_{y \in X^{d-1}} \partial(K^M_{s+1} k(y))}$$

in which X^i is the set of codimension i points on X and ∂ is the boundary map between Milnor K-groups [Mi]. we can sometimes use our knowledge of the groups $K^M_s k(x)$ to make deductions concerning the group $H^d_{Zar}(X, \mathcal{K}^M_{d+s})$.

Specifically, using Steinberg's result that the Milnor K-groups of finite fields (Theorem 3.2) vanish in degree ≥ 2 we have the following result:

Theorem 1.2. *Let X be a smooth quasiprojective variety of dimension d over a finite field k, and suppose $s \geq 2$. Then $CH^{d+s}(X, s) = 0$.*

Obviously this argument does not suffice to investigate the group $CH^{d+1}(X, 1)$, which is the objective of this report. Our main result is:

Theorem 1.3. *Let X be a smooth projective variety of dimension d over a finite field with structure morphism $\sigma : X \longrightarrow Spec\ k$. Then the induced map*

$$\sigma_* : CH^{d+1}(X, 1) \longrightarrow CH^1(Spec\ k, 1)$$

is an isomorphism.

This result formed part of the author's Ph.D. thesis at Brown University. The original proof of this fact relied on a detailed and somewhat lengthy computation involving elements of a particular mixed K-group. In recent months, however, Poonen [Po] has proved a Bertini-type theorem that simplifies the argument considerably. In the interest of elegance, we first give a proof of our result using Poonen's work; we also present the original proof in an appendix. The techniques of the original proof involve manipulations of symbols in mixed K-groups which may be of value to the interested reader; furthermore, Poonen's theorem merely reduces the question to the case of curves, which is then resolved using a nontrivial result of Kahn whose proof is sketched in [Kahn]. Our original proof follows similar lines, and provides some of the details missing from the exposition in [Kahn]. Strictly speaking, one could give a proof of the above theorem without the mediation of mixed K-groups; however, introducing these groups allows us to prove a slightly more general result and furthermore provides the context for the original proof.

The other result of this work concerns higher Chow groups of one-dimensional cycles on varieties over finite fields. While the techniques used to analyze zero-cycles do not apply to this situation, we are able to use the coniveau spectral sequence for higher Chow groups together with known results to deduce that a large class of such groups are torsion.

Throughout this paper, a *variety* X over a field k is a separated integral scheme $\sigma : X \longrightarrow \mathrm{Spec}\ k$. A variety is said to be *defined over* k if it is geometrically integral.

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2 Mixed K-groups

2.1 Definitions

In the following we define the "mixed K-groups" first introduced in [A2].

Let k be a field, and X a smooth quasiprojective variety defined over k. We use the notation $CH_0(X)$ for the group of zero-cycles on X modulo rational equivalence. If G is a group scheme defined over k and A is a k-algebra, we use the notation $G(A)$ for the group of A-rational points, i.e. morphisms $\mathrm{Spec}\ A \longrightarrow G$ which commute with the structure maps.

Let $r \geq 0$ and $s \geq 0$ be integers; let $X_1, \ldots, X_r$ be smooth quasiprojective varieties defined over k and $G_1, \ldots, G_s$ semi-abelian varieties defined over k. Set

$$T = \bigoplus_{E/k \text{ finite}} CH_0((X_1)_E) \otimes \ldots \otimes CH_0((X_r)_E) \otimes G_1(E) \otimes \ldots \otimes G_s(E)$$

We use the notation $(a_1 \otimes \ldots \otimes a_r \otimes b_1 \otimes \ldots \otimes b_s)_E$ to refer to a homogeneous element living in the direct summand of F corresponding to the field E.

Define $R \subseteq T$ as the subgroup generated by the elements of the following type:

- **M1.** For convenience of notation, set $H_i(E) = CH_0((X_i)_E)$ for $i = 1, \ldots, r$ and $H_j(E) = G_{j-r}(E)$ for $j = r+1, \ldots, r+s$.

  For every diagram $k \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ of finite extensions of k, all choices $i_0 \in \{1, \ldots, r+s\}$ and all choices $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ for $i \neq i_0$, the element $R_1(E_1; E_2; i_0; h_1, \ldots, h_{r+s})$ defined to be:

  $$(\phi^*(h_1) \otimes \ldots h_{i_0} \otimes \ldots \otimes \phi^*(h_{r+s}))_{E_2} - (h_1 \otimes \ldots \otimes \phi_*(h_{i_0}) \otimes \ldots \otimes h_{r+s})_{E_1}$$

  Here we have used the notation ϕ^* (ϕ_*) to denote the pullback (pushforward) map for the Chow group structure on H_i (if $1 \leq i \leq r$) or the group scheme structure on H_i (if $s \leq i \leq r+s$).

