Strong complete mappings for 2-groups

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Abstract

A strong complete mapping for a group G is a bijection $\varphi : G \to G$ such that the maps $x \mapsto x\varphi(x)$ and $x \mapsto x^{-1}\varphi(x)$ are also bijections. Groups admitting a strong complete mapping are important to the study of orthogonality problems for Latin squares and group sequencings, among other applications. In previous work we showed that a finite 3-group that contains no cyclic subgroup of index 3 is strongly admissible. In this article, we employ a substantially different strategy to show that a finite 2-group that contains no cyclic subgroup of index 4 is strongly admissible.

1 Introduction

Let G be a group. Given a function $\theta : G \to G$, we define maps $\theta_+, \theta_- : G \to G$ by $\theta_+(x) = x\theta(x)$ and $\theta_-(x) = x^{-1}\theta(x)$. We call θ a *complete mapping* if θ and θ_+ are bijective, an *orthomorphism* if θ and θ_- are bijective, or a *strong complete mapping* if all three maps are bijective. Strong complete mappings are of interest in view of their close relation to orthogonality problems for Latin squares [12]. They have also been used to study group sequencings [2], Knut Vic designs ([15], [16]), strong starters [17], solutions to the toroidal n-queens problem [20], and check digit systems [21].

We call a group *admissible* if it admits a complete mapping and *strongly admissible* if it admits a strong complete mapping. It is immediate that a map θ (as above) is an orthomorphism if and only if θ_- is a complete mapping. The classification of admissible finite groups was initiated in 1955 with the work of Hall and Paige [14] and completed in 2009 by Wilcox, Evans, and Bray et. al. (see [10], [22], [7]). It is now known that a finite group G is admissible if and only if its 2-Sylow subgroup is either trivial or noncyclic. The analogous question for infinite groups was settled by Bateman [3]. In contrast, the classification of strongly admissible finite groups is mostly open. In [11], A. B. Evans completely classified strongly admissible finite *abelian* groups. In [12], he extended the classification to groups of order at most 31 and identified various infinite families of strongly admissible dihedral and quaternion groups.

In earlier work [1], the first and third authors proved that if G is a nonabelian 3-group without a maximal cyclic subgroup, then G is strongly admissible. The proof proceeds by induction. Every noncyclic 3-group G contains a normal subgroup N isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$; hence one argues that if G/N is strongly admissible, then "stitching" a strong complete mapping of G/N together with an appropriate family of orthomorphisms of N yields a strong complete mapping for G. If G/N is cyclic or nonabelian with maximal cyclic subgroup, then one replaces N with an appropriate normal subgroup N' such that G/N' is strongly admissible.

It is natural to wonder whether a similar program might be carried out for 2-groups. As noted in the introduction to [1], two difficulties immediately present themselves. The first is that the most obvious candidate for the group N of the previous paragraph, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$, is too small to admit orthomorphisms of the type needed to carry out the inductive step. The second problem is that, instead of a single infinite family of nonabelian 3-groups with cyclic maximal subgroup, there are are *four* infinite families of nonabelian 2-groups with cyclic maximal subgroup, only one of which has a noncyclic abelian quotient.

In the present article, we address these obstructions, retaining the inductive approach but implementing it in a rather different way. Using a result of Janko, we show that a 2-group G that is neither cyclic itself nor contains a maximal cyclic subgroup contains a normal subgroup N isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with respect to which a stitching construction can be executed. It is not difficult to show that N can be chosen such that G/N is not cyclic, so the principal difficulty occurs when $\overline{K} = G/N$ is a nonabelian group with maximal cyclic subgroup. Although the construction of strong complete mappings on such groups has proved elusive, it is actually rather easy (see Theorem 2.18) to construct an explicit strong complete mapping on a 2-covering of $\overline{K}$, i.e. a 2-group K that does *not* have a maximal cyclic subgroup, and such that $K/\langle z \rangle \cong \overline{K}$ for some $z \in Z(K)$. This observation, together with a version of stitching with respect to subgroups of order 16, can be used to salvage the induction. The program nonetheless requires considerable effort, because there are twelve isomorphism classes of order 16 relevant for our purposes, and the sheer number of automorphisms associated with each of them renders it infeasible to find the required stitchings by hand. Instead, as described in the appendix, we used a `Python` program to construct the necessary stitchings.

Our main result is:

Theorem 3.4: Let G be a 2-group of order 2^r, $r \geq 4$, all of whose elements have order at most 2^{r-3}. Then G is strongly admissible.

Using similar techniques, we also prove:

Theorem 3.5: Suppose G_1 and G_2 are nontrivial 2-groups. Then $G_1 \times G_2$ is strongly admissible.

We begin in Section 2 with preliminaries on 2-groups, a survey of strong admissibility, and various lemmas needed for stitching. Section 3 contains the main results of the paper, along with a discussion of the restrictions imposed on the order of elements of G. Information pertaining to groups of order 16 is collected in an appendix, so as not to interrupt the flow of the main argument.

Some of the work in this article is based on the second author's final master's degree project at Miami University.

