REPRESENTATION NUMBERS OF STARS

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Abstract

A graph G has a representation modulo r if there exists an injective map $f\colon V(G) \to \{0, 1, \ldots, r-1\}$ such that vertices u and v are adjacent if and only if $f(u)-f(v)$ is relatively prime to r. The representation number $rep(G)$ is the smallest positive integer r for which G has a representation modulo r. In this paper we study representation numbers of the stars $K_{1,n}$. We will show that the problem of determining $rep(K_{1,n})$ is equivalent to determining the smallest even k for which $\phi(k) \geq n$: we will solve this problem for "small" n and determine the possible forms of $rep(K_{1,n})$ for sufficiently large n.

1. Introduction

Let G be a finite graph with vertices $v_1, \ldots, v_k$. G is said to be *representable modulo* r if there exists an injective map $f\colon V(G) \to \{0, 1, \ldots, r-1\}$ such that vertices u and v are adjacent if and only if $\gcd(f(u) - f(v), r) = 1$: we refer to f as a *representative labeling* of G. Equivalently, G is representable modulo r if there exists an injective map $f\colon V(G) \to \mathbb{Z}_r$ such that v_i is adjacent to v_j if and only if $f(i) - f(j)$ is a unit of (the ring) $\mathbb{Z}_r$. The *representation number* of G, denoted $rep(G)$, is the smallest positive integer r modulo which G is representable.

The study of representation numbers was initiated by Erdős and Evans in [4]. The main result of [4] was that any finite graph can be represented modulo some positive integer: this was used to give a simpler proof of a result of Lindner et. al. that any finite graph can be realized as an orthogonal Latin square graph [12]. The proof in [4] established an upper bound on the representation number of an arbitrary graph that was later improved by Narayan in [14]. Representation numbers have since been studied for various classes of graphs (see [5], [6], [7], and [15]).

A list of graph representation problems is given in [8]: these include the problem of determining the representation numbers for complete bipartite and complete multipartite graphs. We restrict ourselves in this paper to the problem of determining representation number for a class of complete bipartite graphs, the stars $K_{1,n}$. The representation number of $K_{1,n}$ is bounded below by $2n$ as, by [6, Example (1.1)], the representation number of the edgeless graph on n vertices is $2n$. There are three upper bounds in the literature. In [6, Example (1.3)], it is shown that $rep(K_{1,n}) \leq \min\{2^{\lceil \log_2 n \rceil+1}, 3^{\lceil \log_2 n \rceil+1}, 2p\}$, where p is any prime greater than n. This was improved in [7, Corollary 5.7] to $rep(K_{1,n}) \leq \min\{2^{\lceil \log_2 n \rceil+1}, 2p\}$. One more upper bound is given in [15, Theorem 1]. Let m be a positive integer and let p be the smallest prime divisor of m: if $n \leq p^{k-1}\phi(m)$, then $rep(K_{1,n}) \leq p^k m$. In addition to these bounds, in [7] it was shown that $rep(K_{1,n})$ can never be a power of 3. We will significantly improve on these results.

In Section 2 we give some basic results, and we derive some general results on the representation numbers of complete bipartite graphs that will prove useful to us. In Section 3 we characterize the representation number of the star $K_{1,n}$ using Euler's phi function, and conjecture that this representation number is always of the form 2^a or $2^a p$, where p is a prime: we prove this conjecture true for "small" n in Sections 4 and 5. In Section 6 we prove a weaker version of this conjecture for large n; in particular, we show that for sufficiently large n, the representation number is of the form 2^a, $2^a p$, or $2^a pq$, where p and q are (not necessarily distinct) primes. In a sequel to this paper we will study representation numbers of complete multipartite graphs, with a particular focus on the case of complete bipartite graphs.

2. Preliminaries

We begin with a result concerning the distribution of primes. Although it follows from stronger results (see for example Theorem 6.1), it follows immediately from the Prime Number Theorem and suffices for many applications.

