# Strong complete mappings for 3-groups

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**Abstract**

A strong complete mapping for a group $G$ is a bijection $\varphi : G \to G$ such that the maps $x \mapsto x\varphi(x)$ and $x \mapsto x^{-1}\varphi(x)$ are also bijections. Groups admitting a strong complete mapping are important to the study of orthogonality problems for Latin squares and group sequencings, among other applications. A.B. Evans [6] showed that a finite abelian group admits a strong complete mapping if and only if both its 2-Sylow subgroup and its 3-Sylow subgroup are either trivial or noncyclic. Nilpotent groups resemble abelian groups in that they also possess the property of being the direct product of their Sylow subgroups; therefore, it is natural to begin consideration of the nonabelian case by asking which nilpotent groups admit strong complete mappings. As the function $x \mapsto x^2$ furnishes a strong complete mapping for finite groups of order relatively prime to 6, we need only consider 2-groups and 3-groups. As a step in this direction, we prove that every noncyclic 3-group admits a strong complete mapping, except possibly those in the infinite family $L_r = \left\langle a, b \;\middle|\; a^{3^{r-1}} = b^3 = 1, bab^{-1} = a^{1+3^{r-2}} \right\rangle$, $r \geq 4$.

## 1 Introduction

Let $G$ be a group. A bijection $\theta : G \to G$ is called a *complete mapping* if $x \mapsto x\theta(x)$ is a bijection or an *orthomorphism* if $x \mapsto x^{-1}\theta(x)$ is a bijection. A *strong complete mapping* is a bijection which is both a complete mapping and an orthomorphism. We call $G$ *admissible* if it admits a complete mapping and *strongly admissible* if it admits a strong complete mapping. Strong complete mappings are closely related to orthogonality problems for Latin squares [7] and have been used to study group sequencings [1], Knut Vic designs ([9], [10]), strong starters [11], solutions to the toroidal $n$-queens problem [13], and check digit systems [14].

The classification of admissible finite groups was begun in 1955 with the work of Hall and Paige [8] and completed in 2009 by Wilcox, Evans, and Bray et. al. (see [5], [15], [3]). It is now known that a finite group $G$ is admissible if and only if the 2-Sylow subgroup of $G$ is either trivial or noncyclic. The analogous question for infinite groups was settled by Bateman [2], who proved that every infinite group is admissible. The classification problem for groups admitting an orthomorphism is equivalent to admissibility in view of the elementary fact that $x \mapsto \theta(x)$ is an orthomorphism if and only if $x \mapsto x^{-1}\theta(x)$ is a complete mapping.

In contrast, the classification of strongly admissible finite groups is still open. It has been fully resolved for abelian groups: Evans [6] showed that a finite abelian group is strongly admissible if and only if neither its 2-Sylow subgroup nor its 3-Sylow subgroup is nontrivial and cyclic. In [7] the classification is carried out for (all) groups of order at most 31, and it is also shown that certain infinite families of dihedral and generalized quaternion groups are strongly admissible. However, little is known in general about strong admissibility of nonabelian groups. Part of the difficulty is that, aside from explicit constructions, the primary tools for proving the existence of such maps are inductive. It is relatively easy to show (cf. [8, Cor. 2]) that if $H$ is a normal subgroup of a group $G$ such that both $H$ and $G/H$ are admissible, then $G$ is admissible. However, one needs to assume further that $H$ is contained in the center of $G$ to deduce strong admissibility by the same argument. Such was the strategy adopted by Evans in the case of finite abelian groups (see [6, Lemma 5] or [4, Theorem 3]), ultimately reducing the classification problem to a handful of explicit constructions for groups of a particular form.

It seems appropriate, therefore, to begin consideration of the nonabelian case with the class of nilpotent groups, since these at least have a nontrivial center. As every finite nilpotent group is isomorphic to the direct product of its Sylow subgroups and strong admissibility is preserved under products, we are reduced to studying strong admissibility for $p$-groups. Because the map $x \mapsto x^2$ is a strong complete mapping for every finite group of order relatively prime to 6, the question is only of relevance when $p = 2$ or $p = 3$.

