The Representation Number of Some Sparse Graphs

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Abstract

We study the representation number for some special sparse graphs. For graphs with a single edge and for complete binary trees we give an exact formula, and for hypercubes we improve the known lower bound. We also study the prime factorization of the representation number of graphs with one edge.

1 Introduction

A finite graph G is *representable modulo* r if there exists an injective map $f : V(G) \to \{0, 1, \ldots, r-1\}$ such that for all vertex pairs, $uv \in E(G)$ if and only if $\gcd(f(u) - f(v), r) = 1$. This is equivalent to requiring that there exist an injective map $f : V(G) \to \mathbb{Z}_r$ such that for all $u, v \in V(G)$, $f(u) - f(v)$ is a unit of (the ring) $\mathbb{Z}_r$ if and only if $uv \in E(G)$. The *representation number* of G, denoted $\mathrm{rep}(G)$, is the smallest positive integer r modulo which G is representable. Representation numbers first appeared in [3] and were used by Erdős and Evans to give a simpler proof of a result of Lindner et al. [7] that any finite graph can be realized as an orthogonal Latin square graph – that is, for any graph, there is an assignment of Latin squares (of the same order) to the vertices in such a way that vertices are adjacent if and only if the associated Latin squares are orthogonal. Representation numbers have been determined for complete graphs [5], edgeless graphs [5], and stars [1] ; there are also bounds and partial results for representation numbers of complete multipartite graphs [2], disjoint unions of complete graphs [4], and various other graph families ([6], [10]).

Representations modulo r are closely related to so-called *product representations* of graphs. A product representation of a graph G is a labeling of its vertices by integer k-tuples in such a way that two vertices are adjacent if and only if their labels differ in all coordinates; the *product dimension* of G, denoted $\operatorname{pdim} G$, is the least positive integer k for which this is possible. Now if r is a squarefree integer with prime factorization $p_1 \cdots p_s$, the Chinese Remainder Theorem gives an isomorphism $\mathbb{Z}_r \cong \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_s}$. This provides a natural method to convert mod r representations of a graph into product representations and vice versa. When G is a reduced graph (i.e. no two vertices have the same open neighborhood), $\operatorname{rep}(G)$ will always be squarefree [5]. In this case, the number of prime factors in $\operatorname{rep}(G)$ must be at least $\operatorname{pdim} G$; however, it is not known whether these two quantities are always equal. Even when G is not reduced, product representations are often used to establish upper bounds for the representation number: a general upper bound (for all graphs) is proved by this method [9], as is an upper bound for the representation number of the hypercube [10]. In Section 4 of the present work, we use this same method when studying complete binary trees.

The purpose of this article is to study the representation number on particular families of sparse graphs. We first consider the graph $S_n = K_2 + nK_1$. Representations of S_n were first studied in [5], which established an upper bound of $6n$ for $\operatorname{rep}(S_n)$. Narayan and Urick [10] improved this bound and conjectured that it gave the true value of $\operatorname{rep}(S_n)$. In Section 3, we prove their conjecture for n sufficiently large using bounds on arithmetic functions from [11] and show that $\operatorname{rep}(S_n)$ is close to $2n$. Since the Narayan-Urick formula involves functions that are difficult to compute, we adapt the techniques developed in [1] and [2] to give a partial description of the prime factors of $\operatorname{rep}(S_n)$, reminiscent of the description of $\operatorname{rep}(K_m + nK_1)$ in [6, Section 5] for small values of n. In Section 4, we compute the representation number of complete binary trees, giving an exact formula in all cases except for the tree on 15 vertices. In Section 5, by examining more closely the construction of Narayan and Urick [10] and establishing an easy lower bound, we improve known results about the representation number of the hypercube.

Throughout this paper, various sums and products indexed by a set of prime numbers appear: in all such formulas p or q indicates a prime. The notation γ is reserved for the Euler-Mascheroni constant and $p_1, p_2, \ldots$ for the prime numbers, with $p_1 < p_2 < \cdots$. The *primorial* $\wp_n$ is defined to be $\prod_{i=1}^{n} p_i$. The *radical* of a nonzero integer n, denoted $\operatorname{rad} n$, is the product of the distinct primes dividing it.

2 Preliminaries

In this section we collect several definitions and tools which will be used throughout the article.

Given a graph G, define (following [5]) an equivalence relation on $\sim$ on $V(G)$ by declaring two vertices to be equivalent if they share the same open neighborhood in G. Letting $[u]$ denote the equivalence class of $u \in V(G)$, define the *reduction* of G, denoted $\hat{G}$, by

$$V(\hat{G}) = \{[u] : u \in V(G)\}$$

and

$$[u][v] \in E(\hat{G}) \Leftrightarrow uv \in E(G)$$

It is immediate that the above description of $E(\hat{G})$ is well-defined. A graph is called *reduced* if no two distinct vertices share the same open neighborhood. The main result of relevance to us is the following:

Proposition 2.1. *[5, Lemma 2.4] Suppose $p_1, \ldots, p_s$ are distinct primes and $e_1, \ldots, e_s$ positive integers. If G is representable modulo $p_1^{e_1} \ldots p_s^{e_s}$, then $\hat{G}$ is representable modulo $p_1 \ldots p_s$. In particular, if G is reduced then* $\operatorname{rep}(G)$ *is squarefree.*

For reduced graphs, the product dimension can be used to give a lower bound on the representation number.

