Symmetric linear operator identities in quasigroups

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June 23, 2017

Abstract

Let G be a quasigroup. Associativity of the operation on G can be expressed by the symbolic identity $R_x L_y = L_y R_x$ of left and right multiplication maps; likewise, commutativity can be expressed by the identity $L_x = R_x$. In this article, we investigate symmetric linear identities: these are identities in left and right multiplication symbols in which every indeterminate appears exactly once on each side, and whose sides are mirror images of each other. We determine precisely which identities imply associativity and which imply commutativity, providing counterexamples as appropriate. We apply our results to show that there are exactly eight varieties of quasigroups satisfying such identities, and determine all inclusion relations among them.

1 Introduction

A *quasigroup* is a nonempty set G, equipped with a binary operation (written as juxtaposition), in which the left multiplication maps $L_a : G \to G$, $x \mapsto ax$ and the right multiplication maps $R_a : G \to G$, $x \mapsto xa$ are bijective for all $a \in G$. The group $\mathcal{M}(G)$ generated by these maps (under function composition) is called the *multiplication group* of G, and is a subgroup of the group $P(G)$ of permutations of G. For more generalities on quasigroups, see [10].

It is well-known that an associative quasigroup is in fact a group. Thus, it is natural to ask: which identities, if they hold in a quasigroup, imply associativity? Although this question has received considerable attention in the literature – and recent years have seen greater progress, thanks largely to the advent of automatic theorem-provers (see, for example, [8], [9]) – it is probably too broad a question to be treated in full generality. A sharpening of this question was considered by Niemenmaa and Kepka in [7], who asked: which *linear identities* – identities in which each indeterminate appears exactly once on each side – imply associativity? They showed that for every $n \geq 3$, the "generalized associativity" identity

$$\mathcal{I}_n : x_1(x_2(\ldots(x_{n-1}x_n)\ldots)) = ((\ldots(x_1x_2)\ldots)x_{n-1})x_n \tag{1}$$

is equivalent to associativity for division groupoids. A key insight in their proof is to rewrite $\mathcal{I}_n$ in terms of left and right multiplication maps, working – to the extent possible – with maps rather than elements. This approach was exploited by the present author [1] to show that any division groupoid satisfying the identity $L_{x_1}R_{x_2}\cdots L_{x_{2n-1}}R_{x_{2n}} = R_{x_{2n}}L_{x_{2n-1}}\cdots R_{x_2}L_{x_1}$ must be an abelian group. Considering that common identities such as the associative law $(xa)y = x(ay)$ and the commutative law $xa = ax$ may be expressed as *identities of multiplication operators* by the (respective) formulas $L_xR_y = R_yL_x$ and $L_x = R_x$, it is perhaps natural to define an identity of operators to be *linear* if each symbol appears exactly once on each side of the defining equation, and then to ask which linear identities of operators imply associativity (or commutativity). Of course, not every linear identity (in the sense defined above) can be thus obtained; nevertheless, the ones which do constitute an interesting subfamily of linear identities which is more easily studied than the whole.

In this article, we study a further restriction of the problem to *symmetric* linear identities: these are identities in which the two sides of the defining equation are mirror images of each other. The simplest nontrivial examples are the associative law $L_xR_y = R_yL_x$ and the identities $L_xL_y = L_yL_x$ and $R_xR_y = R_yR_x$. We answer completely the questions of which such identities imply commutativity, and which imply associativity. Furthermore, we give a complete classification of varieties of quasigroups defined by all such identities. Questions of a similar sort – for different families of identities – have been studied by Krapež (see [4], [5]). There are also many papers in the literature (for example, [2] and [3]) which are concerned with functional equations on quasigroups. These papers use terminology and formalism superficially similar to that used in the present article; however, their focus is primarily on finding *operations* that satisfy functional equations of a prescribed type, rather than classifying the varieties of quasigroups defined by a family of equations.

In preparation for stating our results, we introduce some notation and terminology which will be maintained throughout the article. Let $\mathcal{X}$ be a countably infinite set of independent indeterminates. Define the sets of *left multiplication symbols*

$\mathcal{L} = \{L_x : x \in \mathcal{X}\}$ and *right multiplication symbols* $\mathcal{R} = \{R_x : x \in \mathcal{X}\}$, and let $\mathcal{S} = \mathcal{L} \cup \mathcal{R}$. The *handedness* of a symbol $\phi \in \mathcal{S}$, denoted $h(\phi)$, is defined to be L if $\phi \in \mathcal{L}$ or R if $\phi \in \mathcal{R}$. If we wish to emphasize that $x \in \mathcal{X}$ is the (unique) indeterminate appearing in the symbol ϕ, then we write $\phi(x)$ instead of ϕ. A *word* in $\mathcal{S}$ is a formal expression $W = \phi_1 \cdots \phi_d$, where $d \geq 0$ and $\phi_i \in \mathcal{S}$ for $1 \leq i \leq d$; we denote the empty word by 1 and the set of all words in $\mathcal{S}$ by $\mathcal{S}^*$. We write $W = W(x_1, \ldots, x_m)$ to express the fact that $x_1, \ldots, x_m$ are the (distinct) indeterminates appearing in W. We call d the *length* of W and define the *transpose* of W by $W^t = \phi_d \cdots \phi_1$. A word W is called *balanced* if one element of $\{\phi_1, \phi_d\}$ is from $\mathcal{L}$ and the other is from $\mathcal{R}$. Likewise, W is called *heterogeneous* if the symbols in W are drawn from both $\mathcal{L}$ and $\mathcal{R}$, or *homogeneous* otherwise. An *alternating word* is a word $W = \phi_1 \cdots \phi_d$ in which ϕ_i is a left multiplication symbol when i is odd and a right multiplication symbol when i is even (or vice versa). Finally, W is a *palindrome* if for every i, $1 \leq i \leq d$, ϕ_i and ϕ_{d+1-i} are either both in $\mathcal{L}$ or both in $\mathcal{R}$. Note that an alternating word is a palindrome if and only if it has odd length.

We also need some notation to describe the process of substituting elements of a fixed groupoid G for the indeterminates in a word $W \in \mathcal{S}^*$ to obtain a map from G to itself. One might think of this process as *realizing* an abstract word in left and right multiplication symbols as an actual composition of left and right multiplication maps in G. Specifically, if $\phi = \phi(x) \in \mathcal{S}$, then we write $\phi(a)$ to mean L_a if $h(\phi) = L$ or R_a if $h(\phi) = R$. More generally, if $W = W(x_1, \ldots, x_d) = \phi_1(x_1) \cdots \phi_d(x_d) \in S^*$ and $a_1, \ldots, a_d \in G$, we write $W(a_1, \ldots, a_d)$ to mean the composition $\phi_1(a_1) \cdots \phi_d(a_d)$. Although there are no indeterminates appearing in the empty word 1, we stipulate that its realization (under any substitution) be the identity map 1_G.

