

A mod- ℓ vanishing theorem of Beilinson-Soulé type

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Abstract

Let L be a field containing an algebraically closed field and X an equidimensional quasiprojective scheme over L . We prove that $CH^i(X, n; \mathbb{Z}/\ell) = 0$ when $n > 2i$ and $\ell \neq 0$; this was known previously when $i \geq \dim X$ and L is itself algebraically closed. This “mod- ℓ ” version of the Beilinson-Soulé conjecture implies the equivalence of the rational and integral versions of the conjecture for varieties over fields of this type and can be used to prove the vanishing of the (integral) groups $CH^i(X, n)$ (for $n > 2i$) in certain cases.

1 Introduction

Let k be a field and X a smooth scheme over k . The Beilinson-Soulé conjecture asserts that the motivic cohomology groups $H^p(X, \mathbb{Z}(q))$ vanish when $p < 0$; by the work of Suslin and Voevodsky (see [V2]), this is equivalent to requiring that the higher Chow groups $CH^i(X, n)$ vanish when $n > 2i$. It follows from the definition and from the calculations in [B1], Section 7 that the conjecture holds for $i \leq 1$. While the conjecture itself seems very difficult, an analogous conjecture with finite coefficients appears somewhat more tractable; indeed, the work of Suslin [Su2], Geisser [G], and Geisser-Levine [GL] implies immediately that if X is an equidimensional quasiprojective scheme over an algebraically closed field, then $CH^i(X, n; \mathbb{Z}/\ell) = 0$ when $i \geq \dim X$, $n > 2i$, and $\ell \neq 0$.

We extend this vanishing assertion, removing the hypothesis $i \geq \dim X$ and relaxing the hypothesis on k – requiring only that k contain an algebraically closed field. Our main result is:

Theorem.

Let L be a field containing an algebraically closed field, and X an equidimensional quasiprojective scheme over L . Then $CH^i(X, n; \mathbb{Z}/\ell) = 0$ if $\ell \neq 0$ and $n > 2i$, and $CH^i(X, 2i; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^r$, where r is the number of irreducible components of X .

As noted by Voevodsky in the introduction to [V1], this result is a consequence of the Bloch-Kato Conjecture, a proof of which has recently been announced by Voevodsky and Rost. In contrast, the proof of our result is quite elementary; it involves a generalization of Suslin’s rigidity theorem [Su1] due to Panin and Yagunov [PY] and employs reasoning inspired by a result of Coombes [C]. We also show in Corollary 3.4 that $CH^i(X, n)$ is uniquely divisible when $i \geq 1$ and $n \geq 2i$ and hence that it suffices to know that $CH^i(X, n) \otimes \mathbb{Q} = 0$ to deduce $CH^i(X, n) = 0$. Soulé [So] has proved that $CH^i(X, n) \otimes \mathbb{Q} = 0$ when X belongs to a special class of varieties defined over the algebraic closure of a finite field; this class includes products of up to three curves, abelian surfaces, abelian threefolds, and unirational varieties of dimension ≤ 3 and degree ≥ 3 .

Throughout this paper, we use the notation ${}_nA$ for the n -torsion subgroup of an abelian group and A/n for its n -cotorsion. If R is a ring of geometric type, we often write $CH^*(R, \cdot)$ instead of $CH^*(\text{Spec } R, \cdot)$, and similarly for K -theory. If X is an integral scheme over a field L , we write $L(X)$ for its field of rational functions, i.e. the local ring at the generic point of X . If X is a scheme over a ring A and B is an A -algebra, we write X_B for the base extension $X \times_{\text{Spec } A} \text{Spec } B$. If $k \subseteq L$ is an extension of fields and X is a k -scheme, we write $\iota_{L/k}(X) : X_L \rightarrow X$ for the morphism of schemes obtained by base extension.

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2 Preliminaries

It follows from the definition of the higher Chow groups that for any algebraic scheme X over a field k , $CH^*(X, \cdot) \cong CH^*(X_{red}, \cdot)$. We therefore assume for the rest of this article that *all schemes are reduced*.

