

Torsion in Mixed K -groups

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Abstract

We define generalized Milnor K -groups known as *mixed K -groups* and give explicit descriptions of these groups in simple cases. Our main result is a generalization of the theorem of Bass and Tate that the Milnor K -groups $K_n^M(k)$ are uniquely divisible when k is algebraically closed and $n \geq 2$.

1 Introduction

The purpose of this paper is to investigate various properties of *mixed K -groups*, defined in [A1], which constitute in some sense a generalization of Milnor K -groups. The definition of mixed K -groups was formulated to include as special cases the K -groups of Somekawa [So] and the K -groups of Raskind-Spiess [RS]. Somekawa's construction assigns a group $K(k; G_1, \dots, G_s)$ to a family G_1, \dots, G_s of semi-abelian varieties defined over a field k ; he then shows that in the case $G_1 = \dots = G_s = \mathbf{G}_m$ (the multiplicative group scheme), the above group is isomorphic to the Milnor K -group $K_s^M(k)$. The interest in such a generalization of Milnor K -theory seems to stem from a conjectural remark of Somekawa [So] to the effect that $K(k; G_1, \dots, G_s)$ ought to be isomorphic to $\text{Ext}_{M_k}^s(\mathbf{Z}, G_1[-1], \dots, G_s[-1])$ in a (suitably defined) category of mixed motives. It is primarily because of this connection to the philosophy of motives that we study these groups.

The precise objects under study in this paper, mixed K -groups, form a class which include both Somekawa's K -groups and those of Raskind-Spiess [RS]; the latter are defined in a similar spirit and bear a close relation to Chow groups of zero-cycles on products of varieties. Given smooth quasiprojective varieties Y_1, \dots, Y_r defined

over a field k and semi-abelian varieties G_1, \dots, G_s defined over k , we define a group $K(k; \mathcal{CH}_0(Y_1), \dots, \mathcal{CH}_0(Y_r); G_1, \dots, G_s)$. This construction was originally designed to facilitate the proof of the main result of [A1], namely that $K(k; \mathcal{CH}_0(Y); \underbrace{\mathbf{G}_m, \dots, \mathbf{G}_m}_s)$ is isomorphic to Bloch's higher Chow group $CH^{d_Y+s}(Y, s)$, where d_Y is the dimension of Y .

After establishing some preliminaries, the first section of the paper is devoted to the showing that the group $K(k; S)$ (where S is a single semi-abelian variety) is isomorphic to the group $S(k)$ of k -rational points of S . The main result of this paper (Theorem 3.1) appears in the second section, and asserts that when k is an algebraically closed field, X_1, \dots, X_r smooth projective varieties over k and G_1, \dots, G_s semi-abelian varieties over k , the group $K(k; \mathcal{A}_0(X_1), \dots, \mathcal{A}_0(X_r); G_1, \dots, G_s)$ – a slight variant on the mixed K -group defined above – is uniquely divisible. According to Somekawa's conjectural motivic interpretation, this would imply (in the case $r = 0$) unique divisibility of the appropriate corresponding Ext-group. We apply our theorem to give an elementary proof of a result of Suslin, Geisser, and Geisser-Levine on the unique divisibility of the higher Chow groups $CH^{d_X+s}(X, s)$, where $d_X = \dim X$ and $s \geq 2$ is an integer; furthermore, we deduce that the torsion subgroup of $CH^{d_X+1}(X, 1)$ is precisely the group of roots of unity in k^* .

The remainder of the paper is an attempt to place the groups $K(k; A_1, \dots, A_r)$ – where A_1, \dots, A_r are abelian varieties over an algebraically closed field k – in a motivic context. Our primary result in this vein is that this group is a direct summand of the Chow group $CH_0(A_1 \times_k \dots \times_k A_r)$ of zero-cycles on A .

Throughout this paper, a *variety* X over a field k is a separated integral scheme $\sigma : X \rightarrow \text{Spec } k$. A variety is said to be *defined over* k if it is geometrically integral.

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2 Mixed K -groups

2.1 Extension of valuations

Before giving the definition of mixed K -groups, a few auxiliary definitions are in order. The first concerns the extension of a valuation on the multiplicative group of a field to a similar map on points of a semi-abelian variety, following [So].

Suppose k is a field and G is a semi-abelian variety defined over k : that is, there is an exact sequence of group schemes over k :

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

where T is a torus and A is an abelian variety. Fix a finitely generated extension field K of transcendence degree 1 over k and a place v of K such that $v(k) = 0$. Let K_v be the completion of K with respect to v and L/K_v a finite unramified Galois extension such that T splits over the corresponding residue field F of L ; that is, $T \times_k F \cong \mathbf{G}_m^n$ for some n . Finally, let w be the (unique) extension of v to L . Consider the following commutative diagram; the map ord_w is understood as the extension of the usual $\text{ord}_w : L^* \rightarrow \mathbf{Z}$ to a map $T(L) \cong (L^*)^n \rightarrow \mathbf{Z}^n$, and the map $r_w = (r_w^1, \dots, r_w^n)$ is described in the following remarks:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & T(O_w) & \longrightarrow & G(O_w) & \longrightarrow & A(O_w) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong \\
 0 & \longrightarrow & T(L) & \longrightarrow & G(L) & \longrightarrow & A(L) \longrightarrow 0 \\
 & & \downarrow \text{ord}_w & & \downarrow r_w = (r_w^1, \dots, r_w^n) & & \\
 & & \mathbf{Z}^n & \longrightarrow & \mathbf{Z}^n & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The first row is exact: this follows from consideration of the long exact sequence of cohomology over the flat site; the final map is surjective because $H_{\text{fl}}^1(O_w; \mathbf{G}_m^n) \cong \text{Pic}(\text{Spec } O_w)^n = 0$. Exactness of the second row follows from Hilbert's Theorem 90, and the last column is an isomorphism because A is proper. Finally, the cokernel of the first column is clearly isomorphic to \mathbf{Z}^n , so by the snake lemma (applied to the first two rows), the cokernel of the second column is also isomorphic to \mathbf{Z}^n and enables us to define a map $r(w) = (r_w^1, \dots, r_w^n)$.