Theorem 2.1. *Given any $\beta > 1$, there exists a natural number $N(\beta)$ such that for $n > N(\beta)$, there is a prime number in the interval $(n, \beta n)$.*

A weaker bound due to Nagura [13] is often useful in that it gives explicit values for the quantity N mentioned above.

Proposition 2.2.

If $n \geq 25$, then there exists a prime between n and $\frac{6}{5}n$.

Lemma 2.3. *If G is any graph and $\ell \colon V(G) \to \mathbb{Z}_k$ is any representative labeling, then the following are also representative labelings:*

- *For any $a \in \mathbb{Z}_k$, $\tau_a \circ \ell$, where $\tau_a \colon \mathbb{Z}_k \to \mathbb{Z}_k$ is the translation map $x \mapsto x + a$.*
- *$\psi \circ \ell$, where $\psi \colon \mathbb{Z}_k \to \mathbb{Z}_k$ is any group automorphism.*

$K_{m,n}$ denotes the complete bipartite graph with partite sets A, B of respective size m and n. We always assume $m \leq n$ and set $N = m + n$; we use $\phi(n)$ for the Euler totient function. We begin with some basic upper and lower bounds.

Proposition 2.4.

Let p be the smallest prime greater than N. Then $2n \leq rep(K_{m,n}) \leq \min\{4n - 4, 2p\}$. In particular, given any $\varepsilon > 0$, $rep(K_{m,n}) \leq 2(1 + \varepsilon)N$ if N is sufficiently large.

Proof. The lower bound is a simple consequence of the facts that $\overline{K_n}$ is an induced subgraph of $K_{m,n}$ and $rep\left(\overline{K_n}\right) = 2n$ [6, Example 1.1].

For the upper bound, consider a labeling of $K_{m,n}$ which assigns vertices in A labels corresponding to odd integers from $[1, 2|A| - 1]$ and vertices in B even integers from $[0, 2|B| - 2]$. This shows that $K_{m,n}$ is representable modulo the smallest power of 2 greater than $2n - 2$. Since there will always be a power of 2 in the interval $(2n - 2, 4n - 4]$, we have $rep(K_{m,n}) \leq 4n - 4$.

Now suppose p is the smallest prime greater than N and consider a labeling ℓ of $K_{m,n}$ which assigns vertices in B even integers from $[0, 2|B| - 2]$ and vertices in A odd integers from $[|B| - |A|, |B| + |A|]$. If $x \in A$ and $y \in B$, then we have $|\ell(x) - \ell(y)| \leq |A| + |B| = N < p$, so this labeling represents $K_{m,n}$ modulo $2p$. The last statement follows from Theorem 2.1, applied with $\beta = 1 + \varepsilon$. □

Elementary considerations give the following necessary condition:

Lemma 2.5. *$\phi(rep(K_{m,n})) \geq n$.*

Proof. Fix a labeling of $K_{m,n}$ by $\mathbb{Z}_r$ where $r = rep(K_{m,n})$. By Lemma 2.3, we may assume without loss of generality that some vertex $v \in A$ is labeled 0. Since all vertices of B are adjacent to v, the labels on the vertices of B must all be relatively prime to r. Hence, $\phi(r) \geq |B| = n$. □

We write $\alpha(G)$ for the independence number of a graph G. The next result is useful in estimating the size of the smallest prime factor of $rep(G)$ when one has an upper bound for the latter.