Proposition 2.2. *[6, Theorem 2.11] If G is a reduced graph, then*

$$\operatorname{rep}(G) \geq p_\ell p_{\ell+1} \ldots p_{\ell+m-1},$$

where $m =$ pdim G and ℓ is the smallest integer satisfying $p_\ell \geq \chi(G)$.

We also record the following result of Lóvasz et al. as a helpful tool in establishing a lower bound for the product dimension.

Lemma 2.3. *[8] Let $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ be two lists of vertices in a graph G. If u_i is adjacent to v_j when $i = j$ and u_i is not adjacent to v_j when $i < j$, then* $\operatorname{pdim} G \geq \lceil \log_2 r \rceil$.

Finally, we will need some explicit estimates on the values of certain arithmetic functions from the well-known paper of Rosser and Schoenfeld [11].

Lemma 2.4. • *[11, Theorem 4 and Corollary]*

For $x > 1$,

$$e^{x(1-\frac{1}{\log x})} < \prod_{q \le x} q < e^{x(1+\frac{1}{2\log x})}$$

• *[11, Theorem 7 and Corollary] For* $x > 1$,

$$\frac{e^{-\gamma}}{\log x}(1 - \frac{1}{\log^2 x}) < \prod_{q \le x}(1 - \frac{1}{q}) < \frac{e^{-\gamma}}{\log x}(1 + \frac{1}{2\log^2 x})$$

• *[11, p. 72] For* $n \ge 3$,

$$\frac{n}{\phi(n)} < e^{\gamma} \log\log n + 2.50637/\log\log n$$

In particular, if $n \ge 12$,

$$\frac{n}{\phi(n)} < 5\log\log n$$

3 Graphs with a single edge

In this section we study and determine the representation number of the graph $S_n = K_2 + nK_1$, which has $n+2$ vertices and a single edge. To simplify notation, define

$$M_n = \min\{2^k m : \ k \ge 1, \ m \ge 3 \text{ is odd}, \text{ and } 2^{k-1}(m - \phi(m)) \ge n\}.$$

Narayan and Urick proved that M_n is an upper bound for $\text{rep}(S_n)$; we include a proof here for completeness of presentation.

Proposition 3.1. *[10, Corollary 1]*

$$\text{rep}(S_n) \le M_n.$$

Proof.
Given k and m as in the definition of M_n, define a labeling modulo $2^k m$ by assigning to 0 and 1 to the adjacent vertices of S_n and assigning to the isolated vertices elements from the set

$$T = \{l : 0 \le l \le r-1, \ l \equiv i (\text{mod } 2^k), \ l \equiv j (\text{mod } m), \ i \text{ is even, and } \gcd(j, m) \ne 1\}.$$

This is possible because there are 2^{k-1} choices for i and $m - \phi(m)$ choices for j; hence the Chinese Remainder Theorem implies $|T| = 2^{k-1}(m - \phi(m))$. Thus, there are at least n elements of T that can be used to label the isolated vertices. □

Our goal is to prove that when n is sufficiently large, $\text{rep}(S_n) = M_n$.

Lemma 3.2. *For $\varepsilon > 0$ there exists n_0 such that if $n > n_0$, then $2n \leq \operatorname{rep}(S_n) < 2(1+\varepsilon)n$. Furthermore, when n is sufficiently large, 2 divides $\operatorname{rep}(S_n)$.*

Proof.
Since $\overline{K_n}$ is an induced subgraph of S_n, $\operatorname{rep}(S_n) \geq \operatorname{rep}(\overline{K_n}) = 2n$ by [5, Example 1.1]. For $x > 1$, Lemma 2.4 implies

$$\beta_x = \prod_{p \leq x}(1 - \frac{1}{p}) \leq \frac{e^{-\gamma}}{\log x}(1 + \frac{1}{2\log^2 x}).$$

Now fix $\varepsilon > 0$. By choosing x so that $\log x > 8e^{-\gamma}(1 + \frac{1}{2\log^2 x})(1 + \frac{1}{\varepsilon})$, we may ensure that $\delta = (1 - 2\beta_x)^{-1} < (1 + \varepsilon/4)$. Now let $s_x = \prod_{3 \leq p \leq x} p$ and note that by Lemma 2.4, $\frac{1}{2}e^{x(1-\frac{1}{\log x})} \leq s_x \leq \frac{1}{2}e^{x(1+\frac{1}{2\log x})}$. For $n > \frac{4e^{x(1+\frac{1}{2\log x})}}{\varepsilon}$, there is a number of the form ks_x, with k odd, in the interval $(\delta n, (1+\varepsilon/2)n)$. Now let $r = 2ks_x$. Next, $ks_x - \phi(ks_x) = ks_x(1 - \prod_{p|ks_x, p\geq 3}(1-\frac{1}{p})) \geq ks_x(1 - 2\prod_{p \leq x}(1-\frac{1}{p})) \geq n$; thus, Proposition 3.1 implies $\operatorname{rep}(S_n) \leq 2ks_x < 2(1+\varepsilon)n$.