Now fix $g \in G(K_v)$ and $h \in K_v^*$. We are going to construct a map $\partial_v : G(K_v) \otimes K_v^* \longrightarrow G(k(v))$ which simultaneously generalizes the local symbol of Serre (see [Se2] III.1) and the boundary homomorphism $K_2^M(K) \longrightarrow K_1^M(k(v))$ of Milnor K -theory [BT]. For each $i = 1, \dots, n$, we define $h_i \in T(L)$ to be the n -tuple having h in the i th coordinate and 1 elsewhere. Then set

$$\varepsilon(g, h) = ((-1)^{\text{ord}_w(h)r_w^1(g)}, \dots, (-1)^{\text{ord}_w(h)r_w^n(g)}) \in T(O_w) \subseteq G(O_w)$$

and

$$\tilde{\partial}_v(g, h) = \varepsilon(g, h) g^{\text{ord}_w(h)} \prod_{i=1}^n h_i^{-r_w^i(g)} \in G(O_w)$$

We define the ‘‘extended tame symbol’’ $\partial_v(g, h)$ to be the image of $\tilde{\partial}_v(g, h)$ under the canonical map $G(O_w) \longrightarrow G(F)$; we note that since $\tilde{\partial}_v(g, h)$ is invariant under the action of $\text{Gal}(L/K_v)$, $\partial_v(g, h)$ must be invariant under the action of $\text{Gal}(F/k(v))$, so we henceforth interpret $\partial_v(g, h)$ as belonging to $G(k(v))$. Finally, we observe that since L/K_v is unramified, this definition of ∂_v is independent of the choice of L .

We conclude this section by stating an important theorem known as the *Reciprocity Law*.

Theorem 2.1. ([Se2], p.35)

Let k, K, g and h be as above. In the case $G = \mathbf{G}_m$, we have the formula:

$$\prod_v \partial_v(g, h) = 1$$

2.2 Definitions

We are now ready to introduce mixed K -groups. The adjective ‘mixed’ refers to the fact that these groups are defined as class which incorporates both the groups studied by Somekawa in [So] and by Raskind-Spiess in [RS]. The former correspond to the case $r = 0$ and the latter to the case $s = 0$ below.

Let k be a field, and X a smooth quasiprojective variety defined over k . We use the notation $CH_0(X)$ to denote the group of zero-cycles on X modulo rational equivalence. If G is a group scheme defined over k and A is a k -algebra, we use the notation $G(A)$ for the group of A -rational points, i.e. morphisms $\text{Spec } A \longrightarrow G$ which commute with the structure maps $G \longrightarrow \text{Spec } k$.

Now suppose $r \geq 0$ and $s \geq 0$ are integers. Let X_1, \dots, X_r be smooth quasiprojective varieties defined over k and G_1, \dots, G_s semi-abelian varieties defined over k . Set

$$T = \bigoplus_{E/k \text{ finite}} CH_0((X_1)_E) \otimes \dots \otimes CH_0((X_r)_E) \otimes G_1(E) \otimes \dots \otimes G_s(E)$$

We use the notation $(a_1 \otimes \dots \otimes a_r \otimes b_1 \otimes \dots \otimes b_s)_E$ to refer to a homogeneous element living in the direct summand of T corresponding to the field E .

Let $R \subseteq T$ be the subgroup generated by the elements of the following type:

- **M1.** For convenience of notation, set $H_i(E) = CH_0((X_i)_E)$ for $i = 1, \dots, r$ and $H_j(E) = G_{j-r}(E)$ for $j = r+1, \dots, r+s$.

For every diagram $k \hookrightarrow E_1 \xrightarrow{\phi} E_2$ of finite extensions of k , all choices $i_0 \in \{1, \dots, r+s\}$ and all choices $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ for $i \neq i_0$, the element $R_1(E_1; E_2; i_0; h_1, \dots, h_{r+s})$ defined to be:

$$(\phi^*(h_1) \otimes \dots \otimes h_{i_0} \otimes \dots \otimes \phi^*(h_{r+s}))_{E_2} - (h_1 \otimes \dots \otimes \phi_*(h_{i_0}) \otimes \dots \otimes h_{r+s})_{E_1}$$

Here we have used the notation ϕ^* (ϕ_*) to denote the pullback (pushforward) map for the Chow group structure on H_i (if $1 \leq i \leq r$) or the group scheme structure on H_i (if $s \leq i \leq r+s$).

- **M2.** For every K finitely generated of transcendence degree 1 over k , all choices of $h \in K^*$, $f_i \in CH_0((X_i)_K)$ for $i = 1, \dots, r$ and $g_j \in G_j(K)$ for $j = 1, \dots, s$ such that for every place v of K such that $v(k) = 0$, there exists $j_0(v)$ such that $g_j \in G_j(O_v)$ for all $j \neq j_0(v)$, the element $R_2(K, h, f_1, \dots, f_r, g_1, \dots, g_s)$. If $s > 0$, this is defined to be:

$$\sum_v (s_v(f_1) \otimes \dots \otimes s_v(f_r) \otimes g_1(v) \otimes \dots \otimes \partial_v(g_{j_0(v)}, h) \otimes \dots \otimes g_s(v))_{k(v)}$$

Here O_v is the valuation ring of v , $s_v : CH_0((X_i)_K) \rightarrow CH_0((X_i)_{k(v)})$ the specialization map for Chow groups ([F], 20.3), and $g_i(v) \in G_i(k(v))$ the reduction of $g_i \in G_i(O_v)$. The notation ∂_v refers to the ‘‘extended tame symbol’’ as defined above.

If $s = 0$, the element $R_2(K; h; f_1, \dots, f_r; g_1, \dots, g_s)$ is defined to be:

$$\sum_v \text{ord}_v(h) (s_v(f_1) \otimes \dots \otimes s_v(f_r))_{k(v)}$$

We define the *mixed K-group* $K(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_r); G_1, \dots, G_s)$ as the quotient T/R , at least when $r + s > 0$. We denote the class of a generator $(h_1 \otimes \dots \otimes h_{r+s})_E$ by $\{h_1, \dots, h_{r+s}\}_{E/k}$. We refer to the classes of elements of the form **M1** and **M2** as *relations* of the mixed K -group. If $r = s = 0$, we simply define our group to be \mathbf{Z} .

We will mostly be interested in the case $G_1 = \dots = G_s = \mathbf{G}_m$; hence we use $K_s(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_r); \mathbf{G}_m)$ as shorthand for $K(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_r); \underbrace{\mathbf{G}_m, \dots, \mathbf{G}_m}_s)$. We also adopt the practice of omitting superfluous semicolons; for example, if $r = 0$, we simply write $K(k; G_1, \dots, G_s)$ for our group.

Remark.

If $\sigma : Y \rightarrow \text{Spec } k$ is a *projective* variety, we can define the *degree map* $\text{deg} = \sigma_* : CH_0(Y) \rightarrow CH_0(\text{Spec } k) \cong \mathbf{Z}$ by push-forward of cycles. We define $A_0(Y) := \text{Ker } \text{deg}$, and note that if Y contains a k -rational point, or more generally if Y admits a zero-cycle of degree 1, then the degree map splits and we have a direct sum decomposition $CH_0(Y) \cong \mathbf{Z} \oplus A_0(Y)$.