Lemma 2.6. *Let G be any graph, and p the smallest prime divisor of $rep(G)$. Then*

$$\omega(G) \le p \le \frac{rep(G)}{\alpha(G)}.$$

Proof. Let $r = rep(G)$, p the smallest prime dividing r, $t = \alpha(G)$ and $S = \{s_1, \dots, s_t\}$ an independent set in G. Let $f\colon V(G) \to \{0, \dots, r-1\}$ be a representative labeling of G modulo r and $a_i = f(s_i)$, $i = 1, \dots, t$. Then $r = \sum_{i=1}^{t-1}(a_{i+1} - a_i) + (r - a_t + a_1)$. Since S is an independent set, each of the parenthesized expressions in the previous formula is divisible by some prime divisor of r; hence $r \ge tp$. The lower bound is established in [6, Theorem 1.2]. □

3. Characterizing representation numbers of stars

In this section we will characterize the representation numbers of stars using Euler's ϕ-function. We will conjecture the form of $rep(K_{1,n})$: evidence for this conjecture will be given in Sections 4 and 5, where it will be shown to be true for "small" values of n; a proof of a weaker form of this conjecture for large values of n will be given in Section 6.

Theorem 3.1. $rep(K_{1,n}) = \min\{k : 2|k$ *and* $\phi(k) \ge n\}$.

Proof. Suppose that k is even and $\phi(k) \ge n$. Then one may produce a $\mathbb{Z}_k$-labeling of $K_{1,n}$ as follows: label the root 0 and assign labels to the other vertices from $\mathbb{Z}_k^*$, the group of units of $\mathbb{Z}_k$. Notice that the difference between any two labels on leaves is even. Hence $rep(K_{1,n}) \le \min\{k : 2|k \text{ and } \phi(k) \ge n\}$.

Conversely, assume that $rep(K_{1,n}) = k$. We may assume by Lemma 2.3 that the root is labeled 0. Then the remaining n vertices must be labeled with units in $\mathbb{Z}_k^*$, so $n \le \phi(k)$. To complete the proof, we need to show that k is even. The smallest values of n, 1, 2, 3, and 4 are easily checked. The remaining values of n must be dealt with in two cases, $n > 25$ and $4 < n < 25$. If $n \ge 25$, then, by Propositions 2.4 and Proposition 2.2, $k \le \frac{12}{5}n$. If $4 < n < 25$ there is always an even integer $t < 3n$ for which $\phi(t) \ge n$. In either case, $k < 3n$, so by Lemma 2.6, k must be even. □

As an easy corollary we are able to characterize those stars $K_{1,n}$ whose representation numbers are $2n$ and $2n + 2$.

Corollary 3.2.

1. *If* $n \geq 3$*, then* $rep(K_{1,n}) = 2n$ *if and only if* n *is a power of* 2*.*

2. *If* $n \geq 3$*, then* $rep(K_{1,n}) = 2n + 2 = 2p$ *if and only if* n *is not a power of* 2*, and* $p = n + 1$ *is a prime.*

For small values of n, $rep(K_{1,n})$ is a power of 2 or a power of 2 times an odd prime: Table 1 displays the data for $n \leq 66$. This leads us to a conjecture. Let M_n denote the smallest positive integer M of the form 2^{k+1} or $2^{k+1}p$, k a nonnegative integer and p an odd prime, for which $\phi(M) \geq n$.

Table 1: Representation numbers for small stars.

n	$rep(K_{1,n})$	n	$rep(K_{1,n})$	n	$rep(K_{1,n})$
1	2	$19, 20$	$44 = 2^2 \times 11$	$43, 44$	$92 = 2^2 \times 23$
2	$4 = 2^2$	$21, 22$	$46 = 2 \times 23$	$45, 46$	$94 = 2 \times 47$
$3, 4$	$8 = 2^3$	$23, 24$	$52 = 2^2 \times 13$	$47, 48$	$104 = 2^3 \times 13$
$5, 6$	$14 = 2 \times 7$	$25, \ldots, 28$	$58 = 2 \times 29$	$49, \ldots, 52$	$106 = 2 \times 53$
$7, 8$	$16 = 2^4$	$29, 30$	$62 = 2 \times 31$	$53, \ldots, 56$	$116 = 2^2 \times 29$
$9, 10$	$22 = 2 \times 11$	$31, 32$	$64 = 2^6$	$57, 58$	$118 = 2 \times 59$
$11, 12$	$26 = 2 \times 13$	$33, \ldots, 36$	$74 = 2 \times 37$	$59, 60$	$122 = 2 \times 61$
$13, \ldots, 16$	$32 = 2^5$	$37, \ldots, 40$	$82 = 2 \times 41$	$61, \ldots, 64$	$128 = 2^7$
$17, 18$	$38 = 2 \times 19$	$41, 42$	$86 = 2 \times 43$	$65, 66$	$134 = 2 \times 67$