An *identity* (in $\mathcal{S}^*$) is a statement $\mathcal{I} : W_1 = W_2$, where $W_1, W_2 \in \mathcal{S}^*$ are words. Such an identity is called *linear* if every indeterminate present appears exactly once in each of W_1 and W_2. This condition necessitates that W_1 and W_2 have the same length, which we call the length of $\mathcal{I}$. By extension, we call an identity *heterogeneous* if it involves both left and right multiplication symbols or *homogeneous* otherwise. Likewise, an identity is *alternating* (*palindromic*) if both sides are alternating (respectively, palindromes). Finally, an identity is *symmetric* if it takes the form $W = W^t$ for some word W; such an identity is called *balanced* if W is balanced. Since alternating symmetric identities will be of key importance in our results, we reserve certain notation for them. For $n \geq 2$, we define:

$$Alt(2n-1, L) : L_{x_1} R_{x_2} L_{x_3} \cdots L_{x_{2n-3}} R_{x_{2n-2}} L_{x_{2n-1}} = L_{x_{2n-1}} R_{x_{2n-2}} L_{x_{2n-3}} \cdots L_{x_3} R_{x_2} L_{x_1}$$

$$Alt(2n-1, R) : R_{x_1} L_{x_2} R_{x_3} \cdots R_{x_{2n-3}} L_{x_{2n-2}} R_{x_{2n-1}} = R_{x_{2n-1}} L_{x_{2n-2}} R_{x_{2n-3}} \cdots R_{x_3} L_{x_2} R_{x_1}$$

$$Alt(2n) : L_{x_1} R_{x_2} L_{x_3} \cdots R_{x_{2n-2}} L_{x_{2n-1}} R_{x_{2n}} = R_{x_{2n}} L_{x_{2n-1}} R_{x_{2n-2}} \cdots L_{x_3} R_{x_2} L_{x_1}.$$

Similarly, we define the *left homogeneous identities*

$$LHO_n : L_{x_1} \cdots L_{x_n} = L_{x_n} \cdots L_{x_1}$$

and the *right homogeneous identities*

$$RHO_n : R_{x_1} \cdots R_{x_n} = R_{x_n} \cdots R_{x_1}.$$

Strictly speaking, the definitions above correspond to *families* of alternating identities, since there are many possible choices of indeterminates from $\mathcal{X}$. In the interest of convenience, though, we will abuse terminology and refer to particular members of these families as $Alt(2n-1, L)$, etc.

Let $\mathcal{C}$ be a category whose objects are groupoids. An identity $\mathcal{I}$ is *satisfied* in some object G of $\mathcal{C}$ if the two sides of $\mathcal{I}$ are equal upon substitution of any choice of elements of G for the indeterminates appearing in $\mathcal{I}$. We say that an identity $\mathcal{I}$ *implies* an identity $\mathcal{J}$ in $\mathcal{C}$ (and write $\mathcal{I} \Rightarrow_{\mathcal{C}} \mathcal{J}$) if, whenever $\mathcal{I}$ is satisfied in some object of $\mathcal{C}$, $\mathcal{J}$ is also satisfied in G. In this article, $\mathcal{C}$ is almost always the category of quasigroups, so we typically suppress mention of the category and simply write $\mathcal{I} \Rightarrow \mathcal{J}$.

We summarize our main result as follows:

Theorem. *(See Theorem 3.1)*
Let $\mathcal{I}$ be a symmetric linear identity of length at least 2. Then:

- *$\mathcal{I}$ implies commutativity if and only if $\mathcal{I}$ is heterogeneous, of length at least 3, and not an alternating identity of odd length.*
- *$\mathcal{I}$ implies associativity if and only if $\mathcal{I}$ is heterogeneous and of even length.*

In Section 4, we apply this result to give a complete classification and description of the varieties of quasigroups satisfying symmetric linear identities of length at least 2. We then prove:

Theorem. *(See Theorem 4.1)*
There are exactly eight varieties of quasigroups satisfying symmetric linear identities of length at least 2.

When working with quasigroups, we use juxtaposition to denote the principal operation, and the standard notation $/$ and $\backslash$ for the operations defined by $a/b = R_b^{-1}(a)$ and $b\backslash a = L_b^{-1}(a)$, respectively. The identities $(a/b)b = a$, $(ab)/b = a$, $a(a\backslash b) = b$ and $a\backslash(ab) = b$ all follow directly from the definitions and will be used frequently without mention. To control proliferation of parentheses, we write $x \cdot yz$ in place of $x(yz)$, etc. It is understood that $\cdot$ takes precedence over all other operators.

Often, arguments may be shortened by recourse to the following device. If $(G, *)$ is a groupoid, we define its *opposite groupoid* G^{op} to be the groupoid whose underlying set is the same as that of G, but equipped with the operation $\circ$ defined by $a \circ b = b * a$. Clearly, left multiplication in G corresponds to right multiplication in G^{op} and vice versa. If W is a word, we define its *opposite* $\overline{W}$ as the word obtained from W by switching the handedness of every symbol in W, but keeping the same indeterminates throughout. Consequently, an identity $\mathcal{I} : W_1 = W_2$ holds in G if and only the opposite identity $\overline{\mathcal{I}} : \overline{W_1} = \overline{W_2}$ holds in G^{op}. A particularly useful observation is that if $\mathcal{I}$ and $\mathcal{J}$ are identities, then $\mathcal{I} \Rightarrow \mathcal{J}$ if and only if $\overline{\mathcal{I}} \Rightarrow \overline{\mathcal{J}}$. This is especially relevant when working with self-opposite identities like the associative law and the commutative law: if one needs to show that one of these is implied by some other identity $\mathcal{I}$, it is often more convenient to argue instead that it is implied by $\overline{\mathcal{I}}$. We will frequently make use of this principle without explicit mention.

Many results in this article were inspired by computations performed by the automatic theorem prover `Prover9` and its associated model builder `Mace4` [6]. Nevertheless, with the notable exception of Lemma 2.5, all proofs were developed by hand, and are not mere transcriptions of `Prover9` output.

We thank the referee for a careful reading, resulting in a variety of suggestions which helped improve the quality of this paper.

2 Preliminaries

2.1 Balanced identities and the multiplication group

In preparation for studying identities implying commutativity, we study balanced identities, which have particularly pleasant properties. Observe that a balanced identity takes the form $L_x W R_z = R_z W' L_x$ for appropriate words W and W'.