Returning to our situation, suppose that X_1, \dots, X_q and Y_1, \dots, Y_r smooth quasiprojective, with Y_1, \dots, Y_r actually *projective*. By replacing CH_0 with A_0 in the appropriate instances, we may define groups

$K(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_q); \mathcal{A}_0(Y_1), \dots, \mathcal{A}_0(Y_r); G_1, \dots, G_s)$ as was done previously.

Remark.

In the above, we have ordered the “arguments” of the mixed K -groups, putting the \mathcal{CH}_0 -type factors first, followed by the \mathcal{A}_0 -type factors, followed by the semi-abelian varieties. This was done strictly for notational convenience; it is easy to see that the isomorphism class of the mixed K -group remains unchanged when the factors are permuted.

2.3 Some results on mixed K -groups

We collect, for future reference, several results related to mixed K -groups. The first relates mixed K -groups to the higher Chow groups:

Theorem 2.2. (*[A1], Theorem 6.1*) *Let k be a field, X a smooth quasiprojective variety of dimension d over k , and $s \geq 0$ an integer. Then there is a natural map inducing an isomorphism $CH^{d+s}(X, s) \xrightarrow{\cong} K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$.*

In the case of a product of smooth projective varieties, we have the following result of a slightly different flavor:

Theorem 2.3. (*Raskind-Spiess [RS], Theorem 2.4*) *Let k be a field and X_1, \dots, X_n smooth projective varieties over k . Then there are natural isomorphisms:*

$$CH_0(X_1 \times_k \dots \times_k X_n) \cong K(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_n))$$

Finally, we quote the following (slightly simplified) statement of [A2], Corollary 2.9:

Theorem 2.4. *Let k be a field. Suppose X_1, \dots, X_n are smooth projective varieties over k and G_1, \dots, G_s semi-abelian varieties over k . Then there is a natural isomorphism:*

$$\begin{aligned} & K(k; \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_n); G_1, \dots, G_s) \\ & \cong \mathbf{Z} \oplus \bigoplus_{\nu=1}^n \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq n} K(k; \mathcal{A}_0(X_{i_1}), \dots, \mathcal{A}_0(X_{i_\nu}); G_1, \dots, G_s) \end{aligned}$$

2.4 One-variety families

In this section, we give a complete description of the mixed K -group in the case $s = 1$ above, i.e. when the case $K(k; S)$ where S is a semi-abelian variety. This result may be known to experts, but the author has not managed to locate a reference in the literature. The case $r = 1$ has already been treated: Theorem 2.3 implies that $K(k; \mathcal{CH}_0(X)) \cong CH_0(X)$ when X is a smooth projective variety over k , and Theorem 2.2 yields the same result when X is smooth and quasiprojective.

To begin our analysis, observe that every element $\{a\}_{E/k} \in K(k; S)$ is equivalent, via a relation of type **M2**, to $\{N_{E/k}a\}_{k/k}$. Hence the canonical map $c_k : S(k) \rightarrow K(k; S)$ defined by $c_k(x) = \{x\}_{k/k}$, is in fact surjective. The main result is:

Theorem 2.5. *Let k be a field and S a semi-abelian variety defined over k . Then the map $c_k : S(k) \rightarrow K(k; S)$ defined above is an isomorphism.*

Proof.

The following device will simplify our computation considerably; in particular, it will allow us to assume, in the course of our calculations, that all tori are split.

Lemma 2.6. *Let \bar{k} be an algebraic closure of k . If the map $c_{\bar{k}} : S(\bar{k}) \rightarrow K(\bar{k}; S \times_k \bar{k})$ is an isomorphism, then $c_k : S(k) \rightarrow K(k; S)$ is an isomorphism.*

Proof.

By the remarks preceding the statement of the theorem, it suffices to prove that c_k is injective. The following diagram is clearly commutative, in which the first vertical arrow is the canonical inclusion and the second vertical arrow is the base change homomorphism defined in [So]:

$$\begin{array}{ccc} S(k) & \xrightarrow{c_k} & K(k; S) \\ \downarrow & & \downarrow \\ S(\bar{k}) & \xrightarrow{c_{\bar{k}}} & K(\bar{k}, S \times_k \bar{k}) \end{array}$$

If \bar{c} is an isomorphism, then since the first vertical arrow is injective, c must also be injective by commutativity of the diagram.

Returning to the proof of Theorem 2.5, we assume henceforth that k is algebraically closed. We first treat the case of an abelian variety.

If S is an abelian variety, the injectivity of c_k (which we henceforth refer to simply as c) is equivalent to the assertion that all relations of type **M2** are trivial in $K(k; S)$; that is, given a relation $R_2(K, h, g)$,

$$\sum_v \text{ord}_v(h)g(v) = 0$$

Interpreting $g \in S(K)$ as a k -morphism $g : \text{Spec } K \rightarrow S$, we may lift g to a morphism $\tilde{g} : C \rightarrow S$, where C is the smooth proper model for K over k . Observing that we may regard the points of C as corresponding to valuations of K over k , we conclude that the zero-cycle $\sum_v \text{ord}_v(h)[g(v)] = \tilde{g}_*(\text{ord}_v(h)[v])$ on A is rationally equivalent to zero. Since $\sum_v \text{ord}_v(h)g(v)$ is the image of this cycle under the Albanese map $A_0(S) \rightarrow S(k)$, we conclude that this element is indeed 0.

We now assume that S is a semi-abelian variety which is *not* an abelian variety. Since k is algebraically closed, we may interpret S as an extension $0 \rightarrow T \xrightarrow{i} S \xrightarrow{\pi} A \rightarrow 0$ of an abelian variety A by a split torus $T \cong (\mathbf{G}_m)^r$ for some integer $r \geq 1$. In the discussion below, we preserve the notation of Section 2, with the added feature of using multiplicative notation as it is more suggestive in this context. We begin with two elementary lemmas, the first of which is immediate from the definitions.