Conjecture 3.3. *For all* n*,* $rep(K_{1,n}) = M_n$*.*

Corollary 3.2 establishes the truth of Conjecture 3.3 when n is a power of 2 or $n+1$ is an odd prime. We will provide evidence for Conjecture 3.3 for small n: the cases M_n a power of 2, and M_n a power of 2 times an odd prime, will be handled separately in Sections 4 and 5, respectively. In Section 6 we will show that, for n sufficiently large, either $rep(K_{1,n}) = M_n$ or $rep(K_{1,n})$ is of the form $2^{k+1}pq$, where p and q are (not necessarily distinct) odd primes.

4. The case n small and $M_n = 2^{k+1}$.

For q an odd prime there exist unique positive integers a and s for which

$$2^{s-1} < q = 2^s - a < 2^s.$$

We define a mapping β from the set of odd primes to the set of positive rational numbers by $\beta(q) = (a+1)/2^s$, and we say that a prime p' is *β-constrained* if

$$\sqrt{q} < p' < \frac{1}{\beta(q)}$$

for some odd prime q.

As examples, 3 is β-constrained as $\sqrt{7} < 3 < 4 = 1/\beta(7)$; 7 is β-constrained as $\sqrt{31} < 7 < 16 = 1/\beta(31)$; and 11 is β-constrained as $\sqrt{31} < 11 < 16 = 1/\beta(31)$. Note that 5 is not β-constrained.

Conjecture 4.1. *Any odd prime other than* 5 *is β-constrained.*

One approach to trying to prove Conjecture 4.1 is to search for a sequence of odd primes q_i for which the intervals $(\sqrt{q_i}, 1/\beta(q_i))$ and $(\sqrt{q_{i+1}}, 1/\beta(q_{i+1}))$ overlap, and a reasonable candidate is the sequence $q_i = \max\{p : p < 2^i, p \text{ prime}\}$. If we set $I_i = (\sqrt{q_i}, 1/\beta(q_i))$, then any odd prime in I_i is, by definition, β-constrained. Now $I_5 = (\sqrt{31}, 16) \approx (5.6, 16)$ and we were able to verify using magma that I_i and I_{i+1} overlap for $i = 5, \ldots, 412$: magma could not determine the prime q_{414}. Thus any odd prime in the interval $(\sqrt{31}, 1/\beta(q_{413})) \approx (5.6, 9.6 \times 10^{122})$ is β-constrained.

Lemma 4.2. *Let $p_1 < p_2$ be odd primes. If $p_1 = 5$ or p_1 is β-constrained, then there exists an odd prime $q < p_1 p_2$ for which $p_1 < 1/\beta(q)$.*

Proof. p_1 being β-constrained implies that there exists an odd prime $q < p_1^2 < p_1 p_2$ for which $p_1 < 1/\beta(q)$.

If $p_1 = 5$, then $p_2 \geq 7$. Choosing $q = 31$, $q < p_1 p_2$, and $\beta(q) = 1/16$, from which the result follows. □

Theorem 4.3. *If $M_n = 2^{k+1}$ and $rep(K_{1,n}) \neq M_n$, then the smallest odd prime divisor of $rep(K_{1,n})$ is neither* 5 *nor β-constrained.*