Following [7], we make some definitions. Let G be a groupoid and define:

$$AL(G) = \{f \in P(G) : f(xy) = g(x)y \text{ for some } g \in P(G) \text{ and all } x, y \in G\}$$
$$BL(G) = \{f \in P(G) : f(xy) = xg(y) \text{ for some } g \in P(G) \text{ and all } x, y \in G\}$$

We say that G is AL-transitive (BL-transitive) if for all $x, y \in G$ there exists $f \in AL(G)$ (respectively, $f \in BL(G)$) such that $f(x) = y$.

These sets are particularly meaningful when G is a *division groupoid*, a groupoid in which all left and right multiplication maps are surjective. A key property undergirding our arguments is a rigidity principle which appears in [7] as Lemma 2.5. We give a slightly modified version of this below.

Lemma 2.1. *[7, Lemma 2.5] Suppose a division groupoid G is BL-transitive. If $f, f' \in AL(G)$ and $f(a) = f'(a)$ for some $a \in G$, then $f = f'$. The same result holds if G is assumed to be AL-transitive and $f, f' \in BL(G)$.*

Proof.
Suppose that G is BL-transitive and $f, f' \in AL(G)$, $a \in G$ are such that $f(a) = f'(a)$. Select $c \in G$ and then use surjectivity of L_c to find $d \in G$ such that $a = cd$. Next, given $z \in G$, use BL-transitivity to find $h \in BL(G)$ such that $h(a) = z$. Let $g, g', k \in P(G)$ witness that the formulas $f(xy) = g(x)y$, $f'(xy) = g'(x)y$, and $h(xy) = xk(y)$ hold for all $x, y \in G$. Now

$$f(z) = f(h(a)) = f(h(cd)) = f(ck(d)) = g(c)k(d) = h(g(c)d) = h(f(cd)) = hf(a)$$

$$= hf'(a) = hf'(cd) = h(g'(c)d) = g'(c)k(d) = f'(ck(d)) = f'(h(cd)) = f'(h(a)) = f'(z).$$

The proof of the second statement is similar.

□

The relevance of balanced identities is made apparent by the next result.

Corollary 2.2. *Let G be a division groupoid satisfying a balanced linear identity*

$$L_x W(y_1, \ldots, y_m) R_z = R_z W'(y_1, \ldots, y_m) L_x. \tag{2}$$

Then for every $a \in G$, there exist $b_1, \ldots, b_m \in G$ such that

$$L_a W(b_1, \ldots, b_m) = R_a W'(b_1, \ldots, b_m) = 1_G.$$

Proof.
The hypothesis implies that for all $a, c_1, \ldots, c_m \in G$, $R_a W'(c_1, \ldots, c_m) \in BL(G)$ and $L_a W(c_1, \ldots, c_m) \in AL(G)$; it is moreover obvious that $1_G \in AL(G) \cap BL(G)$. We next argue that G is AL-transitive. Given $x, y \in G$, choose $c_1, \ldots, c_m \in G$ arbtirarily, and let $z = W(c_1, \ldots, c_m)x$. Since R_z is surjective, there exists $a \in G$ such that $az = y$, i.e. $L_a W(c_1, \ldots, c_m)x = y$. As $L_a W(c_1, \ldots, c_m) \in AL(G)$, this shows that G is AL-transitive. A similar argument establishes that G is BL-transitive.

Now fix $a \in G$ and use surjectivity of the multiplication maps to select $b_1, \ldots, b_m \in G$ such that $L_a W(b_1, \ldots, b_m)(a) = a$. If we define $f = L_a W(b_1, \ldots, b_m)$, then $f = 1_G$ by Lemma 2.1. Substituting $x = z = a$ and $y_i = b_i$ into (2), we have

$$1_G(aa) = 1_G R_a(a) = L_a W(b_1, \ldots, b_m) R_a(a) = R_a W'(b_1, \ldots, b_m) L_a(a) = [R_a W'(b_1, \ldots, b_m)](aa).$$

Again by Lemma 2.1, $R_a W'(b_1, \ldots, b_m) = 1_G$. $\square$

If G is a quasigroup, then, Under the hypothesis of Corollary 2.2, the inverse of a left or right multiplication map in G is itself a word in left and right multiplication maps. Thus, we have the following interesting consequence:

Corollary 2.3. *If G is a quasigroup satisfying a balanced linear identity, then the multiplication group $\mathcal{M}(G)$ consists of all words in left and right multiplication maps.*

2.2 The Cancellation Principle.

We will often find it convenient to simplify arguments by replacing an unbalanced identity by a balanced identity, using a shortening process which we call the *cancellation principle.*

Suppose $\mathcal{I} : W = W^t$ is a symmetric, unbalanced linear identity. Writing $W = \phi_1 \cdots \phi_d$, with $\phi_i = \phi_i(x_i) \in \mathcal{S}$ for $1 \leq i \leq d$, the condition of being unbalanced implies $h(\phi_1) = h(\phi_d)$. Formally setting $x_d = x_1$, we obtain the identity $\phi_1 \phi_2 \cdots \phi_{d-1} \phi_1 = \phi_1 \phi_{d-1} \cdots \phi_2 \phi_1$. Because the symbols ϕ_i represent formal left and right multiplication maps, which become actual left and right multiplication maps when elements of a particular quasigroup are substituted for the indeterminates, bijectivity of these maps allows us to justify canceling ϕ_1 on the left and right of both sides of the equation to obtain the shorter identity $\phi_2 \cdots \phi_{d-1} = \phi_{d-1} \cdots \phi_2$.

Arguing inductively, we deduce a general principle. Suppose $\mathcal{I} : W = W^t$ is a nonpalindromic symmetric linear identity, with $W = \phi_1 \cdots \phi_d$ as above. Because W is not a palindrome, it must be heterogeneous, and moreover there exists i, $1 \leq i \leq d/2$, such that $h(\phi_i) \neq h(\phi_{d-i+1})$; let $i(W)$ denote the smallest such integer i. Substituting $x_{d+1-i} = x_i$ for i, $1 \leq i \leq i(W)$, and canceling successively, we obtain the shorter identity

$$\phi_{i(W)} \cdots \phi_{d-i(W)+1} = \phi_{d-i(W)+1} \cdots \phi_{i(W)}$$

This identity will still be symmetric and linear, but also has the advantage of being balanced. Moreover, since terms are canceled in pairs, the length of the shortened identity has the same parity as that of the original identity. Of course, even if W

is palindromic, it is still possible (for identities of length at least 3) to cancel terms in pairs to shorten the identity, but this process will never result in a balanced identity. We refer to this general principle of shortening an unbalanced identity as the *cancellation principle.*

As an immediate application of the cancellation principle, we obtain the following result about homogeneous identities.

Proposition 2.4.