Lemma 2.7. *The symbol $\partial_v(g, h)$ is multiplicative in both g and h .*

Lemma 2.8. *For any $g \in S(K)$, $r_v^1(g) = \dots = r_v^r(g) = 0$ for all but finitely many $v \in \mathcal{P}(K/k)$.*

Proof. If g is a closed point, then $g \in S(k)$, and since $k \subseteq O_v$ for all v , we have $g \in S(O_v)$ for all v and hence $r_v^j(g) = 0$ for all v and all $j = 1, \dots, r$. If g is not a closed point, the morphism $\text{Spec } K \xrightarrow{g} S$ corresponds to a nonconstant rational map $X \xrightarrow{\gamma} S$, where X is the smooth projective model for K over k . This map is defined for $v \in X - T$, where $T \subseteq X$ is some finite subset. For such v there is a local homomorphism

$$\mathcal{O}_{S, \gamma(v)} \xrightarrow{\gamma^\#} \mathcal{O}_{X, v} = \mathcal{O}_v$$

and hence a morphism $\text{Spec } \mathcal{O}_v \rightarrow S$ completing a commutative diagram:

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{g} & S \\ \downarrow & \nearrow & \\ \text{Spec } \mathcal{O}_v & & \end{array}$$

Hence $g \in S(O_v)$ and so $r_v^1(g) = \dots = r_v^r(g) = 0$ for all but finitely many v .

Returning to the proof of Theorem 2.5, note that the group $K(k; S)$ may be interpreted as the quotient of $S(k)$ by the subgroup generated by elements of type $R_2(K, g, h)$, i.e. by elements of the form, where v runs through all places of K such that $v(k) = 0$:

$$\prod_v \partial_v(g, h) = \left\{ \prod_v (\varepsilon_v(s, h) s^{\text{ord}_v(h)} \prod_{j=1}^r h_j^{-r_v^j(s)})_v \right\}_{k/k}$$

for all $h \in K^*$ and $s \in S(K)$. (The subscript v serves as a reminder that the parenthesized expression to which it is attached is to be interpreted as an element of $S(k(v))$).

Thus it suffices to prove that the expression within the braces is equal to the identity element 1 of $S(k)$. To this end, fix $s \in S(K)$ and $h \in K^*$.

Case 1: $s \in S(K)$ is a closed point.

Under this assumption, $s \in S(k)$. Since $k \subseteq \bigcap_v O_v$, we have $s \in S(O_v)$ for all v , so $r_v(s) = 0$ and $\varepsilon_v(s, h) = 1$ for all such v , and the assertion reduces to checking that $\prod_v c^{\text{ord}_v(h)} = 1$, which follows from the well-known formula $\sum_v \text{ord}_v(h) = 0$.

Case 2: $\pi_*(s) = 0$

By Hilbert's Theorem 90, we have an exact sequence

$$0 \longrightarrow T(K) \xrightarrow{i_*} S(K) \xrightarrow{\pi_*} A(K) \longrightarrow 0$$

Since $\pi_*(s) = 0$, we must have $s = i_*(t_s)$, where $t_s = (c_1, \dots, c_r) \in T(K)$, and hence $r_v^j(s) = \text{ord}_v(c_j)$.

Therefore we have

$$\begin{aligned}
& \prod_v (\varepsilon_v(s, h) s^{\text{ord}_v(h)} \prod_{j=1}^r h^{-r_v^j(s)})_v \\
&= \prod_v (\varepsilon_v(i_*(t_s), h) i_*(t_s)^{\text{ord}_v(h)} \prod_{j=1}^r h^{-\text{ord}_v(c_j)})_v \\
&= \prod_v i_*((\varepsilon_v(c_1, h), \dots, \varepsilon_v(c_r, h)) \cdot (c_1^{\text{ord}_v(h)} h^{-\text{ord}_v(c_1)}, \dots, c_r^{\text{ord}_v(h)} h^{-\text{ord}_v(c_r)})) \\
&= i_*\left(\prod_v (\partial_v(c_1, h), \dots, \partial_v(c_r, h))\right)
\end{aligned}$$

which is equal to 1 by Theorem 2.1. (The notation ord_v^j refers to the usual map ord_v interpreted as a map on the j th summand of $T(K) \cong (K^*)^r$)

Case 3: s is not closed and is of the form $\text{Spec } k(t) \longrightarrow S$

We may identify s with a nonconstant rational map $\mathbf{P}^1 \longrightarrow S$. Composing with the given morphism $S \xrightarrow{\pi} A$, we get a rational map from \mathbf{P}^1 to the abelian variety A ; but we know ([Mi], Cor. 3.8) that all such maps must be constant. Thus the image of s lies in $\pi^{-1}(a)$ where a is some closed point of a . Choose any closed point $c \in \pi^{-1}(a)$; by Case 1, the assertion holds for c . Since $\pi_*(c^{-1}s) = -a + a = 0$, the assertion holds for $c^{-1}s$ by Case 2. Finally, the assertion for $s = c(c^{-1}s)$ follows from the above two cases in conjunction with Lemma 2.7.

Case 4: $s : \text{Spec } K \longrightarrow S$ is arbitrary

Let v_1, \dots, v_n be the places such that $r_{v_i}^j(s) = a_i^j \neq 0$ for some j . For each $j = 1, \dots, r$, use the weak approximation theorem for valuations (cf. [Bou 85], VI. 7.2, Corollary 1) to choose elements $k_j \in K^*$ such that $\text{ord}_{v_i}^j(k_j) = a_i^j$ for all $i = 1, \dots, n$. Writing $\mathbf{k} = (k_1, k_2, \dots, k_r) \in T(K) \subseteq S(K)$, and $s = (s\mathbf{k}^{-1})\mathbf{k}$, we have $\partial_v(s, h) = \partial_v(s\mathbf{k}^{-1}, h)\partial_v(\mathbf{k}, h)$ by Lemma 2.7. Since $\prod_v \partial_v(\mathbf{k}, h) = 1$ by Case 2, it suffices to check the assertion for s, h under the assumption that $r_v^1(s) = \dots = r_v^r(s) = 0$ for all v such that $\text{ord}_v(h) \neq 0$.

The idea for the remainder of the proof is to reduce the computation to Case 3 by means of “trace maps” which will be described below. Because K is finitely generated of transcendence degree 1 over the algebraically closed field k , the element $h \in K^*$ is either in k or transcendental over k . If $h \in k$, let t be any element transcendental

over k ; otherwise let $t = h$. Then $k(t)$ is a subfield of K of finite index in K ; thus the inclusion map $\pi^* : k(t) \hookrightarrow K$ is in fact the map on function fields associated to a finite morphism

$$\pi : X \longrightarrow \mathbf{P}^1$$

where X is the smooth projective model for K . Note that in either case, h is contained in $\pi^*(k(\mathbf{P}^1))$; we define h' by $h = \pi^*(h')$. We may interpret s as a rational map $X \longrightarrow S$, and use the notation $Tr_\pi s$ to denote the trace of s with respect to the map π (cf. [Se2], III.2); this is a rational map $Tr_\pi s : \mathbf{P}^1 \longrightarrow S$. Finally, we denote by Z the subset of X on which s is not defined; it is a fact that $Tr_\pi s$ is defined away from $Z' = \pi(Z) \subseteq \mathbf{P}^1$. The key result involving local symbols and the trace is

Lemma 2.9. *For every place w of $k(t)$ such that $w(k) = 0$, we have*

$$\partial_w(Tr_\pi s, h') = \prod_{v: v|_{\mathbf{P}^1}=w} \partial_v(s, h)$$

Proof.