In [Su1] Suslin proved the Quillen-Lichtenbaum conjecture for algebraically closed fields by means of a “rigidity” theorem for K' -theory with coefficients. In particular, it follows from his work that if $k \hookrightarrow \Omega$ is an extension of algebraically closed fields and X is a variety over k , then for ℓ relatively prime to $\text{char } k$, the base change map $K'(X; \mathbb{Z}/\ell) \rightarrow K'(X_\Omega; \mathbb{Z}/\ell)$ is an isomorphism. Noting that Suslin uses various formal properties of the K' -theory functor in his argument (but nothing intrinsic to the definition of K' -theory), Panin and Yagunov [PY] axiomatized Suslin’s proof to extend the scope of his result. In particular, they proved that certain homotopy invariant functors on the category Sm/k of smooth varieties over k possessing “weak

transfers” satisfy an analogue of rigidity. The methods they use are related to those employed by Suslin and Voevodsky in Section 4 of [SV]; see also [MVW], Section 7.20.

Theorem 2.1. *(Panin-Yagunov, [PY], Theorem 1.14) Let $k \hookrightarrow \Omega$ be an extension of algebraically closed fields and $\mathcal{F} : Sm/k \rightarrow Ab$ a contravariant, homotopy invariant functor with weak transfers for the class of finite projective morphisms. Let ℓ be an integer relatively prime to $\text{char } k$ such that for every Y , $\ell\mathcal{F}(Y) = 0$. Then the natural map $\Omega \rightarrow \text{Spec } k$ induces an isomorphism $i^* : \mathcal{F}(\text{Spec } k) \xrightarrow{\cong} \mathcal{F}(\text{Spec } \Omega)$.*

Using well-known properties of higher Chow groups, it is a matter of routine to check that for any quasiprojective scheme X over k , the functor $\mathcal{F} : Sm/k \rightarrow Ab$ defined by $\mathcal{F}(Y) = CH^i(X \times_k Y, n; \mathbb{Z}/\ell)$ is indeed a contravariant, homotopy invariant functor with weak transfers for the class of finite projective morphisms.

We therefore deduce the following:

Theorem 2.2. *Let $i : k \hookrightarrow \Omega$ be an extension of algebraically closed fields and X any quasiprojective scheme over k . Let $\ell \neq 0$ be any integer prime to $\text{char } k$. Then $i^* : CH^*(X, \cdot; \mathbb{Z}/\ell) \rightarrow CH^*(X_\Omega, \cdot; \mathbb{Z}/\ell)$ is an isomorphism.*

Since the localization sequence will be used repeatedly in the sequel, we state it here for reference. In the following two lemmas, A is understood to be any abelian group.

Lemma 2.3. *(Bloch, [B2])*

Let k be a field, X an equidimensional quasiprojective scheme over k . Let $Z \subseteq X$ be a closed subscheme of codimension c , and let $U = X - Z$. Then for all $i, n \in \mathbb{Z}$, there is a long exact sequence:

$$\dots \rightarrow CH^{i-c}(Z, n; A) \rightarrow CH^i(X, n; A) \rightarrow CH^i(U, n; A) \rightarrow CH^{i-c}(Z, n-1; A) \rightarrow \dots$$

By taking direct limits, one easily obtains the following:

Lemma 2.4. *Let X be an quasiprojective variety over k . The directed system of dense open sets $U \subseteq X$ with complement $Z = X - U$ of pure codimension 1 in X is cofinal in the directed system of all dense open sets in X . In particular, if $CH^{i-1}(Z, n; A) = CH^{i-1}(Z, n-1; A) = 0$, then $CH^i(X, n; A) \cong CH^i(k(X), n; A)$.*

The next result is due to Suslin in characteristic 0 and Geisser in arbitrary characteristic. Applied to the case $X = \text{Spec } k$, it will be a crucial ingredient in the proof of Theorem 3.1.