In this paper we study strong admissibility for 3-groups. Our strategy is inductive (see Proposition 2.4): given a noncyclic group $G$, we extract a normal subgroup $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and patch together an appropriate strong complete mapping on $H$ together with one on $G/H$ to construct a strong complete mapping on $G$. The "base cases" for the induction come from strong admissibility for noncyclic groups of order 9 ([11, Lemma 2.6]) and order 27 ([11] for the abelian case, [7] for the general case). Difficulties arise when $G/H$ is not strongly admissible; however, in almost all cases these can be circumvented. We were therefore able to prove that *all* noncyclic 3-groups, except possible those in the family $L_r = \left\langle a, b \,\middle|\, a^{3^{r-1}} = b^3 = 1, bab^{-1} = a^{1+3^{r-2}} \right\rangle$, $r \geq 4$ are strongly admissible. The problem with the groups $L_r$ is that they have no normal subgroup $N$ such that both $N$ and $L_r/N$ are noncyclic; thus, the failure of strong admissibility for cyclic 3-groups ([11, Lemma 2.7] or [4, Theorem 2]) renders an inductive strategy hopeless. An explicit construction of a strong complete mapping for $L_r$ seems equally elusive.

It is natural to ask to what extent the techniques we have developed might extend to the case of 2-groups. Unfortunately, the program breaks down on several fronts. The inductive strategy worked for 3-groups in part because every noncyclic 3-group has a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. The analogous statement is no longer true for 2-groups, as witnessed, for example, by the family of dihedral groups. Even worse, the key ingredient in the proof of Proposition 2.4 (the inductive step for 3-groups) is the existence of an orthomorphism $\theta$ of $\mathbb{Z}_3 \times \mathbb{Z}_3$ such that the map $x \mapsto x\theta(x)$, after an appropriate "twisting", becomes bijective. The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is simply too small to admit a similar orthomorphism that might allow this argument to be generalized. These difficulties suggest that a fresh approach is required to address the question of strong admissibility for 2-groups.

In Section 2 we review background results and prove the technical lemmas needed. The main result of the paper is proven in Section 3.

## 2 Preliminaries

We open with a well-known result from the literature (see [7, Theorem 7 and Cor. 2]). Although the argument is elementary, we supply a proof in the interest of completeness, and also to illustrate how our inductive construction (Proposition 2.4) is rooted in this one.

**Lemma 2.1.**

- *Let $H$ be a subgroup of the center of a finite group $G$. If $H$ and $G/H$ are strongly admissible, then $G$ is strongly admissible.*
- *A direct product of strongly admissible groups is strongly admissible.*

**Proof.**
Suppose $H \leq Z(G)$, and let $\alpha : H \to H$ and $\beta : G/H \to G/H$ be strong complete mappings. Fix a choice of right transversal $T \subseteq G$ so that every $g \in G$ may be written uniquely as $g = h_g t_g$, with $h_g \in H$ and $t_g \in T$, and let $\Phi : G/H \to T$ denote the bijection $Hg \mapsto t_g$. We claim that $\gamma : G \to G$ defined by $\gamma(g) = \alpha(h_g)\Phi(\beta(Ht_g))$ is a strong complete mapping. Since $G$ is finite, is suffices to show that the maps $g \mapsto \gamma(g)$, $g \mapsto g\gamma(g)$, and $g \mapsto g^{-1}\gamma(g)$ are injective. To this end, suppose $g, g' \in G$. If $\gamma(g) = \gamma(g')$, then

$$\alpha(h_g)\Phi(\beta(t_g)) = \alpha(h_{g'})\Phi(\beta(Ht_{g'})). \tag{1}$$

Reducing this equation modulo $H$, we have $\beta(Ht_g) = \beta(Ht_{g'})$ in $G/H$. Since $\beta$ is a bijection from $G/H$ to itself, we have $Ht_g = Ht_{g'}$, and because $T$ is a transversal, $t_g = t_{g'}$. Substituting back into (1) and canceling yields $\alpha(h_g) = \alpha(h_{g'})$. Finally, the fact that $\alpha$ is a bijection implies $h_g = h'_g$. Thus, $g = g'$ and so $\gamma$ is injective.

Next, suppose $g\gamma(g) = g'\gamma(g')$, so $h_g t_g \alpha(h_g)\Phi(\beta(Ht_g)) = h_{g'} t_{g'} \alpha(h_{g'})\Phi(\beta(Ht_{g'}))$. Because $H \leq Z(G)$, we have

$$h_g \alpha(h_g) t_g \Phi(\beta(Ht_g)) = h_{g'} \alpha(h_{g'}) t_{g'} \Phi(\beta(Ht_{g'})). \tag{2}$$

As above, reduction modulo $H$ implies $(Ht_g)\beta(Ht_g) = (Ht_{g'})\beta(Ht_{g'})$ in $G/H$. Since $\beta$ is a complete mapping of $G/H$, $t_g = t_{g'}$. Substituting into (2) and canceling yields $h_g\alpha(h_g) = h_{g'}\alpha(h_{g'})$; finally, since $\alpha$ is a strong complete mapping of $H$, $h_g = h_{g'}$.