To prove the second assertion, note that [1, Lemma 2.6] implies that p_0, the smallest prime dividing $\operatorname{rep}(S_n)$, satisfies $p_0 \leq \frac{\operatorname{rep}(S_n)}{n+1}$. Thus, if n is sufficiently large to ensure $\operatorname{rep}(S_n) < 3n$, we have $p_0 < 3$. Thus $p_0 = 2$, as desired. □

We are now in a position to prove our main result.

Theorem 3.3. *For sufficiently large n,*

$$\operatorname{rep}(S_n) = M_n.$$

Proof.
Let $r = \operatorname{rep}(S_n)$ and let x and y denote adjacent vertices of S_n. We know from Lemma 3.2 that $r = 2^k m$ for some positive integers k and m with m odd. If $m = 1$, then $r = 2^k$, which is impossible since the only graphs representable modulo a prime power are complete multipartite graphs. Now fix a labeling of S_n modulo r; for convenience, we consider labels as elements of $\mathbb{Z}_{2^k} \times \mathbb{Z}_m$ and denote by $\pi : \mathbb{Z}_{2^k} \to \mathbb{Z}_2$ the natural quotient map. Without loss of generality (cf. [1, Lemma 2.3]) we may assume that x and y are labeled $(0,0)$ and $(1,1)$, respectively. Let A be the set of vertices in S_n distinct from x that have labels of the form (a,i) where $\pi(a) = 0$ and B the set of vertices distinct from y having labels of the form (a,j) where $\pi(a) = 1$. Observe that

$A \cup B$ is an independent set in S_n. Now for any vertex of B having label (a, i), note that there is no vertex of A having label $(a+1, i+1)$, for then the difference between the two labels would be a unit, forcing the existence of an edge between a vertex of A and a vertex of B. Hence we may relabel all vertices of B, replacing the label (a, i) with $(a+1, i+1)$. This new labeling gives a representation modulo r in which every vertex in $A \cup B$ has a label of the form (a, i) with $\pi(a) = 0$. However, none of these vertices are adjacent to y, which is labeled $(1, 1)$, and since $a - 1 \in \mathbb{Z}_{2^k}^*$, it must be the case that $i - 1 \notin \mathbb{Z}_m^*$. Hence there are 2^{k-1} choices for a and $m - \phi(m)$ choices for i. This forces $n = |A \cup B| \leq 2^{k-1}(m - \phi(m))$, as desired.

□

Calculations of the prime factorization of $\text{rep}(S_n)$ suggest a pattern: most of the exponents on the prime factors of $\text{rep}(S_n)$ are equal to 1, and $\text{rep}(S_n)$ seems to be divisible by a primorial that grows as a function of n. In Proposition 3.5 and Corollary 3.7 we make these observations rigorous; our results are similar in form to those for the representation number of $K_m + nK_1$ (for small n) from [6, Section 5]. The strategy of the proof, however, is quite different from that of [6] but rather similar to that of [1] and [2]: we assume that the the representation number is not of the claimed form, and then prove that it is forced to exceed some known upper bound.

For the balance of this section, we reserve the notation r for $\text{rep}(S_n)$; for sufficiently large n we also define k and s by $r = 2^k s$, where $k \geq 1$ and s is odd. We begin with a result that says essentially that for large enough n we may assume that $\text{rep}(S_n)$ is divisible by some sufficiently large prime.

Lemma 3.4. *For every $x > 0$ there exists n_0 such that for $n > n_0$, $\text{rep}(S_n)$ is divisible by some prime $q \geq x$.*

Proof.
If r is not divisible by any prime $q \geq x$, then

$$n \leq 2^{k-1}(s - \phi(s)) \leq 2^{k-1}s(1 - \prod_{p \leq x}(1 - \frac{1}{p})) \leq 2^{k-1}s(1 - \frac{e^{-\gamma}}{2\log x}) = \frac{r}{2}(1 - \frac{e^{-\gamma}}{2\log x})$$

by Lemma 2.4. Thus,

$$r \geq \frac{2n}{1 - \frac{e^{-\gamma}}{2\log x}}.$$

Since x is fixed, Lemma 3.2 yields a contradiction for sufficiently large n. □

We may now prove that s is close to being squarefree.