Proof. Assume this to be false. That is, for some positive integer n, $M_n = 2^{k+1}$, $m = rep(K_{1,n}) \neq M_n$, the odd prime divisors of m are $p_1 < \cdots < p_r$, and p_1 is either 5 or β-constrained. Let $q < 2^k$ be an arbitrary odd prime, $2^{s-1} < q = 2^s - a < 2^s$, and set $\alpha = 2^{k-s+1} q$. Then $\alpha < M_n$, and by the hypothesis of the Theorem,

$$\begin{aligned} \phi(\alpha) = \; & 2^k \left(1 - \left(\frac{a+1}{2^s}\right)\right) < n \leq \phi(m) \\ = \; & \frac{m}{2}\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) \\ < \; & 2^k \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right). \end{aligned}$$

This reduces to

$$1 - \beta(q) < \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

If $r = 1$, then this inequality is equivalent to $1/p_1 < \beta(q)$, in which case $m = 2^{t+1} p_1^l$ for some $t \geq 0$ and some $l \geq 2$. If $t > 0$ or $l > 2$, then there exists a prime p', $p_1 < p' < 2^t p_1^{l-1}$.

By Lemma 4.2, we may choose $q < p_1 p'$ to satisfy $\beta(q) < 1/p_1$: a contradiction. Thus $m = 2p_1^2$. If $p_1 = 5$, then $m = 50$ and $\phi(50) = 20 = \phi(44)$: a contradiction as $44 < 50$. Thus $p_1 \neq 5$ and p_1 is β-constrained. If $p_1 \neq 5$, then, as p_1 is β-constrained, we may choose $q < p_1^2$, to satisfy $\beta(q) < 1/p_1$: a contradiction.

If $r = 2$, then

$$1 - \beta(q) < 1 - \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_1 p_2}.$$

But, by Lemma 4.2, as p_1 is either 5 or β-constrained, we may choose $q < p_1 p_2$ so that

$$\beta(q) < \frac{1}{p_1} < \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_1 p_2}: \text{ a contradiction.}$$

If $r > 2$, set

$$L_i = 1 - \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_i}\right), \text{ for } i = 1, \ldots, r.$$

Now $L_i < 1$ for all i and $\beta(q) > L_r$. Further, by Lemma 4.2, we may choose q to also satisfy $\beta(q) < L_2$, from which we can inductively show that $\beta(q) < L_i$ for all $i \geq 2$ as

$$L_{i+1} = L_i - \frac{1}{p_{i+1}} L_i + \frac{1}{p_{i+1}} = L_i + \frac{1}{p_{i+1}} (1 - L_i) > L_i,$$

from which it follows that $\beta(q) < L_r$: a contradiction. □

Corollary 4.4. *If $n < 1.8 \times 10^{246}$ and M_n is a power of 2, then $rep(K_{1,n}) = M_n$.*

Proof. If M_n is a power of 2 and $m = rep(K_{1,n}) \neq M_n$, then, by Theorem 4.3, the smallest odd prime divisor p of m cannot be 5 or β-constrained. Thus $p > 9.6 \times 10^{122}$ and $n \geq 2p^2 > 1.8 \times 10^{246}$. □

5. The case n small and $M_n = 2^{k+1}p$.

Considering the possible odd prime divisors of M_n, we see that $M_5 = M_6 = 14$: these are the only cases in which M_n is divisible by 7, and M_n is never divisible by 3, 5, or a Fermat prime.

Lemma 5.1. *If $n \neq 5, 6$, then M_n is not divisible by 3, 5, 7, or a Fermat prime.*

Proof. As $\phi(4) = 2 = \phi(6)$, $\phi(8) = 4 = \phi(10) = \phi(12)$, and $\phi(14) = 6$; $M_2 = 4$, $M_3 = M_4 = 8$, and $M_5 = M_6 = 14$. Thus, we are free to assume that $n \geq 7$ and that $M_n > 14$.