Finally, suppose $g^{-1}\gamma(g) = g'^{-1}\gamma(g')$, so $h_g^{-1}\alpha(h_g)t_g^{-1}\beta(t_g) = h_{g'}^{-1}\alpha(h_{g'})t_{g'}^{-1}\beta(t_{g'})$. Upon reduction modulo $H$ we apply normality of $H$ in $G$ to deduce that $(Ht_g)^{-1}\beta(Ht_g) = (Ht'_g)^{-1}\beta(t_{g'})$ in $G/H$, and then proceed as before. Note that we used the hypothesis $H \leq Z(G)$ in arguing that $\gamma$ is a complete mapping but only normality of $H$ in $G$ to argue that $\gamma$ is an orthomorphism.

For the second statement, suppose $\{G_i\}_{i\in I}$ is a family of strongly admissible groups. For each $i$, fix a choice of strong complete mapping $\varphi_i : G_i \to G_i$. Then a coordinate-wise argument easily establishes that the map $(x_i)_{i\in I} \mapsto (\varphi_i(x_i))_{i\in I}$ is a strong complete mapping of $\prod_{i\in I} G_i$. □

We also recall a key negative result on strong admissibility.

**Theorem 2.2.** *(Evans, [4, Theorem 2] If a finite group $G$ has a nontrivial, cyclic 3-Sylow subgroup $S$ and a normal subgroup $H$ such that $G/H \cong S$, then $G$ is not strongly admissible.*

Finally, we will need the classification of strongly admissible finite abelian groups.

**Theorem 2.3.** *(Evans, [5]) A finite abelian group is strongly admissible if and only if neither its 2-Sylow subgroup nor its 3-Sylow subgroup is nontrivial and cyclic.*

Our own work begins with an appropriate generalization of Lemma 2.1 to 3-groups. Once again, we work with a normal subgroup $H$ of our group $G$, but this time we do *not* assume $H \leq Z(G)$. This requires us to keep track of the conjugation action of $G$ on $H$.

**Proposition 2.4.** *Let $G$ be a 3-group and $N \triangleleft G$, with $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If $G/N$ is strongly admissible, then so is $G$.*

**Proof.**

Since a normal subgroup of $G$ must intersect $Z(G)$ nontrivially, we may assume that $N$ is generated by elements $a, b \in G$, each of order 3, where $a \in Z(G)$. Direct computation shows that $\alpha : N \to N$ defined by $\alpha(a^i b^j) = a^{2j} b^i$ is a strong complete mapping of $N$. For

$\varepsilon \in \{0,1\}$, define $\alpha_\varepsilon : N \to N$ by $\alpha_\varepsilon(a^i b^j) = a^{2j+\varepsilon i} b^i$. Then $\alpha_0 = \alpha$, and it is easy to check that $\alpha_1$ is an orthomorphism.

Fix a choice of right transversal $T \subseteq G$ so that every $g \in G$ may be written uniquely as $g = n_g t_g$, with $n_g \in N$ and $t_g \in T$, and let $\Phi : G/N \to T$ denote the bijection $Ng \mapsto t_g$. Observe that each $g \in G$ may be written uniquely as $g = n_g t_g$, where $n_g \in N$ and $t_g \in T$. Now $G$ acts on $N$ by conjugation, and since $G$ acts trivially on $M = \langle a \rangle$, $G$ therefore acts on $N/M \cong \mathbb{Z}_3$. Because $G$ is a 3-group and $Aut(N/M) \cong \mathbb{Z}_2$, the conjugation action of $G$ on $N/M$ must be trivial, so for every $g \in G$, $t_g b = a^{k(t_g)} b t_g$ for some $k(t_g)$, $0 \leq k(t_g) \leq 2$. For $g \in G$, define

$$\varepsilon(t_g) = \begin{cases} 1 & \text{if } k(t_g) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\beta : G/N \to G/N$ be a strong complete mapping and define $\Gamma : G \to G$ by

$$\Gamma(n_g t_g) = \alpha_{\varepsilon(t_g)}(n_g)\Phi(\beta(Nt_g)).$$

We claim that $\Gamma$ is a strong complete mapping of $G$. To show that $\Gamma$ is bijective, suppose $g, g' \in G$ and $\Gamma(g) = \Gamma(g')$, i.e.

$$\alpha_{\varepsilon(t_g)}(n_g)\Phi(\beta(Nt_g)) = \alpha_{\varepsilon(t_{g'})}(n_{g'})\Phi(\beta(Nt_{g'})). \tag{3}$$

Reducing this equation modulo $N$, we must have $\beta(Nt_g) = \beta(Nt_{g'})$. Because $\beta$ is a strong complete mapping, we have $Nt_g = Nt_{g'}$, and because $T$ is a transversal, $t_g = t_{g'}$. Substituting into (3) and cancelling on the right, we have $\alpha_{\varepsilon(t_g)}(n_g) = \alpha_{\varepsilon(t_g)}(n_{g'})$. Finally, $\alpha_{\varepsilon(t_g)}$ is a bijection, so $n_g = n_{g'}$.