Proposition 3.5. *For sufficiently large n, rad $s > \dfrac{s}{40(\log s)^2 \log\log s}$.*

By Lemma 3.4, we may assume $s \geq 12$. Suppose $t = \text{rad } s$ and let $u = s/t$; observe that u is odd. Let p be the smallest prime number not dividing r; we claim $t \geq \frac{s}{10p^2 \log\log s}$. If $u \leq 3p$, then $t \geq \frac{s}{3p} > \frac{s}{10p^2 \log\log s}$. Otherwise, write $u = \ell p + c$ for some odd integer ℓ and some integer c, $0 < c < 2p$. Now let $s' = t\ell p$ and $r' = 2^k s'$. Since $r' < r = \text{rep}(S_n)$, it follows from Theorem 3.3 that $s' - \phi(s') < s - \phi(s)$ or equivalently $t\ell p - \phi(t\ell p) < tu - \phi(tu)$, which reduces to

$$\phi(tu) - \phi(t(u-c)) < tc. \tag{1}$$

However, p divides $u - c$ and every prime dividing u also divides t, so

$$\phi(t(u-c)) = t(u-c) \prod_{q|t(u-c)} (1 - \frac{1}{q}) \leq \frac{u-c}{u}(1 - \frac{1}{p})\phi(tu),$$

which simplifies to $\frac{\phi(s)}{s} < \frac{c}{u - (u-c)\frac{p-1}{p}}$ after substituting into (1).

Now $s \geq 12$, so combining the above with the estimate $\frac{\phi(s)}{s} > \frac{1}{5\log\log s}$ of Lemma 2.4, we have $\frac{1}{p}u + \frac{c(p-1)}{p} < 5c\log\log s$. Hence $u < 5cp\log\log s < 10p^2\log\log s$ and so $t = \text{rad } s > \frac{s}{10p^2\log\log s}$ in this case also.

From Lemma 2.4 we have $e^{p(1-\frac{1}{\log p})} < \prod_{q\leq p} q \leq 2ps$; thus $p < 2\log s$ and hence

$$\text{rad } s > \frac{s}{40(\log s)^2 \log\log s}.$$

□

Lemma 3.6. *Let p the smallest prime that does not divide s and p' the largest prime divisor of s. For sufficiently large n, $p > \frac{1}{4}\sqrt{\frac{p'}{\log p'}}$.*

Proof.

Observe that if $p' < 3p$, then by Lemma 3.4, $p > \frac{p'}{3} > \frac{1}{4}\sqrt{\frac{p'}{\log p'}}$; we assume henceforth $p' > 3p$. Define a by $s = p'^a m$, where $\gcd(m, p') = 1$, and write $p'^a = \ell p + c$ with ℓ odd and $0 < c < 2p$. Next, define $s' = s\ell\frac{p}{p'^a}$. Since $2^k s' < 2^k s = r$, Proposition 3.3 implies $s' - \phi(s') < s - \phi(s)$. However, $\phi(s') \leq \frac{p-1}{p} \cdot \frac{p'}{p'-1}\phi(s)$; thus we have:

$$\phi(s)(1 - \frac{p'(p-1)}{p(p'-1)}) < s(1 - \frac{\ell p}{p'^a}) = \frac{cs}{p'^a} < \frac{2ps}{p'},$$

which in turn implies $p' < p(2p\frac{s}{\phi(s)} + 1)$. From Lemma 2.4, we have $\operatorname{rad} s \leq e^{p'(1+\frac{1}{2\log p'})}$, so $\log \operatorname{rad} s < \frac{3}{2}p'$. Moreover, Proposition 3.5 gives the bound $\log s < 2 \log \operatorname{rad} s$. Again by Lemma 2.4, $\frac{s}{\phi(s)} < 5 \log\log s < 5\log(3p') < 6\log p'$; thus,

$$p' < p(6p\log p' + 1) < 16p^2 \log p'.$$

□

Remark.
We used very rough bounds to obtain the constants in Proposition 3.5 and Lemma 3.6 in the interest of keeping the arithmetic simple. Their precise values are not important for the proof of our main result on the prime factors of $\operatorname{rep}(S_n)$.

Corollary 3.7. *Fix $\delta > 0$. For sufficiently large n, $\operatorname{rep}(S_n)$ is divisible by all primes less than $(\log n)^{\frac{1}{16}(1-\delta)}$.*

Proof.
Let p and p' be as in Lemma 3.6. By Proposition 3.3, $\frac{r}{2}(1 - \frac{\phi(s)}{s}) \geq n$. Also, by Lemma 2.4,

$$\frac{\phi(s)}{s} = \prod_{q|s}(1 - \frac{1}{q}) \geq \prod_{3\leq q\leq p'}(1 - \frac{1}{q}) \geq \frac{e^{-\gamma}\eta}{\log p'},$$

where $\eta = 1 - \frac{1}{\log^2 p'}$. Thus $\frac{r}{2}(1 - \frac{e^{-\gamma}\eta}{\log p'}) \geq n$, and hence

$$r \geq 2n(1 + \frac{\eta e^{-\gamma}}{\log p' - \eta e^{-\gamma}}). \tag{2}$$

Let $\varepsilon = \frac{\eta e^{-\gamma}}{\log p' - \eta e^{-\gamma}}$. From the argument in Lemma 3.2, we see that $r < 2(1+\varepsilon)n$ if we choose $n > \frac{4e^{x(1+\frac{1}{2\log x})}}{\varepsilon}$, where

$$x > \exp(8e^{-\gamma}(1 + \frac{1}{2\log^2 x})(1 + \frac{1}{\varepsilon})) = \exp(8\frac{\log p'}{\eta}(1 + \frac{1}{2\log^2 x})) = p'^{\frac{8}{\eta}(1+\frac{1}{2\log^2 x})}.$$