M_n cannot be divisible by 3 as then

$$2^{k+2} \times 3 > 2^{k+1} \times 5 \quad \text{and} \quad \phi(2^{k+2} \times 3) = 2^{k+2} = \phi(2^{k+1} \times 5).$$

M_n cannot be divisible by 5 as then

$$2^{k+2} \times 5 > 2^{k+4} \quad \text{and} \quad \phi(2^{k+2} \times 5) = 2^{k+3} = \phi(2^{k+4}).$$

M_n cannot be divisible by 7 as then

$$2^{k+2} \times 7 > 2^{k+1} \times 13 \quad \text{and} \quad \phi(2^{k+2} \times 7) = 2^{k+2} \times 3 = \phi(2^{k+1} \times 13).$$

If $M_n = 2^i p$ and $p = 2^j + 1$ is a prime, and hence a Fermat prime, then $2^{i+j} < M_n$ and $\phi(2^{i+j}) = 2^{i+j-1} = \phi(M_n)$, a contradiction. Hence M_n cannot be divisible by a Fermat prime. □

Lemma 5.2. *Let $p \geq 11$ be a prime. If, for some $i \geq 1$, there exists a prime q satisfying $2^i(p-1) < q < 2^i p$, then $M_n \neq 2^j p$ for all $j > i$.*

Proof. Suppose that the conditions of the lemma are satisfied and that $j > i$. Set $m = 2^{j-i}q$. Then $m < 2^j p$ and $\phi(m) = 2^{j-i-1}(q-1) \geq \phi(2^j p)$, and so $M_n \neq 2^j p$. □

Interestingly, for any given odd prime p there exists some i for which the conditions of Lemma 5.2 hold, and so $M_n = 2^i p$ for only finitely many i.

Lemma 5.3. *For any odd prime p there exists an integer $i \geq 1$ for which the interval $(2^i(p-1), 2^i p)$ contains a prime.*

Proof. Set $x = 2^i(p-1)$ and $\varepsilon = p/(p-1) > 1$. If x is sufficiently large, then the interval $(x, \varepsilon x) = (2^i(p-1), 2^i p)$ will contain a prime by Theorem 2.1. □

We define δ_p to be the smallest integer $k \geq 1$ for which there exists a prime q satisfying $2^k(p-1)+1 < q < 2^k p$. By Lemma 5.3, δ_p is well-defined. As examples $\delta_{11} = 2$, $\delta_{13} = 3$, $\delta_{19} = 1$, $\delta_{23} = 2$, and $\delta_{29} = 2$.

For p and q distinct odd primes there exist a unique nonnegative rational number c and a unique integer s for which

$$2^{s-1} < \frac{q}{p} = 2^s - \frac{c}{p} < 2^s,$$

We define a mapping γ_p from the set of odd primes other than p to the set of positive rational numbers by $\gamma_p(q) = (c+1)/(2^s p)$, if $q \neq p$. In the following theorem we give two tests that can be used to establish that $rep(K_{1,n}) = M_n$.

Theorem 5.4. *Let $M_n = 2^{k+1}p$, p an odd prime, $k \geq 0$.*

1. If $2^t < p = 2^t + b < 2^{t+1}$, and $2^k b^2 < p$, then $rep(K_{1,n}) = M_n$.

2. If $q < 2^k p$ is an odd prime other than p, and

$$\frac{1}{\gamma_p(q)} \geq \sqrt{2^k p},$$

then $rep(K_{1,n}) = M_n$.

Proof.

1. Assume this to be false, and that $m = rep(K_{1,n}) \neq M_n$. Let $p_1 < \cdots < p_r$ be the distinct odd prime divisors of m. Then $2p_1^2 \leq m < M_n = 2^{k+1}p$, from which it follows that $p_1 < \sqrt{2^k p}$. Set $\varepsilon = 2^{k+t+1} < 2^{k+1}p$. Then

$$\begin{aligned} \phi(\varepsilon) = {} & 2^{k+t} < n \leq \phi(m) \\ = {} & \frac{m}{2}\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_r}\right) \\ < {} & 2^k p\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_r}\right) \\ \leq {} & 2^k p\left(1 - \frac{1}{p_1}\right). \end{aligned}$$

Thus

$$p - b = 2^t < p\left(1 - \frac{1}{p_1}\right),$$

and so $p_1 > p/b$. Hence $p/b < \sqrt{2^k p}$, which implies that $p < 2^k b^2$, a contradiction from which the result follows.