Next, we check that $\Gamma$ is a complete mapping. With the same notation as above, suppose $n_g t_g \Gamma(n_g t_g) = n_{g'} t_{g'} \Gamma(n_{g'} t_{g'})$, i.e.

$$n_g t_g \alpha_{\varepsilon(t_g)}(n_g)\Phi(\beta(t_g N)) = n_{g'} t_{g'} \alpha_{\varepsilon(t_{g'})}(n_{g'})\Phi(\beta(t_{g'} N)). \tag{4}$$

Reducing this equation modulo $N$, we have $(Nt_g)\beta(Nt_g) = (Nt_{g'})\beta(Nt_{g'})$. Since $\beta$ is a complete mapping of $G/N$, we must have $Nt_g = Nt_{g'}$. Because $T$ is a transversal, this forces $t_g = t_{g'}$. Thus, (4) becomes

$$n_g t_g \alpha_{\varepsilon(t_g)}(n_g) = n_{g'} t_g \alpha_{\varepsilon(t_g)}(n_{g'}). \tag{5}$$

Writing $n_g = a^i b^j$ and $n_{g'} = a^{i'} b^{j'}$, where $i, j, i', j' \in \{0,1,2\}$ and recalling that $a \in Z(G)$, (5) reads:

$$a^{2j+(1+\varepsilon(t_g))i}b^j t_g b^i = a^{2j'+(1+\varepsilon(t_g))i'}b^{j'} t_g b^{i'}. \tag{6}$$

Because $t_g b = a^{k(t_g)} b t_g$, (6) becomes

$$a^{2j+(1+\varepsilon(t_g)+k(t_g))i}b^{i+j} t_g = a^{2j'+(1+\varepsilon(t_g)+k(t_g))i'}b^{i'+j'} t_g. \tag{7}$$

Canceling $t_g$ from both sides and comparing the exponents on $a$ and $b$, we conclude

$$2(j-j') + (1+\varepsilon(t_g)+k(t_g))(i-i') \equiv 0 (\text{mod } 3)$$

$$(j-j') + (i-i') \equiv 0 (\text{mod } 3).$$

Now from the definition of $\varepsilon(t_g)$, we always have $1+\varepsilon(t_g)+k(t_g) \not\equiv 2(\text{mod } 3)$, so the only solution to the above system is $i-i'=0$, $j-j'=0$, i.e. $n_g = n_{g'}$. Thus, $\Gamma$ is a complete mapping.

Finally, we verify that $\Gamma$ is an orthomorphism. To this end, suppose $(n_g t_g)^{-1}\Gamma(n_g t_g) = (n_{g'} t_{g'})^{-1}\Gamma(n_{g'} t_{g'})$, i.e.

$$t_g^{-1} n_g^{-1} \alpha_{\varepsilon(t_g)}(n_g)\Phi(\beta(Nt_g)) = t_{g'}^{-1} n_{g'}^{-1} \alpha_{\varepsilon(t_{g'})}(n_{g'})\Phi(\beta(Nt_{g'})) \tag{8}$$

Arguing as above, we reduce (8) modulo $N$ and invoke the fact that $\beta$ is an orthomorphism to show that $t_g = t_{g'}$. Substituting and canceling, we obtain $n_g^{-1}\alpha_{\varepsilon(t_g)}(n_g) = n_{g'}^{-1}\alpha_{\varepsilon(t_g)}(n_{g'})$. Finally, $\alpha_{\varepsilon(t_g)}$ is an orthomorphism, so we conclude $n_g = n_{g'}$ and hence that $\Gamma$ is an orthomorphism. □

We conclude this section with some useful results on the structure of 3-groups.

**Lemma 2.5.** *Every noncyclic* 3*-group contains a normal subgroup isomorphic to* $\mathbb{Z}_3 \times \mathbb{Z}_3$*.*

**Proof.**
Let $G$ be a noncyclic 3-group. If $Z(G)$ is noncyclic, then the structure theorem assures the existence of a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, so we assume henceforth that $Z(G)$ is cyclic of order at least 3. Since $G$ is not itself cyclic, $Z(G)$ is a proper, nontrivial, normal subgroup of $G$, so there exists a subgroup $Y \subseteq Z(G)$ of order 3.