Lemma 3.4 implies that for sufficiently large n, p' may be made arbitrarily large; hence we may make η arbitrarily close to 1. Hence in order for (2) to hold, we must have $n < \exp(p'^8(1+\delta))$ or $p' > (\log n)^{\frac{1}{8(1+\delta)}}$. By Lemma 3.6, we may choose n sufficiently large to guarantee $p > p'^{\frac{1}{2}(1-\delta^2)}$, so $p > (\log n)^{\frac{1}{16}(1-\delta)}$, as desired. □

4 The complete binary tree

In this section we compute the representation number of the complete binary tree B_n with $2^n - 1$ vertices. The *level* of a vertex v, denoted $\ell(v)$, is the distance in B_n from v to the root.

It is easy to check that $\text{rep}(B_1) = 1$, $\text{rep}(B_2) = 4 = 2^2$, and $\text{rep}(B_3) = 12 = 2^2 \cdot 3$.

Theorem 4.1. *For $n \geq 5$, $\text{rep}(B_n) = \wp_n$.*

Proof.
Note that B_n is not a reduced graph when $n \geq 2$; its reduction $\hat{B}_n$ is obtained by deleting one leaf from each pair of leaves in B_n with a common parent. Now suppose $n \geq 5$ and let $a_1, a_3, \ldots, a_{2^{n-1}-1}$ denote the leaves of $\hat{B}_n$. For $1 \leq i \leq 2^{n-2}$, let b_{2i-1} denote the parent of a_{2i-1}. For $1 \leq j \leq 2^{n-3}$, let c_{4i+2} denote the common parent of b_{4j-3} and b_{4j-1}. Finally, let d denote the parent of c_2 and e the parent of d.
Now for $1 \leq j \leq 2^{n-3}$, define

$$\begin{aligned} u_{4j-3} &= a_{4j-3} & v_{4j-3} = u_{4j-2} = b_{4j-3} \\ v_{4j-2} = u_{4j-1} &= c_{4j-2} & v_{4j-1} = b_{4j-1} = u_{4j} \\ v_{4j} &= a_{4j-1} & u_{2^{n-1}+1} = d,\ v_{2^{n-1}+1} = e. \end{aligned}$$

For $1 \leq i \leq 2^{n-1}+1$, the vertices u_i and v_i satisfy the hypotheses of Lemma 2.3, so $\text{pdim}\, \hat{B}_n \geq n$. By Proposition 2.2, $\text{rep}(\hat{B}_n) \geq \wp_n$, and hence by Proposition 2.1, $\text{rep}(B_n) \geq \wp_n$.

For the upper bound, we construct a labeling (which is also a product representation) inductively. The labeling for $n = 5$ is given in Figure 2; for convenience of notation, we write $abcd$ to represent the label $(a, b, c, d) \in \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$.
Suppose a representation $g_{n-1} : V(B_{n-1}) \to \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_{p_{n-1}}$ is given. Let r denote the root vertex of B_n, and let x be its left child and y be its right child. Let $B_n(x)$ and $B_n(y)$ denote the subtrees of B_n rooted at x and y, respectively. We will construct a labeling $g_n : V(B_n) \to \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots \mathbb{Z}_{p_n}$; we begin by defining g_n on the subtrees $B_n(x)$ and $B_n(y)$. Fix bijections $h_1 : B_n(x) \to B_{n-1}$ and $h_2 : B_n(y) \to B_{n-1}$. If $v \in B_n(x)$ and $g_{n-1}(h_1(v)) = (a_1, \ldots, a_{n-1})$, define $g_n(v) = (a_1, \ldots, a_{n-1}, 1)$ if $a_1 = 0$ or $g(v) = (a_1, \ldots, a_{n-1}, 0)$ if $a_1 = 1$. Similarly, if $v \in B_n(y)$ and $g_{n-1}(h_2(v)) = (a_1, \ldots, a_{n-1})$, define $g_n(v) = (a_1, \ldots, a_{n-1}, 0)$ if $a_1 = 0$ or $g_n(v) = (a_1, \ldots, a_{n-1}, 1)$ if $a_1 = 1$.