If $w \notin Z'$ (equivalently, $r_w(s) = 0$), the proposition follows from a direct calculation. Let $e_\pi(v|w)$ represent the ramification index with respect to the map π ; thus, if the place v on K restricts to w on $k(t)$, then $\text{ord}_v(h) = e_\pi(v|w)\text{ord}_w(h')$.

Hence,

$$\begin{aligned} & \partial_w(Tr_\pi s, h') \\ &= (Tr_\pi s(w))^{\text{ord}_w(h')} \\ &= \left(\prod_{v: v|_{\mathbf{P}^1}=w} s(v)^{e_\pi(v|w)} \right)^{\text{ord}_w(h')} \\ &= \prod_{v: v|_{\mathbf{P}^1}=w} s(v)^{\text{ord}_v(h)} \\ &= \prod_{v: v|_{\mathbf{P}^1}=w} \partial_v(s, h) \end{aligned}$$

If $w \in Z'$, use the weak approximation theorem for valuations to select $k_1, \dots, k_r \in K^*$ such that $\text{ord}_v(k_j) = r_v^j(s)$ for all $j = 1, \dots, r$ and all v which restrict to w , and $\text{ord}_v(k_j) = 0$ for all $j = 1, \dots, r$ and all v such that $\text{ord}_v(h) \neq 0$ (by hypothesis, these conditions are independent). Writing $\mathbf{k} = (k_1, \dots, k_r)$, we have $r_v^j(s\mathbf{k}^{-1}) = 0$ for all v which restrict to w ; thus $Tr_\pi(s\mathbf{k}^{-1})$ is defined at w and so by the argument above, we have

$$\partial_w(Tr_\pi(s\mathbf{k}^{-1}), h') = \prod_{v: v|_{\mathbf{P}^1}=w} \partial_v(s\mathbf{k}^{-1}, h)$$

Since

$$\partial_w(\mathrm{Tr}_\pi(\mathbf{k}), h') = \prod_{v: v|_{\mathbf{P}^1} = w} \partial_v(\mathbf{k}, h)$$

by [Se1], III. Proposition 3, the multiplicativity of local symbols (Lemma 2.7) on both sides together with the formula $\mathrm{Tr}_\pi ab = \mathrm{Tr}_\pi a \cdot \mathrm{Tr}_\pi b$ allows us to conclude that

$$\partial_w(\mathrm{Tr}_\pi(s, h')) = \prod_{v: v|_{\mathbf{P}^1} = w} \partial_v(s, h)$$

thereby concluding the proof of Lemma 2.9.

To conclude the proof of Theorem 2.5, we observe that

$$\begin{aligned} & \prod_v \partial_v(s, h) \\ &= \prod_w \prod_{v|_{\mathbf{P}^1} = w} \partial_v(s, h) \\ &= \prod_w \partial_w(\mathrm{Tr}_\pi s, h') \text{ by Lemma 2.9} \\ &= 1 \text{ by Case 3} \end{aligned}$$

3 Torsion in mixed K -groups

Throughout this section, k is an algebraically closed field. Let X_1, \dots, X_r be smooth projective varieties over k and G_1, \dots, G_s semi-abelian varieties over k . The main result is:

Theorem 3.1. *If $r + s \geq 2$, the group $U = K(k; \mathcal{A}_0(X_1), \dots, \mathcal{A}_0(X_r); G_1, \dots, G_s)$ is uniquely divisible.*

This theorem may be thought of as a generalization of a theorem of Bass and Tate ([BT], I. Corollary 1.3) that the n th Milnor K -group $K_n^M(k)$ is uniquely divisible for $n \geq 2$. The proof of our theorem starts off in a similar vein, although diverges soon due to the increased complexity of the relations involved.

The proof of Theorem 3.1 requires some background from the dimension theory of fields, which we collect in the next section. A more general treatment of the topic may be found in [Se1], X.7 or [NSW], VI. 5; our discussion follows the latter source closely.

3.1 Dimension of fields

Let k be a field.

Definition 3.2. An n -form in k is a homogeneous polynomial $f \in k[x_1, \dots, x_n]$.

A nontrivial zero of an n -form f is an n -tuple $(\alpha_1, \dots, \alpha_n) \in k^n$ different from $(0, \dots, 0)$ such that $f(\alpha_1, \dots, \alpha_n) = 0$.

Among the most important examples of n -forms are the so-called *norm forms*. Let L/k be a finite extension of degree n and let x_1, \dots, x_n be indeterminates. Since the extension $L(x) = L(x_1, \dots, x_n)/k(x_1, \dots, x_n) = k(x)$ is also finite, fixing an element $\alpha \in L(x)$ yields a $k(x)$ -linear map $m_\alpha : L(x) \rightarrow L(x)$ defined by $m_\alpha(y) = \alpha y$. We may then define the Norm $N : L(x) \rightarrow k(x)$ by the formula $N(\alpha) = \det(m_\alpha)$.

Now choose a basis v_1, \dots, v_n for L/k . Then

$$F_N(x_1, \dots, x_n) = N\left(\sum_{i=1}^n x_i v_i\right)$$

defines an n -form in k of degree n . Evidently, $N(\alpha) = 0$ if and only if $\alpha = 0$, so $F_N(x_1, \dots, x_n)$ has no nontrivial zeros. Furthermore, N induces the norm map $N_{L/k} : L \rightarrow k$ in the sense that for any element $z = \sum_{i=1}^n c_i v_i \in L$ (where $c_i \in k$), $N_{L/k}(z) = F_N(c_1, \dots, c_n)$.

Definition 3.3. The diophantine dimension $d(k)$ of a field k is the smallest number $r \geq 0$ such that whenever $n > d^r$, any n -form f of degree $d \geq 1$ has a nontrivial zero in k . If no such number exists, we set $d(k) = \infty$.

The fields with $d(k) = 0$ are, naturally, the algebraically closed fields. Fields with $d(k) = 1$ are often called quasi-algebraically closed or C^1 , and include several examples of relevance to our work, as described in the following theorem.

Theorem 3.4. (Tsen) Let K be an extension of transcendence degree 1 over an algebraically closed field k . Then $d(K) = 1$.