2. Assume this to be false, and that $m = rep(K_{1,n}) \neq M_n$. Let $p_1 < \cdots < p_r$ be the distinct odd prime divisors of m. For uniquely determined integers s and c, $2^{s-1} < (q/p) = 2^s - (c/p) < 2^s$. Set $\alpha = 2^{k-s+1}q$. Then

$$\alpha < 2^{k+1}\left(\frac{p}{q}\right)q = 2^{k+1}p = M_n.$$

By the hypotheses of the Theorem,

$$\begin{aligned} \phi(\alpha) = {} & 2^k p\left(1 - \left(\frac{c+1}{2^s p}\right)\right) < n \leq \phi(m) \\ = {} & \frac{m}{2}\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_r}\right) \\ < {} & 2^k p\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_r}\right). \end{aligned}$$

This reduces to

$$1-\gamma_p(q) < \left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right) \leq \left(1-\frac{1}{p_1}\right).$$

From this it follows that $p_1 > 1/\gamma_p(q)$. But then

$$1/\gamma_p(q) < p_1 < \sqrt{2^k p},$$

a contradiction from which the result follows. □

We ran a magma program to establish evidence for Conjecture 3.3. For each prime p, $11 \leq p \leq 3181$, that was not a Fermat prime, we computed δ_p. Next, for each $k < \delta_p$, we applied the test of Theorem 5.4.1, and only if this test failed did we apply the test of Theorem 5.4.2, and in this case we searched for the largest odd prime $q \neq p$ for which the test was satisfied. In all cases, we found either the test of Theorem 5.4.1 worked, or the test of Theorem 5.4.2 worked. For $p \leq 229$, the test of Theorem 5.4.1 works for $p = 11$ and $k = 0$, $p = 19$ and $k = 0$, $p = 37$ and $k = 0$, $p = 67$ and $k = 0, 1, 2$, $p = 71$ and $k = 0$, $p = 131$ and $k = 0, 1$, $p = 137$ and $k = 0$, and $p = 139$ and $k = 0$. In Table 2 we list all the instances we found, for $11 \leq p \leq 89$, for which the test of Theorem 5.4.2 worked.

Table 2: The test of Theorem 5.4.2 for $11 \leq p \leq 89$.

p	δ_p	k	q	$\gamma_p(q)$	p	δ_p	k	q	$\gamma_p(q)$	p	δ_p	k	q	$\gamma_p(q)$
11	2	1	19	2/11	43	3	0	41	3/43	61	2	1	113	5/61
13	3	0	11	3/13			1	83	2/43	67	4	3	523	7/268
		1	23	2/13			2	167	3/86	71	2	1	139	2/71
		2	47	3/26	47	3	0	43	5/47	73	3	0	71	3/73
23	2	0	11	3/23			1	89	3/47			1	139	4/73
		1	43	2/23			2	181	2/47			2	283	5/146
29	2	0	13	5/29	53	2	0	47	7/53	79	1	0	73	7/79
		1	53	3/29			1	103	2/53	83	2	0	79	5/83
31	1	0	29	3/31	59	2	0	53	7/59			1	163	2/83
41	2	0	37	5/41			1	113	3/59	89	2	0	83	7/89
		1	79	2/41	61	2	0	59	3/61			1	173	3/89

Corollary 5.5. *If $M_n = 2^{k+1}p$ for some $k \geq 0$ and prime p, $11 \leq p \leq 5693$, then rep$(K_{1,n}) = M_n$.*

6. Stars with many vertices

In this last section we study the prime factorization of the representation number for stars with many vertices, using tools from number theory.

Consider the following statement about the distribution of primes:

$P(\theta)$: For sufficiently large values of x, there exists a prime between x and $x + x^{\theta}$.