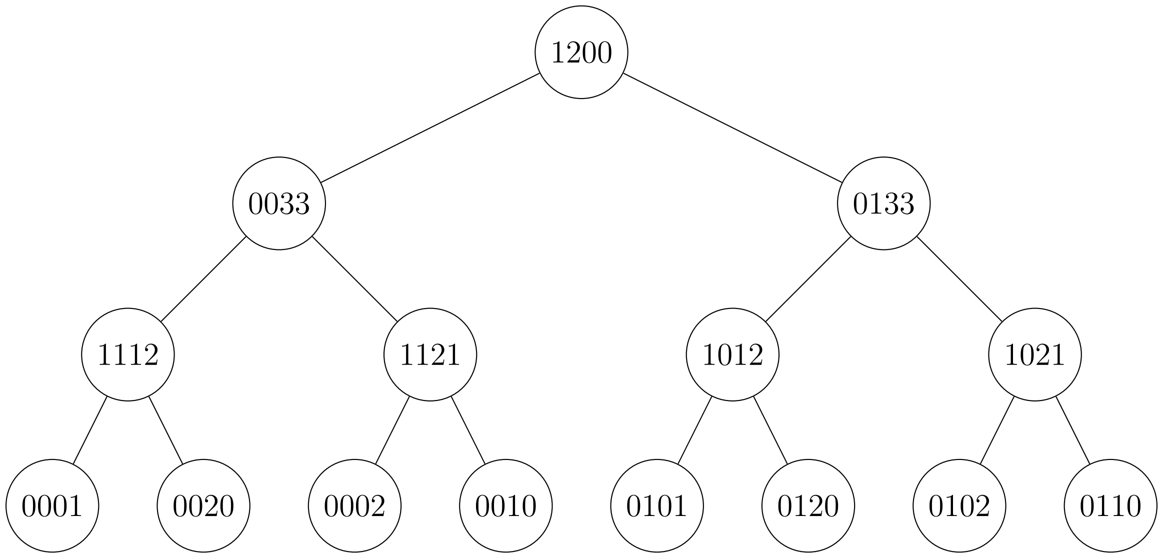

Figure 1: Labeling of B_4

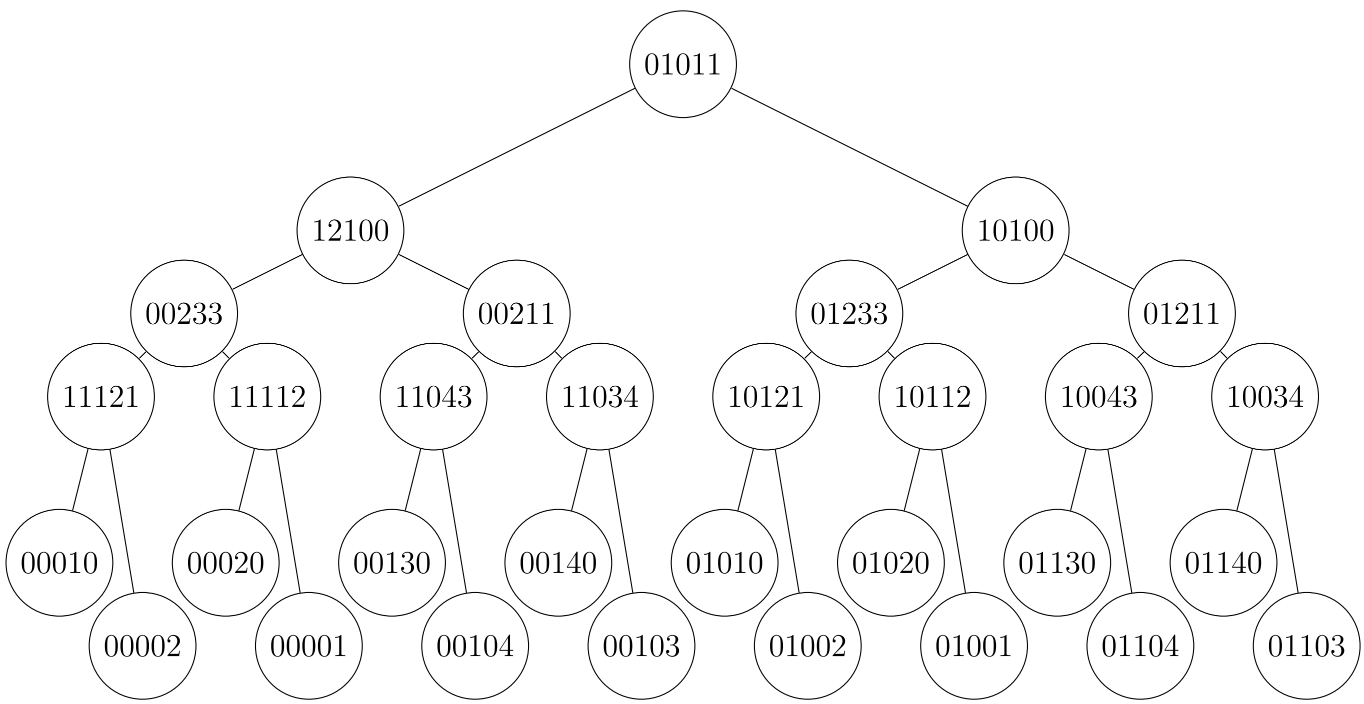

Figure 2: Labeling of B_5

The definition of g_n on the root vertex r is more complicated. First define $g_6(r) = (1, 0, 2, 0, 0, 2)$ and $g_7(r) = (0, 2, 1, 3, 3, 1, 2)$. We now give the definition of $g_n(r) = (c_1, \ldots, c_r)$ when $n \geq 8$.