We will also require the following property enjoyed by fields of finite diophantine dimension:

Theorem 3.5. (Artin-Lang-Nagata) Let k be a field of diophantine dimension r , and let f_1, \dots, f_s be n -forms, each of degree d . If $n > sd^r$, then these forms have a common nontrivial zero in k .

In particular, if k is a C^1 field, then any form in which the number of variables exceeds the degree has a nontrivial zero. This observation will be crucial to the following result of immediate importance to us:

Proposition 3.6. *Let K be a C^1 field. Then for every finite extension L/K , the norm map $N_{L/K} : L^* \rightarrow K^*$ is surjective.*

Proof.

Choose a basis v_1, \dots, v_n for L/K , and fix an element $\alpha \in K^*$. The norm form $F_N(x_1, \dots, x_n) = N(\sum_{i=1}^n x_i v_i)$ is, as discussed above, an n -form of degree n ; thus,

$$f(x_1, \dots, x_n, x) = F_N(x_1, \dots, x_n) - \alpha x^n$$

is an $(n+1)$ -form of degree n . By Theorem 3.5, f has a nontrivial zero $(a_1, \dots, a_n, a) \in K^{n+1}$. It must be the case that $a \neq 0$, since otherwise (a_1, \dots, a_n) would be a nontrivial zero of the norm form F_N , which is impossible. Now set $b_i = a_i/a$, and note that using homogeneity of f , we have:

$$N\left(\sum_{i=1}^n b_i v_i\right) = F_N(b_1, \dots, b_n) = f(b_1, \dots, b_n, 1) + \alpha = \frac{1}{a^{n+1}} f(a_1, \dots, a_n, a) + \alpha = \alpha$$

Thus $N_{L/k}$ is surjective.

Corollary 3.7. *Let k be a C^1 field. Suppose that H_i is either $\mathcal{A}_0(X_i)$ (for some smooth projective variety X_i defined over k) or a semi-abelian variety defined over k . Then $K(k; H_1, \dots, H_r, \mathbf{G}_m)$ is a divisible group.*

Proof.

Fix a positive integer n . Observe that $H_r(\bar{k})$ is a divisible group (cf. Theorem 4.1). Given an element $\{x_1, \dots, x_r, y\}_{L/k}$ of the group, choose a finite extension $i : L \rightarrow M$ such that $i^*(x_r) = nz_r$ for some $z_r \in H_r(M)$. Since an algebraic extension of a C^1 field is still C^1 (cf. [NSW]), we may use Proposition 3.6 to choose $y' \in M^*$ such that $N_{M/L}(y') = y$. Then $\{x_1, \dots, x_r, y\}_{L/k} = \{i^*(x_1), \dots, i^*(x_{r-1}), i^*(x_r), y'\}_{M/k} = n\{i^*(x_1), \dots, i^*(x_{r-1}), z_r, y'\}_{M/k}$.

Combining the above with Theorems 2.2 and 2.4 yields:

Corollary 3.8. *Let k be a C^1 field, $s \geq 2$ an integer and X a smooth projective variety of dimension d , defined over k . Then $CH^{d+s}(X, s)$ is a divisible group.*

3.2 Proof of Theorem 3.1

Since k is an algebraically closed field, U is a quotient of $F = A_0(X_1) \otimes \dots \otimes A_0(X_r) \otimes G_1(k) \otimes \dots \otimes G_s(k)$ by the subgroup R generated by relations of type **M2**. By [B13], Lemma 1.3 (see also Theorem 4.1 below) $A_0(X_i)$ is a divisible group for $i = 1, \dots, r$, and clearly $G_j(k)$ is divisible for $j = 1, \dots, s$. Since the tensor product of two or more divisible groups is uniquely divisible, the multiplication map $F \xrightarrow{n} F$ is an isomorphism for all integers $n > 0$. Consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & U \longrightarrow 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n \\ 0 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & U \longrightarrow 0 \end{array}$$

The second vertical map is an isomorphism, so by the Snake Lemma, we have $U/nU = 0$ and ${}_nU \cong R/nR$. Thus U is divisible, and to prove unique divisibility it suffices to prove that R is divisible.

Consider any generator $r = R_2(K, h, f_1, \dots, f_r, g_1, \dots, g_s) \in R$. Since K has transcendence degree 1 over k , Theorem 3.4 implies that K is a C^1 field and hence by Proposition 3.6 the field norm $N_{L/K} : L^* \rightarrow K^*$ is surjective. Since $r + s \geq 2$, assume without loss of generality that $s \geq 1$ (the proof for the case $r \geq 1$ is similar, in fact easier).

Since $G_1(\bar{K})$ is divisible, we may choose a finite extension $\phi : K \rightarrow L$ such that $\phi^*(g_1) \in G_1(L)$ is divisible by n , say $\phi^*(g_1) = nx_1$. Next, use the surjectivity of $N_{L/K}$ to choose $h' \in L$ such that $N_{L/k}h' = h$. Finally, define

$$r' = R_2(L, h', \phi^*(f_1), \dots, \phi^*(f_r), x_1, \phi^*(g_2), \dots, \phi^*(g_s))$$

Evidently, $\phi^*(g_j) \notin O_w$ for some place w on L if and only if $g_j \notin O_v$ where $v = w|_K$. Likewise, $x_1 \notin O_w$ if and only if $g_1 \notin O_v$ where $v = w|_K$.

Now we compute, using the fact that specialization commutes with pullback ([F], 20.3):

$$\begin{aligned} nr' &= n \sum_v \sum_{w: w|_K=v} (s_w(\phi^*(f_1)) \otimes \dots \otimes s_w(\phi^*(f_r)) \otimes x_1(w) \otimes \dots \otimes \partial_w(\phi^*(g_{j_0(v)}), h') \otimes \dots \otimes \phi^*(g_s)(w))_k \\ &= \sum_v \sum_{w: w|_K=v} (s_w(\phi^*(f_1)) \otimes \dots \otimes s_w(\phi^*(f_r)) \otimes \phi^*(g_1)(w) \otimes \dots \otimes \partial_w(\phi^*(g_{j_0(v)}), h') \otimes \dots \otimes \phi^*(g_s)(w))_k \end{aligned}$$

$$\begin{aligned}
&= \sum_v \sum_{w: w|_K=v} (s_v(f_1) \otimes \dots \otimes s_v(f_r) \otimes g_1(v) \otimes g_2(v) \otimes \dots \otimes \partial_w(\phi^*(g_{j_0(v)}), h') \otimes \dots \otimes g_s(v))_k \\
&= \sum_v (s_v(f_1) \otimes \dots \otimes s_v(f_r) \otimes g_1(v) \otimes g_2(v) \otimes \dots \otimes \prod_{w: w|_K=v} \partial_w(\phi^*(g_{j_0(v)}), h') \otimes \dots \otimes g_s(v))_k \\
&= \sum_v (s_v(f_1) \otimes \dots \otimes s_v(f_r) \otimes g_1(v) \otimes g_2(v) \otimes \dots \otimes \partial_v(g_{j_0(v)}, N_{L/K}h') \otimes \dots \otimes g_s(v))_k = r
\end{aligned}$$

This shows that R is divisible by any integer n and concludes the proof of Theorem 3.1.