Now let $C = \langle a \rangle$ be a cyclic normal subgroup of $G$ of maximum order; define $r$ by $|a| = 3^{r-1}$. Because $C$ intersects $Z(G)$ nontrivially, $C \supseteq Y$, so in fact $Y = \langle a^{3^{r-2}} \rangle$. Let $N$ be a normal subgroup of $G$ containing $C$ such that $[N : C] = 3$. Then $N$ is not itself cyclic but has a maximal cyclic subgroup, so by the classification of such groups (see, for example, [12, 5.3.4]), $N = \langle a, b\rangle$ where $|b| = 3$ and either $bab^{-1} = a$ or $bab^{-1} = a^{1+3^{r-2}}$. If we define $M = \langle a^{3^{r-2}}, b\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then $Y \subseteq M \subseteq N$. By normality of $N$ in $G$, for every $g \in G$ $gMg^{-1}$ is a subgroup of $N$ containing $Y$. However, in either case above, $M$ is the unique subgroup of $N$ which is both isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and contains $Y$, so we must have $gMg^{-1} = M$ for all $g \in G$; hence, $M \lhd G$. □

We will need the following as the base step for the induction in the next section. Although the result can be found in [11] for groups order 9 and in [7] for groups of order 27, the explicit strong complete mappings we discovered might be of interest, so we include them here.

**Lemma 2.6.** *([11], [7]) Every noncyclic group of order* 9 *or* 27 *is strongly admissible.*

**Proof.**
The noncyclic group of order 9 and the abelian noncyclic groups of order 27 are strongly admissible by Theorem 2.3, so the only remaining groups to consider are the Heisenberg group $H_3 = \left\langle x, y, z \;\middle|\; x^3 = y^3 = z^3 = 1, xz = zx, yz = zy, xy = zyx \right\rangle$ and $L_3 = \left\langle a, b \;\middle|\; a^9 = b^3 = 1, bab^{-1} = a^4 \right\rangle$. Every element of $H_3$ may be written (uniquely) as $x^i y^j z^k$, where $0 \le i, j, k \le 2$; a strong complete mapping $\varphi$ of $H_3$ found by `Mace4` is exhibited in Table 1 along with the maps $s \mapsto s\varphi(s)$ and $s \mapsto s^{-1}\varphi(s)$ in Table 2. Using `GAP`, we found a formula for a strong complete mapping of $L_3$. First, we identify $L_3$ with the subgroup $K \le SL_4(\mathbb{F}_3)$ defined by:

$$K = \left\{ \begin{bmatrix} 1 & c & c-c^2 & a \\ 0 & 1 & c & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_3 \right\}$$

A strong complete mapping is then defined by:

$$\begin{bmatrix} 1 & c & c-c^2 & a \\ 0 & 1 & c & c-c^2+b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & b & b-b^2 & -ba+c \\ 0 & 1 & b & -(b+b^2)-a+(1-b^2)c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

□

Table 1: Strong complete mapping for $H_3$

| $s$ | $\varphi(s)$ | $s$ | $\varphi(s)$ | $s$ | $\varphi(s)$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $x$ | $y$ | $x^2$ | $z^2$ |
| $z$ | $yz^2$ | $xz$ | $y^2z^2$ | $x^2z$ | $yz$ |
| $z^2$ | $xz^2$ | $xz^2$ | $x^2y^2$ | $x^2z^2$ | $x^2y^2z^2$ |
| $y$ | $xy^2z$ | $xy$ | $x^2y$ | $x^2y$ | $y^2$ |
| $yz$ | $x^2yz$ | $xyz$ | $z$ | $x^2yz$ | $xy$ |
| $yz^2$ | $x^2y^2z$ | $xyz^2$ | $x$ | $x^2yz^2$ | $x^2z$ |
| $y^2$ | $y^2z$ | $xy^2$ | $xy^2z^2$ | $x^2y^2$ | $x^2$ |
| $y^2z$ | $x^2z^2$ | $xy^2z$ | $x^2yz^2$ | $x^2y^2z$ | $xy^2$ |
| $y^2z^2$ | $xyz^2$ | $xy^2z^2$ | $xz$ | $x^2y^2z^2$ | $xyz$ |