- *All left-homogeneous identities of even (odd) length are equivalent, as are all right-homogeneous identities of even (respectively, odd) length.*
- *Every left (right) homogeneous identity of even length implies every left (respectively, right) homogeneous identity of odd length.*

Proof.
For $n \geq 1$, the implication $LHO_{2n} \Rightarrow LHO_2$ follows from the cancellation principle. The identity LHO_2 is simply the statement that any two left multiplication operators commute, so clearly $LHO_2 \Rightarrow LHO_k$ for all $k \geq 2$. Thus, a left homogeneous identity of even length implies every left homogeneous identity. Similarly, cancellation shows $LHO_{2n+1} \Rightarrow LHO_3$, so it only remains to prove $LHO_3 \Rightarrow LHO_{2n+1}$. We argue by induction on n, the case $n = 1$ being trivial. Assume that $n > 1$ and $LHO_3 \Rightarrow LHO_{2n-1}$. The following argument establishes $LHO_3 \Rightarrow LHO_{2n+1}$, by application of LHO_{2n-1} to the parenthesized expressions and LHO_3 to the expression in square brackets.

$$\begin{aligned}
&(L_{x_1} \cdots L_{x_{2n-1}}) L_{x_{2n}} L_{x_{2n+1}} \\
&= L_{x_{2n-1}} (L_{x_{2n-2}} \cdots L_{x_1} L_{x_{2n}}) L_{x_{2n+1}} \\
&= L_{x_{2n-1}} L_{x_{2n}} (L_{x_1} \cdots L_{x_{2n-2}} L_{x_{2n+1}}) \\
&= [L_{x_{2n-1}} L_{x_{2n}} L_{x_{2n+1}}] L_{x_{2n-2}} \cdots L_{x_2} L_{x_1} \\
&= L_{x_{2n+1}} L_{x_{2n}} L_{x_{2n-1}} L_{x_{2n-2}} \cdots L_{x_2} L_{x_1}.
\end{aligned}$$

By arguing in the opposite groupoid, we see that the analogous statements hold for right-homogeneous identities. □

2.3 Alternating identities

The goal of this section is to prove that all alternating identities of odd length are equivalent. The nontrivial part is to show that $Alt(3, L)$ and $Alt(3, R)$ are equivalent; by arguing in the opposite quasigroup, it is sufficient to show $Alt(3, L) \Rightarrow Alt(3, R)$. The proof of this assertion proceeds in two phases. In the first phase, we show that $Alt(3, L)$ implies the so-called *medial* (or *entropic*) identity $xz \cdot yu = xy \cdot zu$. In the second phase, we derive further consequences of $Alt(3, L)$, which, when combined with the medial identity, imply $Alt(3, R)$.

Lemma 2.5. *A quasigroup satisfies $Alt(3, L)$ if and only if it satisfies $Alt(3, R)$.*

Proof.
Observe that $Alt(3, L)$, which is the identity $x((yz)u) = y((xz)u)$, is equivalent, via the substitutions $x \mapsto x/z$ and $y \mapsto y/z$, to $x/z \cdot yu = y/z \cdot xu$. Renaming variables, we have

$$x/y \cdot zu = z/y \cdot xu. \tag{3}$$

Substitute $z \mapsto zy$ to obtain

$$(x/y)(zy \cdot u) = z \cdot xu. \tag{4}$$

On the other hand, substituting $y \mapsto xy$, $u \mapsto y$ into (3) yields

$$x/(xy) \cdot zy = z. \tag{5}$$

Putting $z \mapsto z/y$ in (5), we obtain

$$(x/(xy))z = z/y \tag{6}$$

and multiplying both sides of (6) on the right by y, we have

$$(x/(xy))z \cdot y = z. \tag{7}$$

Next, we interchange x and z in (6) and rewrite it as $(z/(zy))x = x/y$; then, substituting $y \mapsto z\backslash y$, we deduce $z/(z(z\backslash y)) \cdot x = x/(z\backslash y)$, i.e. $x/(z\backslash y) = z/y \cdot x$. Renaming variables once more, we conclude

$$x/(y\backslash z) = y/z \cdot x. \tag{8}$$

Starting from (7) and substituting $y \mapsto x\backslash y$, we have

$$(x/y)z \cdot (x\backslash y) = z, \tag{9}$$

which, upon making the substitution $x \mapsto xy$, becomes $xz \cdot ((xy)\backslash y) = z$. Renaming variables once again, we obtain

$$xy \cdot ((xz)\backslash z) = y. \tag{10}$$

Using (8), we have

$$xz = (xz)/z \cdot z = z/((xz)\backslash z),$$

whereas replacing y by means of (10) yields

$$yu = (xy \cdot ((xz)\backslash z))u.$$

For convenience, set $w = (xz)\backslash z$. From the formulas immediately above, we obtain

$$xz \cdot yu = (z/w) \cdot ((xy \cdot w)u) \tag{11}$$

Finally, applying (4) to the right side of (11), we deduce the medial identity

$$xz \cdot yu = xy \cdot zu. \tag{12}$$

Returning to (5), substitute $y \mapsto x\backslash y$ to get $x/y \cdot z(x\backslash y) = z$; then substitute $x \mapsto xy$ to obtain $x \cdot z((xy)\backslash y) = z$. Renaming variables gives $x \cdot y((xz)\backslash z) = y$, and dividing each side on the left by x yields

$$y((xz)\backslash z) = x\backslash y. \tag{13}$$

From the medial identity (12), we have $xy \cdot u((zw)\backslash w) = xu \cdot y((zw)\backslash w)$. Applying (13) to both sides of this equation, we conclude

$$xy \cdot (z\backslash u) = xu \cdot (z\backslash y). \tag{14}$$

Now substitute $u \mapsto zu$ and $y \mapsto zy$ to conclude $(x(zy))u = (x(zu))y$, which is precisely $Alt(3, R)$. □

Remark.
Even though $Alt(3, L)$ (or, equivalently, $Alt(3, R)$) implies the medial identity, the converse implication does not hold. A counterexample is furnished by the quasigroup A whose Cayley table is given in Section 3.1.