- **M2.** For every K finitely generated of transcendence degree 1 over k, all choices of $h \in K^*$, $f_i \in CH_0((X_i)_K)$ for $i = 1, \ldots, r$ and $g_j \in G_j(K)$ for $j = 1, \ldots, s$ such that for every place v of K such that $v(k) = 0$, there exists $j_0(v)$ such that $g_j \in G_j(O_v)$ for all $j \neq j_0(v)$, the element $R_2(K, h, f_1, \ldots, f_r, g_1, \ldots, g_s)$. If $s > 0$, this is defined to be:

  $$\sum_v (s_v(f_1) \otimes \ldots \otimes s_v(f_r) \otimes g_1(v) \otimes \ldots \otimes \tilde{T}_v(g_{j_0(v)}, h) \otimes \ldots \otimes g_s(v))_{k(v)}$$

  Here O_v is the valuation ring of v, $s_v : CH_0((X_i)_K) \longrightarrow CH_0((X_i)_{k(v)})$ is the specialization map for Chow groups (cf. [F], 20.3), and $g_i(v) \in G_i(k(v))$ denotes the reduction of $g_i \in G_i(O_v)$. The notation $\tilde{T}_v$ refers to the "extended tame symbol" as defined in [Som]. In the case $G_1 = \ldots = G_s = \mathbf{G}_m$, which is our only concern in this paper, this coincides with the (ordinary) tame v-adic symbol T_v.

  If $s = 0$, the element $R_2(K; h; f_1, \ldots, f_r; g_1, \ldots, g_s)$ is defined to be:

  $$\sum_v \mathrm{ord}_v(h)(s_v(f_1) \otimes \ldots s_v(f_r))_{k(v)}$$

Finally, when $r + s > 0$, we define the *mixed K-group*,
$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r); G_1, \ldots, G_s)$ as the quotient T/R. We denote the class of a generator $(h_1 \otimes \ldots \otimes h_{r+s})_E$ by $\{h_1, \ldots, h_{r+s}\}_{E/k}$. We refer to the classes of elements of the form **M1** and **M2** as *relations* of the mixed K-group.

We complete the picture by defining our group to be $\mathbf{Z}$ in the case $r = s = 0$.

We will mostly be interested in the case $G_1 = \ldots = G_s = \mathbf{G}_m$; hence we use $K_s(k; \mathcal{CH}_0(X_0), \ldots, \mathcal{CH}_{0(}X_r); \mathbf{G}_m)$ as shorthand for $K(k; \mathcal{CH}_0(X_0), \ldots, \mathcal{CH}_{0(}X_r); \underbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}_{s})$. We also adopt the practice of omitting superfluous semicolons; for example, if $r = 0$, we simply write $K(k; G_1, \ldots, G_s)$ for the group above.

Remark.
If $\sigma : Y \longrightarrow \text{Spec } k$ is a *projective* variety, we can define the *degree map* $\deg = \sigma_* : CH_0(Y) \longrightarrow CH_0(\text{Spec } k) \cong \mathbf{Z}$ by push-forward of cycles. We define $A_0(Y) := \text{Ker deg}$, and note that if Y contains a k-rational point, or more generally if Y admits a zero-cycle of degree 1, then the degree map splits and we have a direct sum decomposition $CH_0(Y) \cong \mathbf{Z} \oplus A_0(Y)$.

Returning to our situation, suppose that $X_1, \ldots, X_q$ and $Y_1, \ldots, Y_r$ smooth quasiprojective, with $Y_1, \ldots, Y_r$ actually *projective*. By replacing CH_0 with A_0 in the appropriate instances, we may define groups $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s)$ as was done previously. The aim of the next two sections is to investigate the relationships among these groups.

2.2 Functoriality

In this section, we list the various functorial properties enjoyed by the mixed K-groups introduced above. These properties are exactly analogous to those of [Som], 1.3, so we omit the proofs here and instead refer the interested reader to [A1] for details. The first two properties reflect functoriality in the field, the third functoriality in the semi-abelian varieties, and the latter two functoriality in the quasiprojective varieties.

As before, k is a (base) field; $X_1, \ldots X_q$ are smooth quasiprojective varieties defined over k; $Y_1, \ldots, Y_r$ are smooth projective varieties defined over k, and $G_1, \ldots, G_s$ are semiabelian varieties defined over k.

Proposition 2.1. *(Covariant functoriality in the field) Let $i : k \hookrightarrow k'$ be any field extension. Then there exists a natural homomorphism:*

$$i_* : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s) \longrightarrow$$

$$K(k'; \mathcal{CH}_0((X_1)_{k'}), \ldots, \mathcal{CH}_0((X_q)_{k'}); \mathcal{A}_0((Y_1)_{k'}), \ldots, \mathcal{A}_0((Y_r)_{k'}); (G_1)_{k'}, \ldots, (G_s)_{k'})$$

Proposition 2.2. *(Contravariant functoriality in the field) Let $i : F \hookrightarrow k$ be a* finite *field extension. Then there exists a natural homomorphism:*

$$i^* : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s) \longrightarrow$$

$$K(F; \mathcal{CH}_0({}_FX_1), \ldots, \mathcal{CH}_0({}_FX_q); \mathcal{A}_0({}_FY_1), \ldots, \mathcal{A}_0({}_FY_r); {}_FG_1, \ldots, {}_FG_s)$$

(In the above, ${}_FX_1$ denotes X_1 considered as a scheme over F, etc.)