Table 2: Images for maps $s \mapsto s\varphi(s)$ and $s \mapsto s^{-1}\varphi(s)$

| $s$ | $s\varphi(s)$ | $s^{-1}\varphi(s)$ | $s$ | $s\varphi(s)$ | $s^{-1}\varphi(s)$ | $s$ | $s\varphi(s)$ | $s^{-1}\varphi(s)$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $1$ | $x$ | $xy$ | $x^2y$ | $x^2$ | $x^2z^2$ | $xz^2$ |
| $z$ | $y$ | $yz$ | $xz$ | $xy^2$ | $x^2y^2z$ | $x^2z$ | $x^2yz^2$ | $xy$ |
| $z^2$ | $xz$ | $x$ | $xz^2$ | $y^2z^2$ | $xy^2z$ | $x^2z^2$ | $xy^2z$ | $y^2$ |
| $y$ | $x$ | $xyz^2$ | $xy$ | $y^2z$ | $xz$ | $x^2y$ | $x^2$ | $xyz$ |
| $yz$ | $x^2y^2$ | $x^2z^2$ | $xyz$ | $xyz^2$ | $x^2y^2z^2$ | $x^2yz$ | $y^2$ | $x^2z$ |
| $yz^2$ | $x^2z$ | $x^2yz$ | $xyz^2$ | $x^2yz$ | $y^2z$ | $x^2yz^2$ | $xyz$ | $y^2z^2$ |
| $y^2$ | $yz$ | $z$ | $xy^2$ | $x^2y$ | $z^2$ | $x^2y^2$ | $xy^2z^2$ | $y$ |
| $y^2z$ | $x^2y^2z^2$ | $x^2yz^2$ | $xy^2z$ | $z^2$ | $xy^2$ | $x^2y^2z$ | $yz^2$ | $x^2$ |
| $y^2z^2$ | $xz^2$ | $xy^2z^2$ | $xy^2z^2$ | $x^2y^2z$ | $yz^2$ | $x^2y^2z^2$ | $z$ | $x^2y^2$ |

Finally, we record a result which will allow us to bypass the problem of a cyclic quotient group encountered in the inductive process.

**Lemma 2.7.** *Let $G$ be a noncyclic 3-group of order $3^r$, $r \geq 4$. Let $N = \langle z, b\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ be a normal subgroup of $G$, where $z \in Z(G)$ and $b \notin Z(G)$. If $G/N$ is cyclic, then either $G \cong \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_3$ or $G \cong L_r$, or else there exists $N' \lhd Z(G)$, $N' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that $G/N'$ is not cyclic.*

**Proof.**
Since $G/N$ is cyclic, there exists $x \in G$ which maps to a generator $\bar{x} = xN \in G/N$ under the quotient map, so in fact $G = \langle x, z, b\rangle$. Because $G$ itself is not cyclic, either $|x| = 3^{r-1}$ or $|x| = 3^{r-2}$. If $|x| = 3^{r-1}$, then $G$ has a maximal cyclic subgroup, so by [12, 5.3.4], $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-1}}$ or $G \cong L_r$. If $|x| = 3^{r-2}$, then $\langle z, b\rangle \cap \langle x\rangle = \{1\}$, so let $N' = \langle z, x^{3^{r-3}}\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. We claim that $N' \subseteq Z(G)$. It suffices to show that $x^{3^{r-3}}$ commutes with $b$. Because $N \lhd G$, we have $xbx^{-1} = z^i b$ for some $i \in \{1, 2\}$, i.e. $bxb^{-1} = z^{-i}x$; thus, $bx^{3^{r-3}}b^{-1} = (bxb^{-1})^{3^{r-3}} = z^{-3^{r-3}i}x^{3^{r-3}} = x^{3^{r-3}}$. Finally, the quotient group $G/N'$ contains elements $\bar{x}^{3^{r-4}}$ and $\bar{b}$ of order 3, each of which generates a distinct subgroup of $G/N'$. Therefore, $G/N'$ is not cyclic. □

## 3 Main result

**Theorem 3.1.** *Let $G$ be a nontrivial 3-group which is neither cyclic nor isomorphic to $L_r$, $r \geq 4$. Then $G$ is strongly admissible.*

**Proof.**
Let $G$ be a group of order $3^r$ as in the statement; by Lemma 2.6 we may assume $r \geq 4$. We proceed by induction on $r$. By Lemma 2.5, $G$ has a normal subgroup $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If $G/N$ is noncyclic and $G/N \ncong L_{r-2}$, then $G$ is strongly admissible by the induction hypothesis and Proposition 2.4. If $G/N$ is cyclic and $N \subseteq Z(G)$, then $G$ is abelian and hence strongly admissible by Theorem 2.3. If $G/N$ is cyclic and $N \nsubseteq Z(G)$, then $G$ and $N$ satisfy the hypotheses of Lemma 2.7, so either $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-1}}$ (and hence $G$ is strongly admissible by Theorem 2.3) or we may replace $N$ with a normal subgroup $N' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ such that $G/N'$ is not cyclic and argue as above.