Corollary 2.6. *All alternating identities of odd length are equivalent.*

Proof.
The cancellation principle shows that for all $n \geq 1$, $Alt(2n+1, L)$ implies $Alt(3, L)$ if n is even or $Alt(3, R)$ if n is odd. Likewise, $Alt(2n+1, R)$ implies $Alt(3, R)$ if n is even or $Alt(3, L)$ if n is odd. Because $Alt(3, L)$ and $Alt(3, R)$ are equivalent by Lemma 2.5, we conclude that any alternating identity of odd length implies both alternating identities of length 3. It remains to show that for all $n \geq 1$, $Alt(2n+1, L)$ and $Alt(2n+1, R)$ can be deduced from $Alt(3, L)$ (or equivalently from $Alt(3, R)$). We proceed by induction on n. The base case $n = 1$ is Lemma 2.5, so suppose $n > 1$ and $Alt(3, L)$ holds. By induction, $Alt(2n-1, L)$ and $Alt(2n-1, R)$ hold. Then, reasoning as in the proof of Proposition 2.4:

$$
\begin{aligned}
&(L_{x_1} R_{x_2} \cdots R_{x_{2n-2}} L_{x_{2n-1}}) R_{x_{2n}} L_{x_{2n+1}} \\
&= (L_{x_{2n-1}} R_{x_{2n-2}} \cdots R_{x_2} L_{x_1}) R_{x_{2n}} L_{x_{2n+1}} \\
&= L_{x_{2n-1}} (R_{x_{2n-2}} L_{x_{2n-3}} \cdots L_{x_1} R_{x_{2n}}) L_{x_{2n+1}} \\
&= L_{x_{2n-1}} (R_{x_{2n}} L_{x_1} \cdots L_{x_{2n-3}} R_{x_{2n-2}}) L_{x_{2n+1}} \\
&= L_{x_{2n-1}} R_{x_{2n}} (L_{x_1} R_{x_2} \cdots R_{x_{2n-2}} L_{x_{2n+1}}) \\
&= L_{x_{2n-1}} R_{x_{2n}} (L_{x_{2n+1}} R_{x_{2n-2}} \cdots R_{x_2} L_{x_1}) \\
&= (L_{x_{2n-1}} R_{x_{2n}} L_{x_{2n+1}}) R_{x_{2n-2}} \cdots R_{x_2} L_{x_1} \\
&= (L_{x_{2n+1}} R_{x_{2n}} L_{x_{2n-11}}) R_{x_{2n-2}} \cdots R_{x_2} L_{x_1}
\end{aligned}
$$

Thus, $Alt(3, L) \Rightarrow Alt(2n+1, L)$. By recourse to the opposite quasigroup, we see $Alt(3, R) \Rightarrow Alt(2n+1, R)$. □

3 Main Result

We now come to the statement of our main theorem, which will be proven by stages in Sections 3.1 and 3.2.

Theorem 3.1. *Let $\mathcal{I}$ be a symmetric linear identity of length at least 2. Then:*

- *$\mathcal{I}$ implies commutativity if and only if $\mathcal{I}$ is heterogeneous, of length at least 3, and not an alternating identity of odd length.*
- *$\mathcal{I}$ implies associativity if and only if $\mathcal{I}$ is heterogeneous and of even length.*

3.1 Negative results

In this section, we prove those parts of Theorem 3.1 involving the exhibition of counterexamples. We begin by considering quasigroups represented by the Cayley tables below.

A

$*$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

B

$*$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

C

$*$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(1,0)	(1,1)	(0,1)
(0,1)	(1,1)	(0,1)	(0,0)	(1,0)
(1,0)	(0,1)	(1,1)	(1,0)	(0,0)
(1,1)	(1,0)	(0,0)	(0,1)	(1,1)

D

$*$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(1,0)	(1,1)	(0,1)	(0,0)
(1,0)	(1,1)	(1,0)	(0,0)	(0,1)
(1,1)	(0,1)	(0,0)	(1,1)	(1,0)

To facilitate computation, it is convenient to have formulas for the respective operations $*$ on each of these quasigroups. In each of the following formulas, $+$ denotes either addition modulo 3 (on the underlying set $\{0,1,2\}$ of the quasigroups A and B) or coordinatewise addition modulo 2 (on the underlying set $\{(0,0),(0,1),(1,0),(1,1)\}$ of the quasigroups C and D).

$$\begin{aligned}
A: &\quad x*y = 2x+y+1\\
B: &\quad x*y = 2x+2y\\
C: &\quad (x_1,y_1)*(x_2,y_2) = (x_2+y_1+y_2, x_1+x_2+y_1)\\
D: &\quad (x_1,y_1)*(x_2,y_2) = (x_1+x_2+y_1, x_1+y_2+(x_1+y_1)x_2)
\end{aligned}$$

Proposition 3.2.

- *No homogeneous linear identity implies either commutativity or associativity.*
- *No symmetric linear identity of odd length implies associativity.*
- *No alternating identity of odd length implies either commutativity or associativity.*

Proof.

It is clear that none of these quasigroups are groups, and that B is commutative, whereas A and C are not. It is easy to check that in A, $L_2 = L_0^2$ and $L_1 = L_0^3$, so all left multiplication maps commute and hence every homogeneous linear identity in left multiplication symbols is satisfied in A. Likewise, every homogeneous linear identity in right multiplication symbols is satisfied in A^{op}. Thus, homogeneous linear identities imply neither commutativity nor associativity.

Now consider a symmetric identity $\mathcal{I}$ of odd length. Because B is commutative, we have $L_x = R_x$; hence to verify that $\mathcal{I}$ holds in B, it suffices to show that $L_{x_1} \cdots L_{x_n} = L_{x_n} \cdots L_{x_1}$ holds for n odd. For $i, j \in \{0, 1, 2\}$, $L_i(i) = i$ and $L_i(j) \neq j$ for $j \neq i$, so L_0, L_1, and L_2 are transpositions in the symmetric group $S_{\{0,1,2\}}$. If $b_1, \ldots, b_n \in B$, then $\sigma = L_{b_1} \cdots L_{b_n}$, being the product of an odd number of transpositions in $S_{\{0,1,2\}}$, is an odd permutation and hence must itself be a transposition. Therefore, $L_{b_1} \cdots L_{b_n} = \sigma = \sigma^{-1} = L_{b_n}^{-1} \cdots L_{b_1}^{-1} = L_{b_n} \cdots L_{b_1}$, and so a symmetric linear identity of odd length does not imply associativity.

To prove the last statement, it suffices (by Corollary 2.6) to show that $Alt(3, L)$ holds in the quasigroup C. This can be done by brute force; however, we instead give a more conceptual argument that any alternating identity of odd length holds in C. Direct computation shows that the identity $R_x = L_x^2$ holds in C, and that $x^2 = x$ for all $x \in C$. Since $L_x^3 = 1$, it follows that $R_x = L_x^{-1}$ and $L_x = R_x^{-1}$. By identifying the multiplication group $\mathcal{M}(C)$ with a subgroup of S_4 in the natural manner, the various left and right multiplication maps in $\mathcal{M}(C)$ correspond to the eight elements of order 3 in S_4. Thus, $\mathcal{M}(C) \cong A_4$.