Proposition 2.3. *(Morphisms of semi-abelian varieties) Let G'_j, $j = 1, \ldots, s$ be semi-abelian varieties defined over k and $\psi_j : G_j \longrightarrow G'_j$ morphisms of semi-abelian varieties over k. Then there exists a natural homomorphism:*

$$\psi_* : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s) \longrightarrow$$

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G'_1, \ldots, G'_s)$$

Proposition 2.4. *(Covariant functoriality in varieties) Suppose X'_i $i = 1, \ldots, q$ are smooth quasiprojective varieties defined over k and Y'_j, $j = 1, \ldots, r$ smooth projective varieties defined over k; suppose furthermore that there are proper morphisms $\phi_i : X_i \longrightarrow X'_i$ and $\psi_j : Y_j \longrightarrow Y'_j$ for all $i = 1, \ldots q$ and $j = 1, \ldots, r$. Then pushforward of cycles induces a natural homomorphism:*

$$(\phi, \psi)_* : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s) \longrightarrow$$

$$K(k; \mathcal{CH}_0(X'_1), \ldots, \mathcal{CH}_0(X'_q); \mathcal{A}_0(Y'_1), \ldots, \mathcal{A}_0(Y'_r); G_1, \ldots, G_s)$$

Proposition 2.5. *(Contravariant functoriality in varieties) Suppose X'_i $i = 1, \ldots, q$ are smooth quasiprojective varieties defined over k and Y'_j, $j = 1, \ldots, r$ smooth projective varieties defined over k; suppose furthermore that there are flat morphisms $\phi_i : X_i \longrightarrow X'_i$ and $\psi_j : Y_j \longrightarrow Y'_j$ of relative dimension 0 for all $i = 1, \ldots q$ and $j = 1, \ldots, r$. Then pullback of cycles induces a natural homomorphism:*

$$(\phi, \psi)^* : K(k; \mathcal{CH}_0(X'_1), \ldots, \mathcal{CH}_0(X'_q); \mathcal{A}_0(Y'_1), \ldots, \mathcal{A}_0(Y'_r); G_1, \ldots, G_s) \longrightarrow$$

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s)$$

2.3 Splitting properties

The key result of this section could be viewed as a generalization of the principle that for any proper variety Y over a field k, the exact sequence

$$0 \longrightarrow A_0(Y) \longrightarrow CH_0(Y) \xrightarrow{\deg} \mathbf{Z} \longrightarrow 0$$

splits if and only if Y admits a zero-cycle of degree one. Once again, we omit the proofs of these facts, all of which are routine, and refer the reader to [A1], Section 5.4. We continue to use the notation of the previous section.

We begin with an elementary but useful lemma which says essentially that the argument $\mathcal{CH}_0(k)$ in a mixed K-group acts somewhat like an "identity" in that adding or it removing it yields a K-group isomorphic to the original.

Lemma 2.6. *There is a natural map inducing an isomorphism:*

$$K(k; \mathcal{CH}_0(k), \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s) \xrightarrow{\cong}$$

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s)$$

Our next step is to construct homomorphisms between groups with CH_0-type and A_0-type arguments.

Proposition 2.7. *Suppose $1 \leq j_0 \leq r$, and that Z is a smooth projective variety defined over k admitting a zero-cycle z of degree 1. For convenience of notation, set*

$$K_{CH} = K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q), \mathcal{CH}_0(Z); \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s)$$

and

$$K_A = K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Z), \mathcal{A}_0(Y_1), \ldots, \mathcal{A}_0(Y_r); G_1, \ldots, G_s)$$

Then there are natural homomorphisms $j : K_A \longrightarrow K_{CH}$ and $p_z : K_{CH} \longrightarrow K_A$ defined by the formulae:

$$j(\{x_1, \ldots, x_q; v, y_1, \ldots, y_r; g_1, \ldots g_s\}_{E/k}) =$$

$$\{x_1, \ldots, x_q; v, y_1, \ldots, y_r; g_1, \ldots, g_s\}_{E/k}$$

and

$$p_z(\{x_1,\ldots,x_q;w,y_1,\ldots,y_r;g_1,\ldots,g_s\}_{E/k}) =$$

$$\{x_1,\ldots,x_q;w-deg\ _E(w)\phi^*_{E/k}(z);y_1,\ldots,y_r;g_1,\ldots,g_s\}_{E/k}$$

where $\phi_{E/k}: k\hookrightarrow E$ is the inclusion.

The fundamental result we are after is the following:

Proposition 2.8. *With notation as in the previous proposition, the map j is a split injection (with splitting p_z) inducing a split exact sequence:*

$$0\longrightarrow K_A \xrightarrow{j} K_{CH}\longrightarrow K_{\mathbf{Z}}\longrightarrow 0$$

where

$$K_{\mathbf{Z}} = K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_q);\mathcal{A}_0(Y_1),\ldots,\mathcal{A}_0(Y_r);G_1,\ldots,G_s)$$

The next statement follows by induction on the previous proposition:

Corollary 2.9. *With k, X_i, Y_i and G_i as before, let $Z_1,\ldots,Z_t$ be smooth projective varieties defined over k, each admitting a zero-cycle of degree 1. To ease notation, write $K_{i_1,\ldots,i_\nu}$ for the group*

$$K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_q);\mathcal{A}_0(Z_{i_1}),\ldots,\mathcal{A}_0(Z_{i_\nu}),\mathcal{A}_0(Y_1),\ldots,\mathcal{A}_0(Y_r);G_1,\ldots,G_s)$$

Then there is a natural map inducing an isomorphism:

$$K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_q),\mathcal{CH}_0(Z_1),\ldots,\mathcal{CH}_0(Z_t);\mathcal{A}_0(Y_1),\ldots,\mathcal{A}_0(Y_r);G_1,\ldots,G_s)$$

$$\xrightarrow{\cong}\bigoplus_{0\le\nu\le t}\ \bigoplus_{1\le i_1<\ldots<i_\nu\le r} K_{i_1,\ldots,i_\nu}$$

The following result appears as part of the proof of [RS], Theorem 2.4. We recover it directly from the above without explicit mention of motives; all we need is the (functorial) identification of the group of zero-cycles of degree zero on a smooth projective curve with the group of rational points on its Jacobian.