Theorem 2.5. ([Su2], Corollary 4.3 and [G], Corollary 3.6) *Let k be an algebraically closed field and X an equidimensional quasiprojective scheme over k . Then for any $i \geq \dim X$ and $\ell \neq 0$ relatively prime to $\text{char } k$,*

$$CH^i(X, n; \mathbb{Z}/\ell) = H_c^{2(d-i)+n}(X, \mathbb{Z}/\ell(d-i))^\vee.$$

If X is smooth, then

$$CH^i(X, n; \mathbb{Z}/\ell) = H_{\text{ét}}^{2i-n}(X, \mathbb{Z}/\ell(i)).$$

In fields of positive characteristic, the work of Geisser and Levine enables us to take care of the remaining case:

Theorem 2.6. ([GL], Theorem 1.1) *Let k be a field of characteristic $p > 0$. Then for $n \neq i$ and any $r \geq 1$, $CH^i(k, n; \mathbb{Z}/p^r) = 0$.*

Corollary 2.7. *Let k be a field of characteristic $p > 0$ and X any quasiprojective scheme over k . Then for $n > i$ and any $r \geq 1$, $CH^i(X, n; \mathbb{Z}/p^r) = 0$.*

Proof.

Let $\ell = p^r$. The proof proceeds by induction on $d = \dim X$, the case $d = 0$ following from Theorem 2.6. Now suppose $d > 0$, and let X_1, \dots, X_r be the irreducible components of X . If Z is some subscheme of codimension 1 in X which contains $\cup_{j \neq k} X_j \cap X_k$, then $U = X - Z$ is a disjoint union of irreducible open subschemes U_j , $1 \leq j \leq r$. By induction, $CH^{i-1}(Z, n; \mathbb{Z}/\ell) = CH^{i-1}(Z, n-1; \mathbb{Z}/\ell) = 0$. By Lemmas 2.3 and 2.4, $CH^i(X, n; \mathbb{Z}/\ell) \cong CH^i(U, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(U_j, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(k(U_j), n; \mathbb{Z}/\ell)$.

Now each $k(U_j)$ is a field of characteristic p , so the expression on the far right vanishes by Theorem 2.6.

3 The Main Theorem

Although ultimately we will consider varieties over fields containing an algebraically closed field, the crucial part of the argument involves varieties over algebraically closed fields, as made precise in the following theorem.

Theorem 3.1. *Let k be an algebraically closed field and X an equidimensional quasiprojective scheme over k .*

- *If $n > 2i$ and $\ell \neq 0$, then $CH^i(X, n; \mathbb{Z}/\ell) = 0$.*

- If X is irreducible, then

$$CH^i(X, n; \mathbb{Z}/\ell) = CH^i(k(X), n; \mathbb{Z}/\ell) = 0$$

when $n > 2i$. If moreover $\text{char } k \nmid \ell$, then

$$CH^i(X, 2i; \mathbb{Z}/\ell) \cong CH^i(k(X), 2i; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell.$$

In general, if X_1, \dots, X_r are the irreducible components of X , then for $n \geq 2i$ and ℓ relatively prime to $\text{char } k$,

$$CH^i(X, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(X_j, n; \mathbb{Z}/\ell).$$

Proof of Theorem 3.1

By Corollary 2.7, we may assume without loss of generality that ℓ is relatively prime to $\text{char } k$. Moreover, if $X = \text{Spec } k$, the result follows immediately from Theorem 2.5 and the vanishing of étale cohomology in negative degrees. We assume henceforth that $\dim X > 0$ and let $\sigma : X \rightarrow \text{Spec } k$ denote the structure map.

We prove both assertions of the theorem by induction on i . To dispense with the case $i = 0$, it follows readily from the definition that $CH^0(X, n; \mathbb{Z}/\ell) = 0$ when $n > 0$ and that $CH^0(X, 0; \mathbb{Z}/\ell) = (\mathbb{Z}/\ell)^r$. Now suppose $i > 0$ and that the theorem is true for $i - 1$. Let X_1, \dots, X_r be the irreducible components of X and consider any subscheme Z of codimension 1 in X which contains $\cup_{j \neq k} X_j \cap X_k$. Then $U = X - Z$ is a disjoint union $U = U_1 \cup \dots \cup U_r$, where each U_j is an open subscheme of X_j . By induction, $CH^{i-1}(Z, n; \mathbb{Z}/\ell) = CH^{i-1}(Z, n-1; \mathbb{Z}/\ell) = 0$ when $n \geq 2i$; therefore by Lemma 2.3, $CH^i(X, n; \mathbb{Z}/\ell) \cong CH^i(U, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(U_j, n; \mathbb{Z}/\ell)$. By similar reasoning, we may conclude that for each j , $CH^i(U_j, n; \mathbb{Z}/\ell) \cong CH^i(X_j, n; \mathbb{Z}/\ell)$. We are thus reduced to proving the assertion in the case that X is a variety.