We claim that the identities $L_x R_y = L_y R_x$ and $R_x L_y = R_y L_x$ also hold in C. If $a, b \in C$ are distinct, then $R_b = L_b^{-1} \neq L_a^{-1}$, so $L_a R_b \neq 1_C$. However, $L_a R_b$ fixes neither a nor b, so $L_a R_b$ must have order 2 as a member of $\mathcal{M}(C)$. Thus, $L_a R_b = (L_a R_b)^{-1} = R_b^{-1} L_a^{-1} = L_b R_a$. Since $L_a R_b = L_b R_a$ obviously holds when $a = b$, we have established the identity $L_x R_y = L_y R_x$. From this, we can easily deduce the other identity:

$$R_x L_y = L_y^{-1} L_y R_x L_y = L_y^{-1} L_x R_y L_y = L_y^{-1} L_x = R_y L_x.$$

Using these two identities, it is easy to see that for all $n \geq 1$, both $Alt(2n+1, L)$ and $Alt(2n+1, R)$ hold in C. Therefore, an alternating identity of odd length implies neither commutativity nor associativity. $\square$

3.2 Positive Results

The results of this section will finish the proof of Theorem 3.1.

Lemma 3.3. *A quasigroup satisfying a left- (right-)homogeneous symmetric identity of even length has a left (respectively, right) neutral element.*

Proof.
Suppose G is a quasigroup satisfying $L_{x_1} \cdots L_{x_{2d}} = L_{x_{2d}} \cdots L_{x_1}$. Select $a \in G$ arbitrarily and choose $e \in G$ such that $L_e(a) = a$. Now suppose $b \in G$. Select $a_2, \dots, a_d \in G$ arbitrarily; then use bijectivity of the multiplication maps to select $a_{d+1} \in G$ such that $L_{a_2} \cdots L_{a_d} L_{a_{d+1}} L_{a_d} \cdots L_{a_2} a = b$. Then

$$L_e b = L_e L_{a_2} \cdots L_{a_d} L_{a_{d+1}} L_{a_d} \cdots L_{a_2} a = L_{a_2} \cdots L_{a_d} L_{a_{d+1}} L_{a_d} \cdots L_{a_2} L_e a = b.$$

The assertion for right multiplication maps can be proved by reference to the opposite quasigroup. □

We begin with the case of symmetric balanced identities.

Proposition 3.4. *Every heterogeneous nonpalindromic linear identity implies commutativity.*

Proof.
Let $\mathcal{I}$ be a heterogeneous nonpalindromic linear identity. By the cancellation principle, we may assume without loss of generality that $\mathcal{I}$ is balanced, i.e. $\mathcal{I}$ takes on the form $L_x W(y_1, \dots, y_m) R_z = R_z W'(y_1, \dots, y_m) L_x$. Now fix $a \in G$; by Corollary 2.2 there exist $b_1, \dots, b_m \in G$ such that $W(b_1, \dots, b_m) = L_a^{-1}$ and $W'(b_1, \dots, b_m) = R_a^{-1}$. Then, substitute $y_i = b_i$ in the original identity and apply the above formulae to obtain $L_x L_a^{-1} R_z = R_z R_a^{-1} L_x$. Setting $x = z = a$, we deduce $R_a = L_a$. Since a was arbitrary, we have established the commutative law. □

Proposition 3.5. *Every heterogeneous, palindromic, symmetric linear identity of even length $2d \geq 4$ implies commutativity.*

Proof.
Let $W = \phi_1(x_1) \cdots \phi_d(x_d) \phi_d(y_d) \cdots \phi_1(y_1)$ be a heterogeneous, palindromic word of length at least 4, and consider the identity $\mathcal{I} : W = W^t$. By considering the opposite identity if necessary, we may assume without loss of generality that $h(\phi_d) = R$. Since $\mathcal{I}$ is heterogeneous, there exists i, $1 \leq i < d$, such that $h(\phi_i) = L$. By choosing i as large as possible, we see that our identity takes on the form

$$\phi_1(x_1) \cdots \phi_{i-1}(x_{i-1}) L_{x_i} R_{x_{i+1}} \cdots R_{x_d} R_{y_d} \cdots R_{y_{i+1}} L_{y_i} \phi_{i-1}(y_{i-1}) \cdots \phi_1(y_1)$$

$$= \phi_1(y_1)\cdots\phi_{i-1}(y_{i-1})L_{y_i}R_{y_{i+1}}\cdots R_{y_d}R_{x_d}\cdots R_{x_{i+1}}L_{x_i}\phi_{i-1}(x_{i-1})\cdots\phi_1(x_1),$$

which after cancellation implies

$$L_{x_i}R_{x_{i+1}}\cdots R_{x_d}R_{y_d}\cdots R_{y_{i+1}}L_{y_i} = L_{y_i}R_{y_{i+1}}\cdots R_{y_d}R_{x_d}\cdots R_{x_{i+1}}L_{x_i}. \tag{15}$$

Applying further cancellation to (15) yields the identity:

$$R_{x_d}R_{y_d} = R_{y_d}R_{x_d}. \tag{16}$$

By Lemma 3.3, G has a right neutral element $e \in G$. Now substitute $x_j = e$ and $y_j = e$ in (15) for j, $i+1 \leq j \leq d$, to obtain $L_{x_i}L_{y_i} = L_{y_i}L_{x_i}$. Applying both sides of this identity to e, we have $x_iy_i = y_ix_i$, which is the commutative law. □

The analogous statement for identities of odd length is more difficult to prove.

Proposition 3.6. *Every heterogeneous, palindromic, nonalternating, symmetric linear identity of odd length* $2d+1 \geq 3$ *implies commutativity.*

Proof.
Let $W = \phi_1(x_1)\cdots\phi_d(x_d)\phi_{d+1}(x_{d+1})\phi_d(x_{d+2})\cdots\phi_1(x_{2d+1})$ be a heterogeneous, palindromic, nonalternating word of length at least 3, and consider the identity $\mathcal{I} : W = W^t$. By substituting $x_{2d+2-i} = x_i$ for $1 \leq i \leq d-1$ into $\mathcal{I}$ and applying the cancellation principle, we obtain the shorter identity

$$\phi_d(x_d)\phi_{d+1}(x_{d+1})\phi_d(x_{d+2}) = \phi_d(x_{d+2})\phi_{d+1}(x_{d+1})\phi_d(x_d) \tag{17}$$

which we call the *core* of $\mathcal{I}$. We separate the proof into two cases, according to whether the core is homogeneous or heterogeneous.