Corollary 2.10. *Let $C_1,\ldots,C_t$ be smooth projective curves over a field k, each admitting a zero-cycle of degree 1. Let J_i denote their respective Jacobians. Then there is an isomorphism:*

$$K(k;\mathcal{CH}_0(C_1),\ldots,\mathcal{CH}_0(C_t))\cong\bigoplus_{0\le\nu\le t}\ \bigoplus_{1\le i_1<\ldots<i_\nu\le t} K(k;J_{i_1},\ldots,J_{i_t})$$

3 Zero-cycles over finite fields

Our main result is:

Theorem 3.1. *Let X be a smooth projective variety of dimension d over a finite field with structure morphism $\sigma : X \longrightarrow Spec\ k$. Then the map*

$$\sigma_* : CH^{d+1}(X, 1) \longrightarrow CH^1(Spec\ k, 1)$$

induces an isomorphism of $CH^{d+1}(X, 1)$ with $CH^1(Spec\ k, 1) \cong k^*$.

We begin with an overview of some established results:

Theorem 3.2. *(Steinberg, [Mi]) Let k be a finite field. For $s \geq 2$, $K_s^M(k) = 0$.*

The analogue for Kato-Somekawa groups is:

Theorem 3.3. *(Kahn, [Kahn]) Let k be a finite field and $G_1, \ldots, G_s$ semi-abelian varieties defined over k. Then for $s \geq 2$, $K(k; G_1, \ldots, G_s) = 0$.*

We also recall the main theorem of [A2]:

Theorem 3.4. *Let k be a field, $s \geq 0$ an integer, and X a smooth quasiprojective variety of dimension d defined over k. Then there is a map inducing an isomorphism:*

$$\alpha = \alpha_X : CH^{d+s}(X, s) \xrightarrow{\cong} K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$$

An important result of Lang and Weil is of special relevance to our situation.

Theorem 3.5. *([Sou], 1.5.3 Lemme 1) Let $\sigma : X \longrightarrow Spec\ k$ be a smooth projective variety where k is a finite field. Then X admits a zero-cycle of degree 1.*

With notation as in Theorem 3.5, let z be a zero-cycle of degree 1 on X. Furthermore, let $d = \dim X$ and $s \geq 0$ be a integer. Using the product structure on the higher Chow groups, we may define a map $B_z : CH^s(k, s) \longrightarrow CH^{d+s}(X, s)$ by $B_z(x) = z \times x$. Evidently we have $\sigma_* \circ B_z = id$, so B_z furnishes a splitting of σ_*.

Having fixed a choice of zero-cycle z, we have the following consequence of Theorem 3.4.

Corollary 3.6. *Let X be a smooth projective variety of dimension d defined over a field k such that X admits a zero-cycle of degree 1. Let $s \geq 0$ be an integer. Then all squares in the diagram below commute, the map on kernels being induced by the vertical maps:*

$$\begin{tikzcd}
0 \arrow[r] & Ker\ (\sigma_*) \arrow[r] \arrow[d] & CH^{d+s}(X,s) \arrow[r, "\sigma_*"] \arrow[d, "\alpha_X"] & CH^s(k,s) \arrow[r] \arrow[d, "\alpha_k"] & 0 \\
0 \arrow[r] & K_s(k;\mathcal{A}_0(X);\mathbf{G}_m) \arrow[r] & K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m) \arrow[r, "\sigma_*"] & K_s(k;\mathbf{G}_m) \arrow[r] & 0
\end{tikzcd}$$

Furthermore, the vertical arrows are isomorphisms and both the top and bottom rows are split exact.

Note in particular that (using Theorem 3.5) Corollary 3.6 applies in the case that k is a finite field.

We are now in a position to prove Theorem 3.1. By the preceding Corollary, it suffices to prove that the group $K(k;\mathcal{A}_0(X);\mathbf{G}_m)$ is zero. We will prove a more general result:

Proposition 3.7. *Let k be a finite field. Suppose $X_1,\dots,X_r$ are smooth projective varieties defined over k and $G_1,\dots,G_s$ semi-abelian varieties defined over k. If $r+s \geq 2$, then*

$$K(k;\mathcal{A}_0(X_1),\dots,\mathcal{A}_0(X_r);G_1,\dots,G_s) = 0.$$

Before embarking on the proof of Proposition 3.7, it is worth noting that when $X_1,\dots,X_r$ are curves, the desired result follows from Theorem 3.3. In this context, the group $K(k;\mathcal{A}_0(X_1),\dots,\mathcal{A}_0(X_r);G_1,\dots,G_s)$ may be identified with the (Kato-Somekawa) group $K(k;J_1,\dots,J_r,G_1,\dots,G_s)$ (where J_i is the Jacobian of X_1), which is known to vanish.