If n is even, then set

$$c_1 = 1,\ c_2 = 1,\ c_3 = 2,\ c_4 = 0,\ c_5 = 0,\ c_{n-2} = 0,\ c_{n-1} = 1,\ c_n = 2.$$

Now if m is odd and $7 \leq m \leq n-3$, then set $c_m = 2$. If m is even and $6 \leq m \leq n-4$, then set $c_m = 0$.

If n is odd, then set

$$c_1 = 0,\ c_2 = 2,\ c_3 = 1,\ c_4 = 3,\ c_5 = 3,\ c_{n-2} = 0,\ c_{n-1} = 1,\ c_n = 2.$$

Now if m is even and $6 \leq m \leq n-3$, then set $c_m = 2$. If m is odd and $7 \leq m \leq n-4$, then set $c_m = 0$.

It remains to verify that g_n is a representation of B_n modulo $\wp_n$. To this end, select distinct vertices $u, v \in B_n$ and let $g_n(u) = (a_1, \ldots, a_n)$, $g_n(v) = (b_1, \ldots, b_n)$. We divide the argument into several cases.

First, suppose both vertices both belong to $B_n(x)$. Suppose that both vertices belong to $B_n(x)$. If they are adjacent, then since g_n (restricted to $B_n(x)$) is defined in terms of g_{n-1}, which is a representation of B_{n-1}, we have $a_i \neq b_i$ for $1 \leq i \leq n-1$; moreover, since $a_1 \neq b_1$, we must have $a_n \neq b_n$. If u and v are not adjacent, then $a_i = b_i$ for some i, $1 \leq i \leq n-1$. The same reasoning applies if both vertices belong to $B_n(y)$.

Next, suppose (without loss of generality) that u belongs to $B_n(x)$ and v belongs to $B_n(y)$. Because u is not adjacent to v in B_n, we need to show that $a_i = b_i$ for some i, $1 \leq i \leq n$. If $a_1 \neq b_1$, then either $a_1 = 0$ and $b_1 = 1$ or $a_1 = 1$, $b_1 = 0$. The construction of g_n implies that $a_n = b_n = 1$ in the first case and that $a_n = b_n = 0$ in the second.

Finally, suppose that u is the root vertex, $v \in B_n(x)$, and $n \geq 6$. (If $v \in B_n(y)$, a similar argument applies.) We divide this case into two subcases: $v = x$ and $v \neq x$. If $v = x$ and $n = 6$, then each coordinate of $g_n(u) = (1, 0, 2, 0, 0, 2)$ differs from $g_n(v) = (0, 1, 0, 1, 1, 0)$. Suppose n is odd; then $(a_1, \ldots, a_5) = (0, 2, 1, 3, 3)$. Moreover, if $6 \leq m \leq n-3$, then $a_m = 2$ if and only if m is even, and furthermore we have $a_{n-2} = 0$, $a_{n-1} = 1$, $a_n = 2$. On the other hand, $(b_1, \ldots, b_5) = (1, 1, 2, 0, 0)$. If $6 \leq m \leq n-4$, then $b_m = 2$ if and only if m is odd, and furthermore we have $b_{n-3} = 0$, $b_{n-2} = 1$, $b_{n-1} = 2$, and $b_n = 0$ (the latter since $v \in B_n(x)$). Therefore, $a_i \neq b_i$ for all i. If n is even and at least 8, then a similar argument can be made.

In the remaining case, $v \neq x$ and $\ell(v) \leq n-3$. Thus $u = r$ is not adjacent to v, and we must show that $a_i = b_i$ for some i with $1 \leq i \leq n$. If $n = 6$, then $g_n(u)$ has at least one coordinate in common with any vertex of level 1 or 3. Next suppose that n is odd. If $\ell(v)$ is odd, then $a_1 = b_1$, so suppose $\ell(v)$ is even. If $\ell(v) = 4$, then $a_3 = b_3 = 1$ regardless of the choice of v. If $\ell(v) = 2$, then at least one of the statements $b_3 = 1$, $b_4 = 3$, $b_5 = 3$ is true; hence $a_k = b_k$ for some k with $3 \leq k \leq 5$. If $\ell(v) \geq 6$, then $b_{\ell(v)} = 2$, and since $\ell(v)$ is even and n is odd, $a_{\ell(v)} = 2$ by construction. Finally, suppose n is even and at least 8. If $\ell(v)$ is even, then $a_1 = b_1$, so suppose $\ell(v)$ is odd. If $\ell(v) = 5$, then $a_2 = b_2 = 1$; if $\ell(v) = 3$, then $a_3 = b_3 = 2$. If $\ell(v) = 1$, then either $b_4 = 0$ or $b_5 = 0$; hence $a_k = b_k$ for some k with $4 \leq k \leq 5$. □