To this end, let $j_X : CH^i(X, n; \mathbb{Z}/\ell) \rightarrow CH^i(k(X), n; \mathbb{Z}/\ell)$ denote the isomorphism of Lemma 2.4. Let Ω be an algebraic closure of $k(X)$, and consider the following commutative diagram, in which (to simplify notation), we write $\alpha = \iota_{k(X)/k}(\text{Spec } k)^*$, $\beta = \iota_{\Omega/k(X)}(\text{Spec } k(X))^*$, $\gamma = \iota_{\Omega(X_\Omega)/\Omega}(\text{Spec } \Omega)^*$, $\delta = \iota_{\Omega/k(X)}^*$.

$$\begin{array}{ccc}
CH^i(k, n; \mathbb{Z}/\ell) & & \\
\downarrow \alpha & & \\
CH^i(k(X), n; \mathbb{Z}/\ell) & \xrightarrow{j_X^{-1} \cong} & CH^i(X, n; \mathbb{Z}/\ell) \\
\downarrow \beta & & \downarrow \delta \\
CH^i(\Omega, n; \mathbb{Z}/\ell) & & \\
\downarrow \gamma & & \\
CH^i(\Omega(X_\Omega), n; \mathbb{Z}/\ell) & \xrightarrow{j_{X_\Omega}^{-1} \cong} & CH^i(X_\Omega, n; \mathbb{Z}/\ell)
\end{array}$$

By Theorem 2.2, the map $\beta \circ \alpha$ is an isomorphism, so α is injective and β is surjective. Again by Theorem 2.2, δ is an isomorphism, so by a diagram chase, $\delta \circ j_X^{-1} = j_{X_\Omega}^{-1} \circ \gamma \circ \beta$ is an isomorphism, and hence β is injective. Thus, β is an isomorphism, so α is an isomorphism, too. This implies that $\sigma^* = j_X^{-1} \circ \alpha : CH^i(k, n; \mathbb{Z}/\ell) \rightarrow CH^i(X, n; \mathbb{Z}/\ell)$ is an isomorphism. By Theorem 2.5, $CH^i(k, n; \mathbb{Z}/\ell) \cong H_{\acute{e}t}^{2i-n}(k, \mathbb{Z}/\ell(i))$ which equals 0 when $n > 2i$ and \mathbb{Z}/ℓ when $n = 2i$. This completes the inductive step.

Corollary 3.2. *Let X be an equidimensional quasiprojective scheme over an algebraically closed field k . Then $CH^i(X, n)$ is uniquely divisible when $n > 2i$ and torsion-free when $n = 2i$. If $i \geq 1$, $CH^i(X, 2i)$ is uniquely divisible.*

Proof.

Let $\ell \neq 0$ be any integer. From Theorem 3.1 and the universal coefficients exact sequence

$$0 \rightarrow CH^i(X, n)/\ell \rightarrow CH^i(X, n; \mathbb{Z}/\ell) \rightarrow {}_\ell CH^i(X, n-1) \rightarrow 0$$

the only assertion that requires proof is the divisibility of $CH^i(X, 2i)$ when $i \geq 1$, and for this we may assume without loss of generality that ℓ is prime. In the following, we follow closely the arguments of [PW], Section 1 – the only difference being that we need to modify the argument slightly for fields of positive characteristic.

Let X_1, \dots, X_r be the irreducible components of X . When $\ell = \text{char } k > 0$, then by Theorem 2.6, $CH^i(X, 2i; \mathbb{Z}/\ell) = 0$, so $CH^i(X, 2i)/\ell = 0$. When $\ell \neq \text{char } k$, $CH^i(X, 2i; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(X_j, 2i; \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^r$ by Theorem 3.1. By the universal coefficients sequence, it suffices to show that for each j , ${}_\ell CH^i(X_j, 2i-1) \cong \mathbb{Z}/\ell$. Indeed, since k is algebraically closed, the structure map $\sigma_j : X_j \rightarrow \text{Spec } k$ has a section, so by functoriality it suffices to show ${}_\ell CH^i(k, 2i-1) \cong \mathbb{Z}/\ell$.