The statement $P(1)$ is Bertrand's postulate. This was first proved by Chebyshev; simpler proofs were given later by Ramanujan and by Erdös [3]. The first proof of the validity of $P(\theta)$ for $\theta < 1$ was given by Hoheisel [10], for $\theta = 32999/33000$. This result was improved upon by Heilbronn [9] for $\theta = 249/250$ and Tchudakoff [16] for $\theta = (3/4) + \varepsilon$; a major breakthrough was made by Ingham [11], who established the validity of $P(5/8)$. The best result to date is for $\theta = .525$; this is due to Baker, Harman, and Pintz:

Theorem 6.1 ([1]). *There exists N_0 such that for all $x > N_0$, there is a prime between x and $x + x^{.525}$.*

For the remainder of the article, we will use N_0 to denote the smallest such integer which makes Theorem 6.1 true. In practice, we will not use the full strength of the theorem; for most of our applications, the case $\theta = 2/3$ is sufficient.

The following statements is of interest, in that its resolution would lead to interesting results about representation numbers.

Hypothesis 6.2. *For sufficiently large x, there is a prime between x and $x + x^{1/2}$.*

In the absence of compelling evidence to suggest its truth, we hesitate to frame this statement as a conjecture. However, slightly weaker statements appear in the literature: letting p_n denote the nth prime, Cramér [2] proved, under the assumption of the Riemann hypothesis, that $p_{n+1} - p_n = O(\sqrt{p_n}\log p_n)$.

Before stating our main result, we need a tool for establishing the existence of a prime number in certain intervals:

Lemma 6.3. *Let $\phi(k) > N_0$ be an integer.*

- *If k is even and has at least three (not necessarily distinct) odd prime factors, then there exists a prime number in $(\phi(k), \frac{k}{2})$.*
- *Suppose Hypothesis 6.2 holds. If k is even and has at least two (not necessarily distinct) odd prime factors, then there exists a prime number in $(\phi(k), \frac{k}{2})$.*

Proof. Suppose first that k is even and has at least three odd prime factors, and let p be the smallest such prime factor. Writing $k = 2pm$, we have $\phi(k) \leq (p-1)m$. By

Theorem 6.1, the result will hold except possibly if $\phi(k) + \phi(k)^{2/3} > \frac{k}{2}$. However, this implies $(p-1)m + (p-1)^{2/3}m^{2/3} > pm$, which in turn implies $(p-1)^{2/3} > m^{1/3}$. This, however, yields $(p-1)^2 > m$; so since k has at least three odd prime factors, the smallest of which is p, $k > 2p(p-1)^2 > 2pm = k$, which is a contradiction.

If Hypothesis 6.2 holds and k has at least two odd prime factors, there is a prime between $\phi(k)$ and $\phi(k) + (\phi(k))^{1/2}$, so the result holds except possibly if $\phi(k) + (\phi(k))^{1/2} > \frac{k}{2}$. In the above notation, this implies $(p-1)m + (p-1)^{1/2}m^{1/2} > pm$, which in turn implies $(p-1)^{1/2} > m^{1/2}$, or $p - 1 > m$. Then $k > 2p(p-1) > 2pm = k$, a contradiction. □

We now come to our result on the form of the representation number of stars with many vertices.

Theorem 6.4. *For n sufficiently large, $rep(K_{1,n})$ is of one of the following forms: 2^a, 2^ap, 2^apq, where $a \geq 1$ and p, q are (not necessarily distinct) primes. If Hypothesis 6.2 is assumed, then the last possibility cannot occur and so $rep(K_{1,n}) = M_n$.*

Proof. Let $r = rep(K_{1,n})$. Since r is even, Lemma 6.3 guarantees (for sufficiently large n) the existence of a prime $\ell \in (\phi(r), \frac{r}{2})$ if r is not of any of the forms listed above. Then $2\ell < r$ but $\phi(2\ell) = \ell - 1 \geq \phi(r) \geq n$, which contradicts Theorem 3.1.

□

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