Remark.
The labeling in Figure 1 shows that B_4 is representable modulo $\wp_4$. However, the methods used in the proof of Theorem 4.1 only guarantee a lower bound of 3 for $\operatorname{pdim} \hat{B}_4$. Hence, we may only conclude $30 \leq \operatorname{rep}(B_4) \leq 210$. It is easy to show $\operatorname{pdim} B_4 = 4$ and it seems likely that $\operatorname{pdim} \hat{B}_4 = 4$ also, although we do not know how to prove this at present.

5 The hypercube

In this section, we consider the problem of determining the representation number of the hypercube Q_n; this is the graph whose vertices are 0,1-strings of length n, with two strings adjacent if and only if they differ in exactly one bit. Note that Q_n is a reduced graph, so by Proposition 2.1, $\operatorname{rep}(Q_n)$ is squarefree. Narayan and Urick studied this problem; by considering product representations, they established the following bounds:

Theorem 5.1. *[10, Theorem 5 and Corollary 4]*

$$\textit{For } n \geq 3,\ \operatorname{rep}(Q_n) \leq \wp_n/3 \textit{ and } \operatorname{pdim} Q_n \leq n-1.$$

The proof of Theorem 5.1 is inductive: given a representation of Q_n modulo $\wp_n/3$, the authors explicitly construct a representation of Q_{n+1} modulo $\wp_{n+1}/3$. An important aspect of their construction is that for each odd prime p with $5 \leq p \leq p_n$, every label used is congruent modulo p to one of four residue classes. Thus, their proof actually shows the following:

Lemma 5.2. *If $n \geq 3$ and Q_n is representable modulo r, then Q_{n+1} is representable modulo rp, where p is any prime at least 5 that does not divide r.*

Towards a lower bound for $\mathrm{rep}(Q_n)$, we record the following result which is surely known to experts. In the interest of completeness, we provide a proof.

Proposition 5.3. *For $n \geq 3$, $\mathrm{pdim}\, Q_n = n-1$ and $\mathrm{rep}(Q_n) \geq \wp_{n-1}$.*

Proof.
From Theorem 5.1 we have $\mathrm{pdim}\, Q_n \leq n-1$. For the other inequality, we use Lemma 2.3. Let $r = 2^{n-1}$ and let $x_1, \ldots, x_r$ be any ordering of the $0,1$-strings of length n beginning with 0. Now for $1 \leq i \leq r$, construct y_i from x_i by changing the first bit to 1. It is clear from the construction that x_i is adjacent to y_j in Q_n if and only if $i = j$. Thus, $\mathrm{pdim}\, Q_n \geq \log_2 r = n-1$. The second statement follows from Proposition 2.2. □

We may now narrow the value of $\mathrm{rep}(Q_n)$ down to two possibilities.

Corollary 5.4. *If there exists $n_0 \geq 3$ such that $\mathrm{rep}(Q_{n_0}) = \wp_{n_0-1}$, then $\mathrm{rep}(Q_n) = \wp_{n-1}$ for all $n \geq n_0$; otherwise, $\mathrm{rep}(Q_n) = \wp_n/3$ for all $n \geq 3$.*

Proof.
First note that $\mathrm{rep}(Q_n)$ must be divisible by 2: if this were not the case, then Proposition 5.3 would imply $\mathrm{rep}(Q_n) \geq \frac{\wp_n}{2}$, contradicting Theorem 5.1. If $\mathrm{rep}(Q_n)$ is not divisible by 3, then Proposition 5.3 forces $\mathrm{rep}(Q_n) \geq \frac{\wp_n}{3}$, which when combined with Theorem 5.1 yields $\mathrm{rep}(Q_n) = \frac{\wp_n}{3}$. On the other hand, if there exists n_0 such that $r = \mathrm{rep}(Q_{n_0})$ is divisible by 3, then r cannot be the product of more than $n-1$ distinct primes, since this would imply $r \geq \wp_n$, a contradiction to Theorem 5.1. Thus, $r = 2 \cdot 3 \cdot q_3 \cdot \ldots \cdot q_{n-1}$, where $q_3, \ldots, q_{n-1}$ are primes and $5 \leq q_3 < \ldots < q_{n-1}$. Let k be the number of primes in $[5, q_{n-1})$ distinct from $q_3, \ldots, q_{n-1}$. By iterated application of Lemma 5.2, we see that for $\ell \geq k$, $Q_{n+\ell}$ is representable modulo $\wp_{n+\ell-1}$. Since $\mathrm{rep}(Q_{n+\ell}) \geq \wp_{n+\ell-1}$ by Proposition 5.3, it follows that $\mathrm{rep}(Q_{n+\ell}) = \wp_{n+\ell-1}$ for all $\ell \geq k$. □

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