By Theorem 2.5, $CH^i(k, n; \mathbb{Z}/\ell) = H_{\acute{e}t}^{2i-n}(k, \mathbb{Z}/\ell(i))$, which vanishes unless $n = 2i$. Thus, by the universal coefficients sequence, $CH^i(k, n)$ is ℓ -torsion-free as long as $n \neq 2i - 1$. As argued in [PW], Proposition 1.3 and Lemma 1.4, all differentials in the Bloch-Lichtenbaum spectral sequence

$$E_2^{p,q} = CH^{-q}(k, -p - q) \Rightarrow K_{-p-q}(k)$$

are zero; thus, $CH^i(k, 2i - 1)_{tors} \cong K_{2i-1}(k)_{tors}$. Finally, by the calculation of the torsion in the K -theory of algebraically closed fields, we see that ${}_{\ell}CH^i(k, 2i - 1) \cong {}_{\ell}K_{2i-1}(k) \cong \mathbb{Z}/\ell$.

It is now relatively easy to develop analogous results for schemes over any field which contains an algebraically closed field.

Corollary 3.3. *Let L be an field containing an algebraically closed field k , and X an equidimensional quasiprojective scheme over L .*

- *If $n > 2i$ and $\ell \neq 0$, then $CH^i(X, n; \mathbb{Z}/\ell) = 0$.*
- *If X is irreducible, then*

$$CH^i(X, n; \mathbb{Z}/\ell) = CH^i(L(X), n; \mathbb{Z}/\ell) = 0$$

when $n > 2i$. If moreover $\text{char } k \nmid \ell$, then

$$CH^i(X, 2i; \mathbb{Z}/\ell) \cong CH^i(L(X), 2i; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell.$$

In general, if X_1, \dots, X_r are the irreducible components of X , then for $n \geq 2i$ and ℓ relatively prime to $\text{char } k$,

$$CH^i(X, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(X_j, n; \mathbb{Z}/\ell).$$

Proof.

First suppose $X = \text{Spec } M$ for some field $M \supseteq L$. If M is finitely generated over k , then $M = k(X)$ for some variety X over k , so the conclusion follows immediately from Theorem 3.1. The general case follows since any such M may be written as a direct limit of finitely generated extensions of k .

Now suppose X is any equidimensional quasiprojective scheme over L ; we follow the reasoning adopted in the proof of Theorem 3.1. As before, the case $i = 0$ follows immediately from the definitions, so we may assume $i > 0$. We proceed

to prove the theorem by induction on $d = \dim X$. The case $d = 0$ is covered by the previous paragraph. If $d > 0$, choose, as in the proof of Theorem 3.1, a closed subscheme $Z \subseteq X$ of codimension 1 such that $U = X - Z$ is a disjoint union of irreducible subschemes U_1, \dots, U_r , where each U_j is open in X_j . By induction, $CH^{i-1}(Z, n; \mathbb{Z}/\ell) = CH^{i-1}(Z, n-1; \mathbb{Z}/\ell) = 0$ when $n \geq 2i$. Thus, by Lemma 2.3, $CH^i(X, n; \mathbb{Z}/\ell) \cong CH^i(U, n; \mathbb{Z}/\ell) \cong \bigoplus_{j=1}^r CH^i(U_j, n; \mathbb{Z}/\ell)$. By Lemma 2.4, $CH^i(U_j, n; \mathbb{Z}/\ell) = CH^i(L(U_j), n; \mathbb{Z}/\ell)$. By the previous paragraph, $CH^i(L(U_j), n; \mathbb{Z}/\ell)$ equals 0 when $n > 2i$ or \mathbb{Z}/ℓ when $n = 2i$.

Corollary 3.4. *Let X be an equidimensional quasiprojective scheme over a field L containing an algebraically closed field k . Then $CH^i(X, n)$ is torsion-free when $n = 2i$ and uniquely divisible when $n > 2i$. When $i \geq 1$, $CH^i(X, 2i)$ is uniquely divisible.*

Proof.