From Theorem 3.1 we are able to recover a special case of a result due to Suslin ([Su], Corollary 4.3) in characteristic 0, and to Geisser ([G], Theorem 3.6) and Geisser-Levine ([GL]) in positive characteristic.

Corollary 3.9. *Let k be an algebraically closed field and X a smooth projective variety of dimension d over k . Then $CH^{d+s}(X, s)$ is uniquely divisible for $s \geq 2$.*

Proof.

By Theorem 2.2, there is an isomorphism $CH^{d+s}(X, s) \cong K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$. Next, Theorem 2.4 provides a direct sum decomposition

$$K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \cong K_s(k; \mathbf{G}_m) \oplus K_s(k; \mathcal{A}_0(X); \mathbf{G}_m)$$

Applying Theorem 3.1 to each factor yields the desired result.

The same method allows us to compute the torsion in $CH^{d+1}(X, 1)$:

Corollary 3.10. *The torsion subgroup of $CH^{d+1}(X, 1)$ is isomorphic to the group μ of roots of unity in k^* .*

Proof.

As above, we have $CH^{d+1}(X, 1) \cong K_1(k; \mathbf{G}_m) \oplus K_1(k; \mathcal{A}_0(X); \mathbf{G}_m)$. The second factor is uniquely divisible by Theorem 3.1, and the first factor is isomorphic to k^* by Theorem 2.5.

Remark.

The techniques used here are not sufficient to treat the case $s = 0$ in the statement of the corollary. Fortunately, this is covered by a result of Roitman ([Ro], [Bl2], also Theorem 4.1 below) stating that the Albanese map induces an isomorphism between the torsion subgroup of $CH^d(X) = CH_0(X)$ and the torsion subgroup of $\text{Alb}(X)(k)$.

4 Families of abelian varieties

In [So], Somekawa suggested that the groups $K(k; G_1, \dots, G_s)$ (where the G_i are semi-abelian varieties) might be interpreted as having a motivic meaning. There have been several results relating such groups (more generally, mixed K -groups) to the higher Chow groups, for example:

- $K_s(k; \mathbf{G}_m)$ is isomorphic to $K_s^M(k)$ ([So], Theorem 1.4) and hence to $CH^s(k, s)$. ([To], Theorem 1; [NS], Theorem 4.9)
- If J is the Jacobian of a smooth projective curve C with a k -rational point, then $K(k; J, \mathbf{G}_m) \cong V(C) = \text{Ker}(CH^2(C, 1) \xrightarrow{\sigma^*} CH^1(k, 1) = k^*)$, where $\sigma : C \rightarrow \text{Spec } k$ is the structure map. ([So], Theorem 2.1)
- More generally, with notation as in the previous example, $K_s(k; J, \mathbf{G}_m) = \text{Ker}(CH^{s+1}(C, s) \xrightarrow{\sigma^*} CH^s(k, s))$ ([A1], Theorem 6.1; [A2], Corollary 3.5)
- If C_1, \dots, C_n are curves over k such that $C_i(k) \neq \emptyset$ for all i , and J_1, \dots, J_n are their respective Jacobians, then
$$CH_0(C_1 \times_k \dots \times_k C_n) \cong \mathbf{Z} \oplus \bigoplus_{\nu=1}^n \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq n} K(k; J_{i_1}, \dots, J_{i_\nu}).$$
 ([RS], Theorem 2.4)
- If A is an abelian variety over an algebraically closed field k , then $K(k; A) \cong A(k)$ by Theorem 2.5, and $A(k) \cong I/I^{*2}$, where $I = A_0(A)$ is the ideal of $CH_0(A)$ considered as a ring with respect to the Pontryagin product $*$. (cf. [B1])

As we will make use of the Pontryagin product in the following, we briefly recall its definition. Let A be an abelian variety over a field k , $m : A \times A \rightarrow A$ the “multiplication” morphism, and $CH_*(A) = \bigoplus_{n \geq 0} CH_n(A)$ the group of cycles on A modulo rational equivalence, graded by dimension. Let $p_i : A \times_k A \rightarrow A$ ($i = 1, 2$) be the projection on the i th factor. Then for $x, y \in CH_*(A)$, the map $CH_*(A) \otimes CH_*(A) \rightarrow CH_*(A)$ given by $x \otimes y \mapsto m_*(p_1^*(x) \cdot p_2^*(y))$ is a homomorphism of graded groups known as the *Pontryagin product*; we typically denote by $x * y$ the image of $x \otimes y$ under this map.

Having investigated the mixed K -groups associated to a single semi-abelian variety in Section 2, and in light of the fact that those associated to families of Jacobians have already been identified in a motivic context, it seems natural to study families of (general) abelian varieties. In the following, we prove a result identifying such groups,

valid over an algebraically closed base field. Although the statement looks similar, the proof differs in a fundamental way from that of the fourth example discussed above.

Before stating our theorem, we quote three nontrivial results which we will need in the proof:

Theorem 4.1. *Let A be an abelian variety over an algebraically closed field k ; consider $CH_0(A)$ as a ring under Pontryagin product. Let $I \subseteq CH_0(A)$ denote the ideal consisting of cycles of degree 0 and I^{*m} its m th power as an ideal.*

- (Bloch, [Bl3] Lemma 1.3) I is a divisible group.
- (Roitman-Bloch, [Ro] and [Bl2]) I^{*2} (and hence I^{*m} for all $m \geq 2$) is uniquely divisible.
- (Bloch, [Bl1] Theorem 0.1), $I^{*m} = 0$ for $m > \dim A$.

Our main result is the following:

Theorem 4.2. *Let k be an algebraically closed field and B an abelian variety over k . Then there is a natural split surjective map*

$$S : K(k; \mathcal{A}_0(B), C_1, \dots, C_n) \longrightarrow K(k; B, C_1, \dots, C_n)$$

induced by the Albanese map $G_k : A_0(B) \longrightarrow B(k)$ on the j th factor and the identity on the other factors.