We are therefore reduced to the case in which $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ is a normal subgroup of $G$ and $G/N \cong L_{r-2}$. If $r = 4$ or $r = 5$, then $L_{r-2}$ is strongly admissible by Lemma 2.6, and so $G$ is strongly admissible by Proposition 2.4. We assume henceforth $r \geq 6$. Fix generators $z, b$ of order 3 for $N$, where $z \in Z(G)$, and let $\pi : G \to G/N = L_{r-2}$ denote the quotient map.

Choose generators $\bar{x}, \bar{y} \in L_{r-2}$ such that $\bar{x}^{3^{r-3}} = \bar{y}^3 = 1$ and $\bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^{1+3^{r-4}}$ and then select $x, y \in G$ such that $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$; evidently, $G = \langle x, y, z, b\rangle$. From normality of $N$ in $G$ we have $xbx^{-1} = z^{-e_x}b$, $yby^{-1} = z^{-e_y}b$, i.e.

$$bxb^{-1} = z^{e_x}x \text{ and } byb^{-1} = z^{e_y}y \text{ for some } e_x, e_y, 0 \le e_x, e_y \le 2. \tag{9}$$

Since $G/N = \langle \bar{x}, \bar{y}\rangle = \langle \overline{yx}, \bar{y}\rangle = \langle \overline{y^2x}, \bar{y}\rangle$, we may replace $x$ by $yx$ or $y^2x$ without disturbing anything claimed henceforth. In particular, if $e_y \neq 0$, then replacing $x$ by $y^{-e_x}x$, we may assume $e_x = 0$. Replacing $z$ with $z^2$ if needed, we assume without loss of generality that at least one of $e_x$, $e_y$ is 0 and the other is either 0 or 1. We claim furthermore that $x^3 \in Z(G)$; to prove this, it suffices to show that $x^3$ commutes with $y$ and $b$. From normality of $N$ in $G$ we have $xbx^{-1} = z^ib$ for some $i$, $0 \le i \le 2$, i.e. $b$ and $x$ commute modulo $\langle z\rangle$. Then $bx^3b^{-1} = (bxb^{-1})^3 = (z^{-i}x)^3 = x^3$. From the relation $\bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^{1+3^{r-4}}$ in $G/N$ we have $yxy^{-1} = z^ib^jx^{1+3^{r-4}}$ for some $i, j$, $0 \le i, j \le 2$. Again, since $b$ and $x$ commute modulo $\langle z\rangle$, it follows that $yx^3y^{-1} = (yxy^{-1})^3 = x^3$. Now define $c = x^{3^{r-4}}$ and $M = \langle z, c\rangle \subseteq Z(G)$. Observe that either $|c| = 3$ or $|c| = 9$.

If $|c| = 3$, then $|x| = 3^{r-3}$. Since $r \ge 6$, the residues of $x^{3^{r-5}}$ and $b$ are (respectively) central elements of order 3 in $G/M$ that generate distinct subgroups. For this reason $G/M$ cannot be cyclic, and because $Z(L_{r-2}) \cong \mathbb{Z}_3$, $G/M \ncong L_{r-2}$. By induction, $G/M$ is strongly admissible, and therefore $G$ is strongly admissible by Lemma 2.1.

If $|c| = 9$, then $c^3 \in \operatorname{Ker} \pi = N$, so $c^3 = z^ib^j$ for some $i, j$, $0 \le i, j \le 2$. If $j \neq 0$, then $b \in M \cong \mathbb{Z}_3 \times \mathbb{Z}_9$, and $G/M \cong (G/N)/\langle \pi(c)\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-4}}$ is admissible by Theorem 2.3 and hence $G$ is strongly admissible by Lemma 2.1. If $j = 0$ then we must have $i \neq 0$, so $z \in M' = \langle c, b\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. Note that $M' \triangleleft G$ and $G/M' \cong \mathbb{Z}_3 \times \mathbb{Z}_{3^{r-4}}$ is strongly admissible. We now turn to the construction of a strong complete mapping for $G$. If $e_x = e_y = 0$, then $M' \subseteq Z(G)$, so we may conclude as before by Lemma 2.1. We therefore assume that one of $e_x, e_y$ is 0 and the other is 1; we will use a technique similar to that used in Proposition 2.4 to construct a strong complete mapping for $G$. Observe that every member of $M'$ may be written uniquely as $z^ib^jc^k$, where $0 \le i, j, k \le 2$. Now consider the transversal $T = \{x^ly^m : 0 \le l \le 3^{r-4} - 1,\ 0 \le m \le 2\}$ for $M'$ in $G$, and let $\Phi : G/M' \to T$ be the bijection which assigns to each (right) coset $M'g$ the unique element $t_g \in T$ such that $M'g = M't_g$. By Theorem 2.3, $G/M'$ admits a strong complete mapping $\beta : G/M' \to G/M'$, and $M'$ likewise admits a strong complete mapping $\alpha_0 : M' \to M'$. In Table 3 we define maps $\alpha_s : M' \to M'$, $s = 1, 2$, in which (for brevity) we use the string $ijk$ to represent the element $z^ib^jc^k \in M'$.