If the core of $\mathcal{I}$ is homogeneous, then, since $\mathcal{I}$ is heterogeneous and palindromic, there exists some largest value i, $1 \leq i < d$, such that $h(\phi_i) \neq h(\phi_d)$. By considering the opposite groupoid, we may assume without loss of generality that $h(\phi_i) = h(\phi_{2d-i+2}) = L$, and that $h(\phi_j) = R$ for j, $i+1 \leq j \leq 2d-i+1$. Rewriting the core (17) as $R_xR_yR_z = R_zR_yR_x$, or $((uz)y)x = ((ux)y)z$, substitute $x \mapsto u\backslash x$ and $z \mapsto u\backslash z$ to obtain $(zy)(u\backslash x) = (xy)(u\backslash z)$, which can be recast as

$$R_{u\backslash x}L_z = R_{u\backslash z}L_x. \tag{18}$$

Applying the cancellation principle to $\mathcal{I}$ again, but this time cancelling off only $i-1$ pairs, we obtain the identity:

$$L_{x_i}R_{x_{i+1}}\cdots R_{x_{2d-i+1}}L_{x_{2d-i+2}} = L_{x_{2d-i+2}}R_{x_{2d-i+1}}\cdots R_{x_{i+1}}L_{x_i}. \tag{19}$$

Apply both sides to a new indeterminate v and write $w = R_{x_{2d-i}} \cdots R_{x_{i+1}} L_{x_i} v$ to yield

$$L_{x_i} R_{x_{i+1}} \cdots R_{x_{2d-i+1}} L_{x_{2d-i+2}} v = x_{2d-i+2}(w x_{2d-i+1}) \tag{20}$$

Next, substitute $x_{2d-i+1} \mapsto w \backslash x_{2d-i+1}$ to obtain

$$L_{x_i} R_{x_{i+1}} \cdots R_{x_{2d-i}} R_{w \backslash x_{2d-i+1}} L_{x_{2d-i+2}} v = x_{2d-i+2} x_{2d-i+1} \tag{21}$$

In light of (18), we see that the left side of (21) is invariant upon permutation of x_{2d-i+1} and x_{2d-i+2}. The same must be true of the right side; hence, $x_{2d-i+2}x_{2d-i+1} = x_{2d-i+1}x_{2d-i+2}$, and so commutativity holds.

Now suppose the core of $\mathcal{I}$ is heterogeneous. Since $\mathcal{I}$ is assumed to be palindromic and not alternating, it must contain two consecutive symbols of the same handedness. By considering the opposite quasigroup, we may assume without loss of generality that $h(\phi_i) = h(\phi_{i+1}) = L$ for some i, $1 \leq i < d$. The core of $\mathcal{I}$ is then either $Alt(3, L)$ or $Alt(3, R)$; however, since these two are logically equivalent by Lemma 2.5, we may assume in either case that $Alt(3, L)$ holds, i.e. $L_x R_y L_z = L_z R_y L_x$. Thus $x((zu)y) = z((xu)y)$, which, upon making the substitutions $x \mapsto x/u$, $z \mapsto z/u$, may be recast as $(x/u)(zy) = (z/u)(xy)$, or

$$L_{x/u} L_z = L_{z/u} L_x. \tag{22}$$

Now applying the cancellation principle to $\mathcal{I}$, we obtain

$$L_{x_i} L_{x_{i+1}} \phi_{i+2}(x_{i+2}) \cdots \phi_{2d-i}(x_{2d-i}) L_{x_{2d-i+1}} L_{x_{2d-i+2}} = L_{x_{2d-i+2}} L_{x_{2d-i+1}} \phi_{2d-i}(x_{2d-i}) \cdots \phi_{i+2}(x_{i+2}) L_{x_{i+1}} L_{x_i}. \tag{23}$$

Apply both sides to a new indeterminate v, and write $w = \phi_{2d-i}(x_{2d-i}) \cdots \phi_{x_{i+2}} L_{x_{i+1}} L_{x_i} v$ to deduce

$$L_{x_i} L_{x_{i+1}} \phi_{i+2}(x_{i+2}) \cdots \phi_{2d-i}(x_{2d-i}) L_{x_{2d-i+1}} L_{x_{2d-i+2}} v = L_{x_{2d-i+2}} L_{x_{2d-i+1}} w = x_{2d-i+2}(x_{2d-i+1} w) \tag{24}$$

Now substitute $x_{2d-i+1} \mapsto x_{2d-i+1}/w$ to obtain:

$$L_{x_i} L_{x_{i+1}} \cdots L_{x_d} R_{x_{d+1}} L_{x_{d+2}} \cdots L_{x_{2d-i+1}/w} L_{x_{2d-i+2}} v = x_{2d-i+2} x_{2d-i+1} \tag{25}$$

By (22), the left side of (25) is invariant under permutation of x_{2d-i+1} and x_{2d-i+2}. Thus, $x_{2d-i+2}x_{2d-i+1} = x_{2d-i+1}x_{2d-i+2}$, and commutativity holds in this case also. $\square$

We have now proved all assertions of Theorem 3.1 involving commutativity. The remaining statements now follow readily.

Proposition 3.7. *Every heterogeneous symmetric identity of even length implies associativity.*

Proof.
Suppose $\mathcal{I}$ takes on the form

$$\phi_1(x_1)\ldots\phi_d(x_{2d}) = \phi_d(x_{2d})\ldots\phi_1(x_1). \tag{26}$$

If $d = 1$, there is nothing to prove, so assume $d \geq 2$. If G is a quasigroup in which $\mathcal{I}$ is satisfied, then by Proposition 3.4 or 3.5, the commutative law $L_x = R_x$ holds in G. Replacing every right multiplication symbol in $\mathcal{I}$ with its corresponding left multiplication symbol, we observe that G satisfies a homogeneous symmetric identity of even length. By Lemma 3.3, G has a left neutral element e, which, by commutativity, must be a two-sided neutral element. Now choose integers i and j, $1 \leq i, j \leq d$, such that $h(\phi_i) = L$ and $h(\phi_j) = R$; without loss of generality, we may assume $i < j$. Then, set $x_k = e$ for all $k \neq i, j$ in (26) to conclude $L_{x_i}R_{x_j} = R_{x_j}L_{x_i}$. This shows that associativity holds in G. $\square$

Table 1:

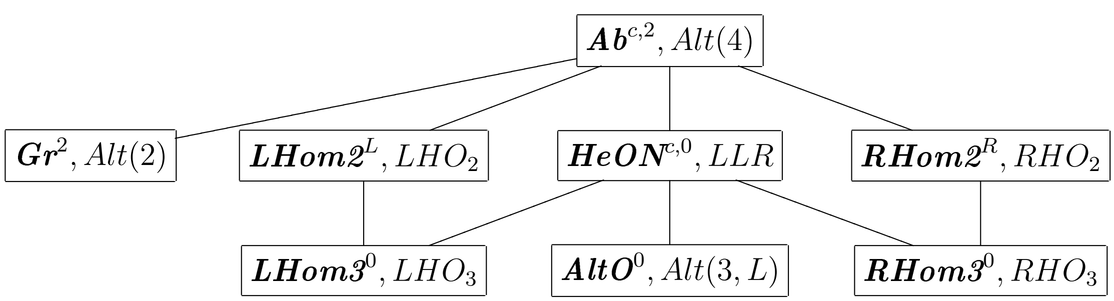
Name	*Identity*	*Description*
Ab	$Alt(4)$	Abelian groups: heterogeneous, of even length ≥ 4
Gr	$Alt(2)$	Groups: heterogeneous, of length 2
AltO	$Alt(3, L)$	Alternating, of odd length ≥ 3
LHom(2)	LHO_2	Homogeneous in L, of even length ≥ 2
LHom(3)	LHO_3	Homogeneous in L, of odd length ≥ 3
RHom(2)	RHO_2	Homogeneous in R, of even length ≥ 2
RHom(3)	RHO_3	Homogeneous in R, of odd length ≥ 3
HeON	LLR	Heterogeneous, nonalternating, of odd length ≥ 3

4 Varieties

We apply the results of Section 3 to classify varieties of quasigroups satisfying symmetric linear identities. For convenience of reference, we abbreviate by LLR the identity $L_xL_yR_z = R_zL_yL_x$.