In the general case, we will use the following Bertini-type theorem of Poonen to reduce our computation to the (known) case of curves.

Theorem 3.8. *(Poonen [Po], Corollary 3.4)*
Let X be a smooth, projective, geometrically integral variety of dimension $m \geq 1$ over a finite field. Suppose $F \subseteq X$ is a finite set of points and y an integer $1 \leq y \leq m$. Then there exists a smooth, projective, geometrically integral subvariety $Y \subseteq X$ of dimension y such that $F \subseteq Y$.

Proof of Proposition 3.7.

When $r = 0$, this theorem is simply Theorem 3.3; we therefore assume $r > 0$. Pick a generator $g = \{z_1,\dots,z_r,a_1,\dots,a_s\}_{E/k} \in K(k;\mathcal{A}_0(X_1),\dots,\mathcal{A}_0(X_r);G_1,\dots,G_s)$, where $\phi : k \hookrightarrow E$ is a finite field extension, $z_i \in A_0((X_i)_E)$ for $i = 1,\dots,r$ and

$a_j \in A_0((G_j)_E)$ for $j = 1, \ldots, s$. Evidently g lies in the image of the "norm" map (cf. Proposition 2.2)

$$\begin{aligned}\phi^* : K(E; \mathcal{A}_0((X_1)_E), \ldots, \mathcal{A}_0((X_r)_E); (G_1)_E, \ldots, (G_s)_E) \\ \longrightarrow K(k; \mathcal{A}_0(X_1), \ldots, \mathcal{A}_0(X_r); G_1, \ldots, G_s)\end{aligned}$$

Thus to prove that $g = 0$, we may assume that $E = k$.

For each i, $1 \le i \le r$, z_i is a zero-cycle of degree 0 on X_i; z_i represents the class (in $CH_0(X_i)$) of a formal sum $\sum n_{i_j} P_{i_j}$, where P_{i_j} runs through all closed points of X_i and all but finitely many of the n_{i_j} are zero. For each such i, set

$$F_i = \{P_{i_j} : n_{i_j} \neq 0\}.$$

By Theorem 3.8, there is a curve $C_i \subseteq X_i$ containing the set F_i. Thus the element $g = \{z_1, \ldots, z_r, a_1, \ldots, a_s\}_{k/k}$ is in the image of the natural map (cf. Proposition 2.4)

$$K(k; \mathcal{A}_0(C_1), \ldots, \mathcal{A}_0(C_r); G_1, \ldots, G_s) \longrightarrow K(k; \mathcal{A}_0(X_1), \ldots, \mathcal{A}_0(X_r); G_1, \ldots, G_s)$$

induced by the various inclusions $C_i \hookrightarrow X_i$. Since the group on the left is zero by Theorem 3.3, this forces g to be zero, as desired.

4 One-dimensional cycles

It is natural to ask whether the techniques of the previous section may be extended to study higher-dimensional cycles on varieties over a finite field. This does not seem likely: one of the key facts implicit in the above treatment is that (at least in the case of an algebraically closed base field) every element (zero-cycle) of $CH^{d+s}(X, s)$ is decomposable, i.e. lies in the image of the product map

$$CH_0(X) \otimes (k^*)^{\otimes s} = CH^d(X) \otimes CH^1(k, 1)^{\otimes s} \longrightarrow CH^{d+s}(X, s)$$

where the notation is as before. For higher-dimensional cycles, it is simply not true that every cycle is thus decomposable. Nevertheless, it is possible to deduce certain results of a weaker nature. The following result is due (in a different guise) to Bass and Tate [BT]:

Proposition 4.1. *Let k be a finite field and C a smooth curve defined over k. Then $CH^2(C, 2)$ is finite.*

Proof.
Let K be the function field of C. A slightly generalized version of Bloch's localization theorem ([A2], Corollary 4.4) yields an exact sequence

$$\bigoplus_{z\in C^1} CH^1(k(z),2) \longrightarrow CH^2(C,2) \longrightarrow CH^2(K,2) \xrightarrow{\partial} \bigoplus_{z\in C^1} CH^1(k(z),1) \longrightarrow \dots$$

Each group in the direct sum on the left vanishes by [Bl3], Lemma 6.3; furthermore, the following diagram commutes, in which the vertical arrows are the isomorphisms of [To], Theorem 1, and the lower horizontal arrow ∂^M is the boundary map of Milnor K-groups, cf. [A2], 2.2.

$$\begin{tikzcd} CH^2(K,2) \arrow[r, "\partial"] \arrow[d, "\cong"] & \bigoplus_{z\in C^1} CH^1(k(z),1) \arrow[d, "\cong"] \\ K_2^M(K) \arrow[r, "\partial^M"] & \bigoplus_{z\in C^1} k(z)^* \end{tikzcd}$$

Thus we may identify $CH^2(C,2)$ with Ker $(\partial^M : K_2^M(K) \longrightarrow \bigoplus_{z\in C^1} k(z)^*)$, which is finite by [BT], II. Theorem 2.1.

A more general result may be deduced using the coniveau spectral sequence for higher Chow groups introduced by Bloch in [Bl3]. The second assertion below was added subsequent to a suggestion made by the referee.