These maps were found by `Mace4`; they are orthomorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_9$ satisfying the following additional condition: for $\ell \in \{1, 2\}$, the map

$$z^ib^jc^k \mapsto z^{i+\ell j}b^jc^k\alpha_\ell(z^ib^jc^k) \tag{10}$$

Table 3: Definition of the maps $\alpha_1$ and $\alpha_2$

| $ijk$ | $\alpha_1(ijk)$ | $\alpha_2(ijk)$ | $ijk$ | $\alpha_1(ijk)$ | $\alpha_2(ijk)$ | $ijk$ | $\alpha_1(ijk)$ | $\alpha_2(ijk)$ |
|---|---|---|---|---|---|---|---|---|
| 000 | 000 | 000 | 100 | 210 | 022 | 200 | 222 | 011 |
| 001 | 002 | 002 | 101 | 120 | 012 | 201 | 110 | 122 |
| 002 | 220 | 220 | 102 | 001 | 001 | 202 | 112 | 112 |
| 010 | 100 | 100 | 110 | 211 | 210 | 210 | 122 | 202 |
| 011 | 202 | 021 | 111 | 101 | 101 | 211 | 221 | 120 |
| 012 | 011 | 212 | 112 | 012 | 010 | 212 | 020 | 020 |
| 020 | 022 | 221 | 120 | 111 | 111 | 220 | 121 | 222 |
| 021 | 102 | 211 | 121 | 010 | 200 | 221 | 021 | 110 |
| 022 | 212 | 102 | 122 | 201 | 121 | 222 | 200 | 201 |

is bijective.

Now every $g \in G$ may be rewritten uniquely as $g = m'_g t_g$, with $m'_g = z^i b^j c^k \in M'$ and $t_g = x^s y^t \in T$, where $0 \le i, j, k, t \le 2$ and $0 \le s \le 3^{r-3} - 1$. Define $\Gamma : G \to G$ by

$$\Gamma(g) = \alpha_u(m_g)\Phi(\beta(M' t_g)).$$

where

$$u = \begin{cases} s \pmod 3 & \text{if } e_y = 0 \\ t \pmod 3 & \text{if } e_x = 0 \end{cases}$$

The argument of Proposition 2.4, *mutatis mutandis*, shows that $\Gamma$ is an orthomorphism. To show that $\Gamma$ is a complete mapping, suppose $g\Gamma(g) = g'\Gamma(g')$, i.e.

$$(z^i b^j c^k)(x^s y^t)\Gamma((z^i b^j c^k)(x^s y^t)) = (z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'})\Gamma((z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'})).$$

Then

$$(z^i b^j c^k)(x^s y^t)\alpha_u(z^i b^j c^k)\Phi(\beta(M' x^s y^t)) = (z^{i'} b^{j'} c^{k'})(x^{s'} y^{t'})\alpha_u(z^{i'} b^{j'} c^{k'})\Phi(\beta(M' x^{s'} y^{t'})). \quad (11)$$

Reducing modulo $M'$, we have $x^s y^t \beta(M' x^s y^t) = x^{s'} y^{t'} \beta(M' x^{s'} y^{t'})$. Since $\beta$ is a complete mapping, $M' x^s y^t = M' x^{s'} y^{t'}$, and because $T$ is a transversal, $s = s'$ and $t = t'$. Substituting into Equation (11) and canceling, we are left with:

$$(z^i b^j c^k)(x^s y^t)\alpha_u(z^i b^j c^k) = (z^{i'} b^{j'} c^{k'})(x^s y^t)\alpha_u(z^{i'} b^{j'} c^{k'}). \quad (12)$$

If $e_y = 0$, then $e_x = 1$ and $u = s$, so Equation (12) may be rewritten as:

$$z^{i+sj} b^j c^k \alpha_s(z^i b^j c^k) = z^{i'+sj'} b^{j'} c^{k'} \alpha_s(z^{i'} b^{j'} c^{k'}).$$

From bijectivity of the maps $\alpha_i$ (by construction when $i = 0$, or by Equation (10) when $i = 1, 2$), it follows that $i = i'$, $j = j'$, $k = k'$. A similar argument applies when $e_x = 0$ and $e_y = 1$. □

**Corollary 3.2.** *Let $G$ be a nilpotent group of odd order whose 3-Sylow subgroup is neither nontrivial and cyclic, nor isomorphic to $L_r$, $r \geq 4$. Then $G$ is strongly admissible.*

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