In the above, C_1, \dots, C_n are any arguments of a mixed K -group, i.e. each C_i is either a semi-abelian variety or is equal to $\mathcal{CH}_0(X)$ for some smooth quasiprojective variety X or $\mathcal{A}_0(Y)$ for some smooth projective variety Y .

Proof.

We give the proof in the case $n = 1$ (and write $C = C_1$) for the sake of notational convenience; the general case is formally identical.

Let $G_k : A_0(B) \longrightarrow B(k)$ be the Albanese map.

Construct a map $S : K(k; \mathcal{A}_0(B), C) \longrightarrow K(k; B, C)$ by setting $S(\{x, c\}_{k/k}) = \{G_k(x), c\}_{k/k}$.

We need to check that S kills relations of the form $R_2(K, h, f, g)$, where $h \in K^*$, $f \in A_0(B \times_k K)$ and $g \in C(K)$. By linearity, we may assume that f is of the form $(P) - (0)$, where P is a point of $B \times_k K$. We may also assume that P is a K -rational

point of $B \times_k K$: if not, let L be the field of definition of P and $\phi : K \hookrightarrow L$ the inclusion map. Then, since K is an extension field of transcendence degree 1 over an algebraically closed field, we may use Proposition 3.6 to choose $h' \in L$ such that $N_{L/k}h' = h$. Finally we note, by an argument similar to that used at the end of the proof of Theorem 3.1, that $R_2(L, h', \phi^*(f), \phi^*(g))$ is the same relation as $R_2(K, h, f, g)$. Under these assumptions, let $G_K : A_0(B \times_k K) \longrightarrow (B \times_k K)(K)$ be the corresponding Albanese map. Then $S(R_2(K, h, f, g)) = R_2(K, h, G_K(f), g)$ and so S is well-defined.

It remains to construct a map in the other direction; the idea used here is based on the work of Beauville [Be]. In view of Theorem 4.1, the rule

$$\log(a) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} ([a] - [0])^{*j}$$

yields a (well-defined) function $\log : B(k) \longrightarrow A_0(B)$

Note in particular that this is a finite sum, since all the terms of index greater than $g = \dim A$ are zero. Furthermore, for $a, b \in B(k)$, the formal identity $\log(ab) = \log(a) + \log(b)$ shows that \log is a homomorphism. Finally, direct computation shows that $G_k(\log(a)) = a$; thus, G_k is a split surjection and $B(k)$ is a direct summand of $A_0(B)$.

Now construct a map $T : K(k; B, C) \longrightarrow K(k; A_0(B), C)$ by setting $T(\{b, c\}_{k/k}) = \{\log(b), c\}_{k/k}$. We must check that relations of type $R_2(K, h, z, d)$, where $h \in K^*$, $z \in B(K)$ and $d \in C(K)$ are killed by T .

First, we may assume that $[z] - [0] \in A_0(B \times_k K)$ is divisible by $g!$. If not, choose a finite extension $\phi : K \hookrightarrow L$ such that the pullback of the cycle $[z] - [0]$, i.e. $\phi^*([z] - [0]) = (g!)y$ for some element $y \in A_0(B \times_k L)$. Use Proposition 3.6 to choose $h' \in L^*$ such that $N_{L/k}h' = h$; then we may replace our relation with $R_2(L, h', \phi^*(z), \phi^*(d))$.

Now consider the element $l_z = \sum_{j=1}^g (-1)^j \frac{(g!)^j}{j} y^{*j} \in A_0(B \times_k K)$. We claim that $T(R_2(K, h, z, d)) = R_2(K, h, l_z, d)$, and prove this by explicit calculation. In all the relevant summations below, v runs through all places of K fixing k . We implicitly use the fact that specialization commutes with pullbacks, pushforwards, and intersection products ([F], 20.3) and thus also with Pontryagin products; we also implicitly use Roitman's theorem (second part of Theorem 4.1).

$$T(R_2(K, h, z, d)) = T\left(\sum_v \text{ord}_v(h) \{z(v), d(v)\}_{k/k}\right)$$

$$\begin{aligned}
&= \sum_v \text{ord}_v(h) \{ \log(z(v)), d(v) \}_{k/k} \\
&= \sum_v \text{ord}_v(h) \left\{ \sum_{j=1}^n (-1)^j \frac{([z(v)] - [0])^{*j}}{j}, d(v) \right\}_{k/k} \\
&= \sum_v \text{ord}_v(h) \left\{ \sum_{j=1}^n (-1)^j s_v \left(\frac{(g!)^j}{j} y^{*j} \right), d(v) \right\}_{k/k} \\
&= \sum_v \text{ord}_v(h) \{ s_v(l_z), d(v) \}_{k/k} = R_2(K, h, l_z, d)
\end{aligned}$$

This concludes the proof that T is well-defined. It now follows directly from the formula $N_k \circ \log = id$ that $S \circ T = id$, and hence that S is a split surjection, concluding the proof of Theorem 4.2.

Corollary 4.3. *Let k be an algebraically closed field and B an abelian variety of dimension d over k . Then $K_s(k; B, \mathbf{G}_m)$ is a direct summand of $CH^{d+s}(B, s)$.*

Proof.

By Theorem 4.2, $K_s(k; B, \mathbf{G}_m)$ is a direct summand of $K_s(k; \mathcal{A}_0(B), \mathbf{G}_m)$, and by Theorem 2.4 the latter is a direct summand of $K_s(k; \mathcal{CH}_0(B), \mathbf{G}_m)$. Finally, by Theorem 2.2, $K_s(K; \mathcal{CH}_0(B), \mathbf{G}_m)$ is isomorphic to $CH^{d+s}(B, s)$.

Corollary 4.4. *Let k be an algebraically closed field and B_1, B_2, \dots, B_n abelian varieties over k . Then $K(k; B_1, \dots, B_n)$ is a direct summand of $CH_0(B_1 \times_k \dots \times_k B_n)$.*

Proof.

First note that by Theorem 2.3, we may identify $CH_0(B_1 \times_k \dots \times_k B_n)$ with $K(k; \mathcal{CH}_0(B_1), \dots, \mathcal{CH}_0(B_n))$, and by Theorem 2.4, the latter is isomorphic to $\mathbf{Z} \oplus \bigoplus_{\nu=1}^n \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq n} K(k; \mathcal{A}_0(B_{i_1}), \dots, \mathcal{A}_0(B_{i_\nu}))$. Thus to prove the above theorem, it suffices to show that $K(k; B_1, \dots, B_n)$ is a direct summand of $K(k; \mathcal{A}_0(B_1), \dots, \mathcal{A}_0(B_n))$; this is accomplished by induction on Theorem 4.2.

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