Proposition 4.2. *Let k be a finite field and X a smooth quasiprojective variety of dimension d over k.*

1. *When $s \geq 2$, the group $CH^{d+s-1}(X,s)$ of 1-cycles is a torsion group.*
2. *When $s \geq 3$, $CH^{d+s-1}(X,s)$ is ℓ-divisible, where $\ell =$ char k. If X is a curve, $CH^{d+s-1}(X,s)$ is uniquely ℓ-divisible.*

Proof.
The coniveau spectral sequence reads:

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{CH}^{d+s-1}(-q)) \Longrightarrow CH^{d+s-1}(X,-p-q)$$

We are interested in the case $-p-q = s$, or $q = -s-p$. To prove that $CH^{d+s-1}(X,s)$ is torsion, it is sufficient to show that the terms $E_\infty^{p,-s-p}$ are torsion for all p. Furthermore, since $E_{n+1}^{p,q}$ is a subquotient of $E_n^{p,q}$ for all p, q, and n, it is sufficient to show that

$$E_2^{p,-s-p} = H^p_{Zar}(X, \mathcal{CH}^{d+s-1}(p+s))$$

is torsion for all p.

If $p > d$, this (cohomology) group is zero for reasons of dimension. If $p < d-1$, the Zariski sheaf $\mathcal{CH}^{d+s-1}(p+s)$ is zero by the Gersten resolution for higher Chow groups ([Bl3], Theorem 10.1). It remains to consider the cases $p = d-1$ and $p = d$. When $p = d-1$, we have $E_2^{d-1,-s-d+1} = H^{d-1}_{Zar}(X, \mathcal{CH}^{d+s-1}(d+s-1))$. By the Gersten resolution, this is a subquotient of $\bigoplus_{x \in X^{d-1}} CH^s(k(x), s) = \bigoplus_{x \in X^{d-1}} K_s^M(k(x))$; however, since $s \geq 2$ and $k(x)$ is a global field (of positive characteristic), this latter group (and hence its subquotient, too) is a torsion group by [BT], II. Theorem 2.1. In the case $p = d$, the group to consider is $E_2^{d,-s-d} = H^d_{Zar}(X, \mathcal{CH}^{d+s-1}(d+s))$. By the Gersten resolution, this is a subquotient of $\bigoplus_{x \in X^d} CH^{s-1}(k(x), s)$. Each term $CH^{s-1}(k(x), s)$ is, modulo torsion, isomorphic via the Riemann-Roch Theorem for higher algebraic K-theory ([Bl3]) to a direct summand of the (algebraic) K-group $K_s(k(x))$. Since $k(x)$ is a finite field and $s \geq 2$, the latter group is torsion by Quillen's computations of the K-theory of finite fields [Qu]. Thus the term $E_2^{d,-s-d}$ is itself torsion.

For the second assertion, it suffices, as above, to study the terms $E_2^{d-1,-s-d+1}$ and $E_2^{d,-s-d}$. Note that when $s \geq 3$, $K_s^M(k(x)) = 0$ for $x \in X^{d-1}$ ([BT], II. Theorem 2.1), so $E_2^{d-1,-s-d+1} = E_\infty^{d-1,-s-d+1}$ is zero. To analyze the term $E_2^{d,-s-d}$, we need the following result of Geisser and Levine:

Theorem 4.3. *(Geisser-Levine, [GL]) If F be a field of characteristic ℓ. Then the motivic cohomology group $H^i(k, \mathbf{Z}(n))$ is uniquely ℓ-divisible when $i \neq n$.*

In terms of motivic cohomology, $CH^r(X, s) \cong H^{2r-s}(X, \mathbf{Z}(r))$, so

$$E_2^{d,-s-d} \cong \frac{\bigoplus_{x \in X^d} CH^{s-1}(k(x), s)}{\partial(\bigoplus_{y \in X^{d-1}} CH^s(k(y), s+1))} \cong \frac{\bigoplus_{x \in X^d} H^{s-2}(k(x), \mathbf{Z}(s-1))}{\partial(\bigoplus_{y \in X^{d-1}} H^{s-1}(k(y), \mathbf{Z}(s)))}$$

is uniquely ℓ-divisible. Next,

$$E_{r+1}^{d,-s-d} \cong \frac{\mathrm{Ker}\ (E_r^{d,-s-d} \longrightarrow E_r^{d+r,-s-d-r+1})}{\mathrm{Im}\ (E_r^{d-r,-s-d+r-1} \longrightarrow E_r^{d,-s-d})} = \frac{E_r^{d,-s-d}}{\mathrm{Im}\ (E_r^{d-r,-s-d+r-1} \longrightarrow E_r^{d,-s-d})}$$

From this formula, it is evident that $E_\infty^{d,-s-d}$ is divisible.

If X is a curve, then $d = 1$ and the term $E_2^{d-2,-s-d+r-1}$ is zero for all $r \geq 2$; thus, $E_\infty^{d,-s-d} = E_2^{d,-s-d}$ is uniquely divisible. This concludes the proof of Proposition 4.2.