Arguing as in Corollary 3.2, we see from Corollary 3.3 and the universal coefficients sequence that it remains only to show that $CH^i(X, 2i)$ is divisible when $i \geq 1$.

First suppose $X = \text{Spec } M$ for some field M containing L . Because M may be written as a direct limit of fields finitely generated over k , we may assume without loss of generality that M is finitely generated over k , that is, $M = k(Y)$ for some variety over k . Then $CH^i(M, 2i) = \varinjlim_U CH^i(U, 2i)$, where U runs through the dense open subsets of Y . Since each $CH^i(U, 2i)$ is (uniquely) divisible by Corollary 3.3, the same is true of $CH^i(M, 2i)$.

Now suppose $\dim X > 0$. Pick a closed subscheme $Z \subseteq X$ of codimension 1 as in the proof of Theorem 3.1 such that $U = X - Z$ is a disjoint union of subschemes U_1, \dots, U_r , each open in X . Since $CH^{i-1}(Z, 2i)$ and $CH^{i-1}(Z, 2i-1)$ are (uniquely) divisible by the assertion already proved, Lemmas 2.3 and 2.4 show that divisibility of $CH^i(X, 2i)$ is equivalent to divisibility of $CH^i(U, 2i) \cong \bigoplus_{j=1}^r CH^i(U_j, 2i) \cong \bigoplus_{j=1}^r CH^i(k(U_j), 2i)$. By the previous paragraph, each $CH^i(k(U_j), 2i)$ is divisible, which completes the proof.

4 Connection to the Beilinson-Soulé conjecture

A well-known conjecture due to Beilinson and Soulé might be phrased as stating that for any smooth scheme X over a field k , $CH^i(X, n) = 0$ when $n > 2i$. Corollary 3.3 can then be interpreted as a “mod ℓ ” version of the Beilinson-Soulé conjecture. In particular, it implies:

Proposition 4.1. *Let L be a field containing an algebraically closed field and X an equidimensional quasiprojective scheme over L . Suppose $n > 2i$. Then*

$$CH^i(X, n) = 0 \text{ if and only if } CH^i(X, n; \mathbb{Q}) = 0$$

This is of interest because the Riemann-Roch theorem in higher K -theory [B1] identifies $CH^i(X, n; \mathbb{Q})$ with the graded piece $K_n^{(m)}(X) \otimes \mathbb{Q}$ arising from the γ -filtration on $K_n(X) \otimes \mathbb{Q}$. In particular, if the groups $K_n(X)$ are known to be torsion for $n > 0$, one may deduce the vanishing of the (integral) higher Chow groups. This is the case, for example, if p is a prime and X is a variety of dimension ≤ 3 in the class $A(\bar{\mathbb{F}}_p)$ defined by Soulé in [So], 3.3; this class is closed under products and includes all curves, abelian varieties, and unirational varieties of degree at least 3. Parshin conjectured that $K_i(X)$ is torsion for $i > 0$ and any smooth projective X over $\bar{\mathbb{F}}_p$ – hence that all such X belong to $A(\bar{\mathbb{F}}_p)$. Thus we have the following:

Proposition 4.2. *If X satisfies Parshin’s conjecture, then $CH^i(X, n) = 0$ for all $i > 0$ and $n \geq 2i$.*

Proof.

The hypothesis implies that $K_n(X)$ is torsion for all $n > 0$, so by the Riemann-Roch Theorem, the groups $CH^i(X, n)$ are torsion when $n > 0$. By Corollary 3.2, these groups are torsion-free when $i > 0$ and $n \geq 2i$, so we conclude that $CH^i(X, n) = 0$ in this case.

We remark that if X is a variety satisfying Parshin’s conjecture, it also follows that $CH^i(X, 2i - 1)$ is a torsion group for $i > 0$. Since by Corollary 3.2 we have ${}_\ell CH^i(X, 2i - 1) \cong \mathbb{Z}/\ell$ for primes $\ell \neq p$, we deduce $CH^i(X, 2i - 1) \cong \bigoplus_{\ell \neq p} \mathbb{Q}_\ell / \mathbb{Z}_\ell$.

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