Theorem 4.1. *There are exactly eight varieties of quasigroups satisfying symmetric linear identities of length at least 2. Their names and descriptions, along with a representative identity from each, is given in Table 1.*

The inclusions among these varieties are described by the following Hasse Diagram. Superscripts indicate properties enjoyed by quasigroups in that variety (c=commutative, 2=two-sided neutral element, L=left neutral element only, R=right neutral element only, 0=no neutral element).



The first step in the proof is to show that the list of varieties in Table 1 is exhaustive, and that all the implications in the above Hasse diagram are valid. Observe that $LHO_2 \Rightarrow LHO_3$ and $RHO_2 \Rightarrow RHO_3$ by Proposition 2.4. All left (respectively, right) homogeneous identities of even (respectively, odd length) are equivalent by Proposition 2.4 and all alternating identities of odd length are equivalent by Corollary 2.6. The remaining implications are proven below.

Lemma 4.2.

- *Let G be a quasigroup. Then*

  G is an abelian group $\Leftrightarrow$ $Alt(2n)$ holds in G for some $n \geq 2 \Leftrightarrow Alt(2n)$ holds in G for all $n \geq 2$.

- *Every nonalternating heterogeneous identity of odd length implies LHO_3, RHO_3, and $Alt(3, L)$. All nonalternating heterogeneous identities of odd length are equivalent.*

Proof.
In an abelian group, the identities $L_x R_y = R_y L_x$, $L_x L_y = L_y L_x$ and $R_x R_y = R_y R_x$ are all satisfied, so any symmetric linear identity will hold. Conversely, if G is a quasigroup in which $Alt(2n)$ holds for some $n \geq 2$, then by Theorem 3.1 the operation on G must be both commutative and associative.

Now suppose $m, n \geq 1$, and $\mathcal{I}$, $\mathcal{I}'$ are nonalternating heterogeneous identities of respective lengths $2m + 1$ and $2n + 1$. By Theorem 3.1, $\mathcal{I}$ implies commutativity, i.e. $L_x = R_x$ holds; this in turn implies LHO_{2m+1}. By Proposition 2.4, LHO_3 holds; in conjunction with commutativity, this means that any symmetric identity of length 3 holds. On the other hand, Proposition 2.4 shows $LHO_3 \Rightarrow LHO_{2n+1}$; in conjunction with commutativity, this implies that any symmetric identity of length $2n + 1$ holds. In particular, $\mathcal{I} \Rightarrow \mathcal{I}'$. □

It remains to show that there are no further inclusions among the varieties. To this end, we construct a table summarizing which defining identities hold in each of the quasigroups A, B, C, and D defined in Section 3.1. A bullet in an entry means that the identity is satisfied in that quasigroup. A tuple of elements represents data constituting a counterexample: these are the elements to be substituted for the indeterminates appearing in the defining identity (read in order of appearance from left to right), the last coordinate being the element to which each side of that identity is to be applied. For instance, the entry $(1, 0, 0, 0)$ for the identity $Alt(3, L)$ and the quasigroup A means that $(L_1 R_0 L_0)0 \neq (L_0 R_0 L_1)0$. As none of the quasigroups in

Table 2:

Quasigroup	$Alt(3, L)$	LHO_2	LHO_3	RHO_2	RHO_3	LLR
A	$(1, 0, 0, 0)$	•	•	$(1, 0, 0)$	•	NC
B	•	$(1, 0, 0)$	•	$(1, 0, 0)$	•	•
C	•	$(01, 00, 00)$	$(01, 00, 00, 00)$	$(01, 00, 00)$	(01,00,00,00)	NC
D	$(10, 00, 00, 00)$	•	•	$(10, 00, 00)$	$(10, 00, 00, 00)$	NC

question are associative, we omit the columns corresponding to the varieties **Ab** and **Gr**. Since LLR implies commutativity by Theorem 3.1, we use the symbol 'NC' (noncommutative) to indicate the reason that LLR is not satisfied in the quasigroups A, C, and D. Finally, for brevity we use the notation ab in place of (a, b) for elements of the quasigroups C and D.

Lemma 4.3. *The only implications among the varieties in Table 1 are those shown in the Hasse diagram.*

None of the quasigroups in Table 2 satisfies associativity, yet each of the identities LHO_2, RHO_2, LLR is satisfied in at least one of these quasigroups. Thus $Alt(2)$, which defines the variety of groups, cannot be implied by any of these identities. Furthermore, direct computation in the group S_3 shows that $L_{(1\ 3\ 2)}L_{(1\ 2)}L_{(1\ 2\ 3)} \neq L_{(1\ 2\ 3)}L_{(1\ 2)}L_{(1\ 3\ 2)}$, so LHO_3 is not satisfied in S_3. Since S_3 is the unique non-abelian group of order 6, it is isomorphic to its opposite group, and thus RHO_3 is not satisfied in it, either. Finally, letting e denote the identity element of S_3, we have $L_{(1\ 2)}R_eL_{(1\ 3)} \neq L_{(1\ 3)}R_eL_{(1\ 2)}$; hence, $Alt(3, L)$ is not satisfied in S_3. Thus, $Alt(2)$ does not imply any among LHO_3, $Alt(3, L)$, and RHO_3.

From the table, we see that LHO_2 implies neither $Alt(3, L)$ nor RHO_3. Since $Alt(3, L) \Leftrightarrow Alt(3, R)$ by Lemma 2.5, it follows (by consideration of opposite structures) that RHO_2 implies neither $Alt(3, L)$ nor LHO_3. The quasigroup B witnesses that LLR implies neither LHO_2 nor RHO_2. Likewise, B shows that LHO_3 does not imply LHO_2; D shows that LHO_3 does not imply RHO_3, and A shows that LHO_3 does not imply $Alt(3, L)$. By consideration of opposite structures, we see that RHO_3 does not imply any identity among RHO_2, $Alt(3, L)$, and LHO_3. Finally, C shows that $Alt(3, L)$ does not any among LHO_3, RHO_3, and LLR. This concludes the proof that there can be no further containment relations among the varieties in the Hasse diagram. □

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