The case $s = 1$ has been excluded from the statement of Proposition 4.2; however, one still expect the group $CH^d(X, 1)$ to be torsion when X is *projective* (and smooth). For curves this is clear, as we have $CH^1(X, 1) \cong k^*$, which is finite. A conjecture due to Parshin asserts that for $i \geq 1$, the groups $K_i(X)$ (and hence also $CH^j(X, i)$) are torsion for X a smooth projective variety defined over a finite field. The validity of this conjecture is known when X is a member of a special class of varieties of dimension at most 3 including products of curves, abelian varieties, unirational varieties and Fermat curves [Sou]. However, even in the case of a general surface X, we do not yet know that $CH^2(X,1)$ is torsion. Such a result would be of interest, as it would automatically imply that $CH^2(X,1)$ were *finite*: using a spectral sequence argument (cf. [MSEV], Corollary 5.2) we may identify $CH^2(X,1)$ with $H^1_{Zar}(X,\mathcal{K}_2)$, and the latter is known by a theorem of Gros and Suwa ([GS], Théorème 4.19) to be an extension of a finite group by a uniquely divisible group.

Appendix: Alternate proof of Theorem 3.1

As noted in Section 3, it suffices to prove that $K(k;\mathcal{A}_0(X);\mathbf{G}_m)=0$.
This follows readily from the next Lemma; the idea for the proof is based on that of Kahn [Kahn].

Lemma 4.4. *For convenience of notation, let $A=K(k;\mathcal{A}_0(X);\mathbf{G}_m)$.*

1. *A is torsion*
2. *A is divisible*
3. *For every prime l, there exists an integer $m_l>0$ which annihilates the l-primary torsion subgroup $A\{l\}\subseteq A$.*

Proof.
Throughout this proof we reserve the notation $\phi_{E/k}$ for the map Spec $E\longrightarrow$ Spec k associated to a field extension E/k.

Since A is a quotient of $\bigoplus_{E/k \text{ finite}} A_0(X_E)\otimes E^*$ and E^* is a finite group for each E/k, A must be torsion.

Now let $\{x,y\}_{E/k}$ be a homogeneous element of the group A and suppose $n>0$ is an integer. Since $x\in A_0(X_E)$ and $A_0(\bar{X})$ is divisible by a result of Bloch ([Bl2], Lemma 1.3), we may choose a finite extension E'/E such that $\phi^*_{E'/E}(x)=nx'$ for some $x'=A_0(X_{E'})$. Since E is a finite field, it is C^1 (cf. [Se], 10.7), which implies that the field norm map $F^*\longrightarrow E^*$ is surjective for any finite extension F/E. Thus we may choose $z\in(E')^*$ such that $N_{E'/E}(z)=y$. By a relation of type **M1**, we have

$$\{x,y\}_{E/k}=\{x,N_{E'/E}(z)\}_{E/k}=\{nx',z\}_{F/k}=n\{x',z\}_{F/k}$$

and so A is divisible.

Towards a proof of the third assertion, suppose $l = p = \text{char } k$. Then a typical generator of A is a symbol $\{x, y\}_{E/k}$, where $y = E^*$. Since y has order prime to p, so does $\{x, y\}_{E/k}$; hence, $A\{p\} = 0$.

To treat the case $l \neq \text{char } k$, we will need to make use of a number of powerful results, which we collect together in the next theorem. The proof continues afterward, via a sequence of lemmas.

Theorem 4.5. *1. (Kato-Saito, [KS])*

Let F be a finite field and X a smooth projective variety defined over F. Then $A_0(X)$ is a finite group.

2. (Weil conjectures)

Let F be a finite field and A an abelian variety defined over F. Let $\bar{F}$ denote the algebraic closure of F and $\bar{A}$ the extension of A to $\bar{F}$. Suppose l is a prime not equal to char F. Then the eigenvalues of the action of the Frobenius automorphism $\phi \in Gal\ (\bar{F}/F)$ on $\mathbf{T}_l(\bar{A})$ are algebraic integers of absolute value greater than 1.

3. (Rojtman-Bloch, [Ro], [Bl1])

Let L be an algebraically closed field and X a smooth projective variety defined over L. Then the Albanese map $A_0(X) \longrightarrow Alb(X)(L)$ induces an isomorphism on torsion subgroups.

Lemma 4.6. *The group $A = K(k; \mathcal{A}_0(X); \mathbf{G}_m)$ is a quotient of $\bigoplus_{E/k \text{ finite}} H_0(Gal\ (E/k), A_0(X_E) \otimes E^*)$.*

Proof.
By construction, A is a quotient of $\bigoplus_{E/k \text{ finite}} A_0(X_E) \otimes E^*$. Furthermore, for every $x \in A_0(X_E)$, $y \in E^*$ and every automorphism $\sigma : E \longrightarrow E$ of E over k, the element $x \otimes y = x \otimes \sigma_*(\sigma^*(y))$ is equivalent in A (modulo a relation of type **M1**) to $\sigma^*(x) \otimes \sigma^*(y)$. These are exactly the relations by which one defines the zeroth homology group.

Lemma 4.7. *Let $G = Gal\ (\bar{k}/k)$. Then $H_0(G, \mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m))$ is a finite group.*

Proof.
First, we claim that the Frobenius automorphism $\phi \in G$ acts (diagonally) without fixed points on $\mathbf{T}(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}(\mathbf{G}_m)$. To analyze the action of ϕ on the first

factor, we need to use Theorem 4.5, part 3, which gives a functorial isomorphism $A_0(\bar{X})[l^n] \cong \mathrm{Alb}(\bar{X})(\bar{k})[l^n]$ for all n. Taking inverse limits, we have a natural isomorphism $\mathbf{T}(A_0(\bar{X})) \cong \mathbf{T}(\mathrm{Alb}(\bar{X}))$. The action of ϕ on $\mathbf{T}(\mathbf{G}_m) \cong \mathbf{Z}_l$ is simply multiplication by l. Thus, the Weil conjectures (Theorem 4.5) show that the eigenvalues of the action of ϕ on $M = \mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m)$ are of absolute value greater than 1; therefore, the action has no fixed points.

Therefore, $\phi - id$ is an invertible $\mathbf{Q}_l$-linear automorphism of the $\mathbf{Q}_l$-vector space $M \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$. Since $H_0(G, M) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ is a quotient of $\dfrac{M}{(\phi - id)M} \otimes_{\mathbf{Z}_l} \mathbf{Q}_l = 0$, we conclude that $H_0(G, M) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ must itself be 0. Hence $H_0(G, M)$ is a torsion $\mathbf{Z}_l$-module; furthermore, being a quotient of a finitely generated $\mathbf{Z}_l$-module, it is finitely generated. We conclude that $H_0(G, M)$ is finite.

Lemma 4.8. *For every finite extension E/k, $(A_0(X_E) \otimes E^*)\{l\}$ is a quotient of a finite number of copies of M.*

Proof.
Clearly E^* is a finite group, and $A_0(X_E)$ is finite by Theorem 4.5, Part 1. Thus $A_0(X_E)\{l\}$ be may written as a finite direct sum $\mathbf{Z}/l^{c_1}\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/l^{c_m}\mathbf{Z}$ of cyclic l-groups. By choosing $a \geq \max(c_1, \ldots, c_m)$ and n sufficiently large, we may thus realize $A_0(X_E)\{l\}$ as a subgroup of $((\mathbf{Z}/l^a\mathbf{Z})^{2g})^n \cong (\mathrm{Alb}(\bar{X})[l^a])^n \cong (A_0(\bar{X})[l^a])^n$. Using duality, $A_0(X_E)\{l\}$ may also be realized as a quotient of $(A_0(\bar{X})[l^a])^n$; since the latter is a quotient of $\mathbf{T}(A_0(\bar{X}))^n$, we conclude that $A_0(X_E)\{l\}$ is a quotient of some power of $\mathbf{T}_l(A_0(\bar{X}))$.

An analogous (simpler) argument shows that $E^*\{l\}$ is a quotient of some power of $\mathbf{T}_l(\mathbf{G}_m)$. Hence, we conclude that $A_0(X_E)\{l\} \otimes_{\mathbf{Z}_l} E^*\{l\} \cong (A_0(X_E) \otimes_{\mathbf{Z}_l} E^*)\{l\}$ is a quotient of some power of $M = \mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m)$.

Finally, since $A_0(X_E)\{l\}$ and $E^*\{l\}$ are finite abelian l-groups, their tensor product as $\mathbf{Z}_l$-modules coincides (as an abelian group) with their tensor product as $\mathbf{Z}$-modules; therefore we conclude that $(A_0(X_E) \otimes E^*)\{l\}$ is a quotient of some power of M.

Lemma 4.9. *There exists an integer m_l which kills $\bigoplus_{E/k \text{ finite}} H_0(G_{E/k}, A_0(X_E) \otimes E^*)\{l\}$.*

From Lemma 4.8 and the right exactness of the functor H_0, we deduce the existence of a surjective map $H_0(G, M^n) \longrightarrow H_0(G, (A_0(X_E) \otimes E^*)\{l\})$. By Shapiro's Lemma we may identify the latter group with $H_0(\mathrm{Gal}\ (E/k), (A_0(X_E) \otimes E^*)\{l\}) = H_0(\mathrm{Gal}\ (E/k), A_0(X_E) \otimes E^*)\{l\}$.

Set $m_l = \#H_0(G, M)$. By Lemma 4.7 and the fact that H_0 commutes with direct sums, m_l kills $H_0(G, M^n)$, and therefore (by the above argument) m_l kills $H_0(\text{Gal }(E/k), A_0(X_E) \otimes E^*)\{l\}$ for *any* finite extension E/k. We conclude that m_l kills the direct sum in the statement of the lemma.

The proof of Lemma 4.4 is finished by the next and last lemma:

Lemma 4.10. *$A\{l\}$ is killed by m_l.*

Fix $a \in A\{l\}$. Then a is represented by some element $a' \in H_0(\text{Gal }(E/k), A_0(X_E) \otimes E^*)$ where E/k is some finite extension. If the order of a^{prime} is prime to l, then the order of a (which divides the order of a') must also be prime to l. The assumption that $a \in A\{l\}$ forces $a = 0$, which is trivially killed by m_l. If l divides the order of a', then for some $c > 1$, $\gcd(c, l) = 1$, $ca' \in H_0(\text{Gal }(E/k), A_0(X_E) \otimes E^*)\{l\}$; by Lemma 4.9, $m_l ca' = 0$, and hence $m_l ca = 0$. Finally, the assumption that a is killed by some power l^b enables us to conclude that a is killed by $g = \gcd(l^b, m_l c)$. Since $g|l$, $g|m_l c$, and $gcd(c, l) = 1$, this forces $g|m_l$. Thus a is killed by m_l.

This concludes the proof of the third assertion of Lemma 4.4 and hence also of Theorem 3.1.

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