2 Preliminaries

2.1 2-groups

For a finite 2-group G, define

$$c(G) = \min\left\{\log_2\left(\frac{|G|}{|x|}\right) :\ x \in G\right\} = \log_2|G| - \max\left\{\log_2|x| :\ x \in G\right\}.$$

Thus, $c(G) = 0$ if G is cyclic; $c(G) = 1$ if G is not cyclic but has a maximal cyclic subgroup, etc. In general, $2^{c(G)}$ is the smallest index of a cyclic subgroup of G. We denote by $\mathcal{C}_k$ the class of 2-groups such that $c(G) = k$.

Theorem 2.1. *[19, 5.3.4] Suppose $n \geq 2$. A 2-group in $\mathcal{C}_1$ that has order 2^n is isomorphic to one of the following:*

1. *The direct product of $\mathbb{Z}_{2^{n-1}}$ and $\mathbb{Z}_2$: $T_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^2,\ bab^{-1} = a\rangle$*
2. $M_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^2,\ bab^{-1} = a^{1+2^{n-2}}\rangle$
3. *The dihedral group $D_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^2,\ bab^{-1} = a^{-1}\rangle$*
4. *The generalized quaternion group*
   $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^4, a^{2^{n-2}} = b^2,\ bab^{-1} = a^{-1}\rangle$
5. *The semidihedral group $S_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1 = b^2,\ bab^{-1} = a^{-1+2^{n-2}}\rangle$.*

Remark. In each of the five families of groups listed in Theorem 2.1, the element $a^{2^{n-2}}$ is central and of order 2. For each such group G, we denote by $q_G : G \to G/\langle a^{2^{n-2}} \rangle = \widehat{G}$ the quotient map. Note that if $n \geq 3$, then $\widehat{G}$ is also in $\mathcal{C}_1$.

Definition 2.2. *A* 2-covering *of a 2-group $\overline{G}$ is a surjective homomorphism $\pi : G \to \overline{G}$ such that $|Ker\ \pi| = 2$ and $c(G) = c(\overline{G}) + 1$*

The last condition of the definition requires that the maximum order of an element be the same in G as in $\overline{G}$. We record an important property of 2-coverings that follows quickly from the definitions:

Lemma 2.3. *Suppose a 2-group $\overline{G}$ of order 2^n, $n \geq 2$, is isomorphic to one of the groups in Theorem 2.1, and that there exists a 2-covering $\pi : G \to \overline{G}$. Let $\overline{a}$ and $\overline{b}$ denote generators of $\overline{G}$ of order 2^{n-1} and 2 respectively; next, choose $a, b \in G$ such that $\pi(a) = \overline{a}$ and $\pi(b) = \overline{b}$. Then $a^{2^{n-2}} \in Z(G)$, and, denoting by $q_G : G \to G/\langle a^{2^{n-2}} \rangle = \widehat{G}$ the quotient map, there exists a 2-covering $\widehat{\pi} : \widehat{G} \to \widehat{\overline{G}}$ such that the diagram below commutes:*

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "q_G"] & \overline{G} \arrow[d, "q_{\overline{G}}"] \\ \widehat{G} \arrow[r, dotted, "\widehat{\pi}"] & \widehat{\overline{G}} \end{tikzcd}$$

Proof.
The hypothesis implies that Ker $\pi = \langle z \rangle$, where $z \in G$ has order 2. Moreover, $c(\overline{G}) = 1$, so $c(G) = 2$, and thus $|a| = 2^{n-1}$. By assumption $\overline{b}\overline{a}\overline{b}^{-1} = \overline{a}^{\pm 1 + \varepsilon 2^{n-2}}$ for some $\varepsilon \in \{0, 1\}$, so $bab^{-1} = z^i a^{\pm 1 + \varepsilon 2^{n-2}}$ for some $i \in \{0, 1\}$. In all cases, $ba^{2^{n-2}}b^{-1} = a^{2^{n-2}}$, and since G is generated by a, b, and z, it follows that $a^{2^{n-2}} \in Z(G)$ is an element of order 2. However, $\pi(a^{2^{n-2}}) \neq 1$, so $z \notin \langle a \rangle$. It follows that Ker $(q_{\overline{G}} \circ \pi) = \langle z, a^{2^{n-2}} \rangle$, from which the conclusion follows. $\square$

For the balance of this paper, we will reserve the "hat" notation for this construction and the letter q for the associated quotient maps.

For a group G we define $Z_0(G) = \{1\}$ and for $n \geq 1$, we define $Z_{n+1}(G)$ by $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$. Then we have a chain $1 = Z_0(G) \leq Z_1(G) \leq \cdots \leq Z_k(G)$. The group G is *nilpotent* if $Z_c(G) = G$ for some $c \neq 0$; in that case, the *nilpotence class* of G is the smallest such nonnegative integer c. A group of p-power order (where p is prime) has a nontrivial center, so a group of order p^n has nilpotence class at most $n - 1$. A group of order p^n, $n \neq 3$, with nilpotence class equal to $n - 1$ is said to be of *maximal class.*

Theorem 2.4. *[4, Corollary 1.7] Every 2-group of maximal class is isomorphic to either D_{2^n}, Q_{2^n}, or S_{2^n} for some $n \geq 3$.*

2.2 Strong admissibility: known results

We first establish some notation for the balance of the paper. Suppose N is a normal subgroup of a group G and $T \subseteq G$ a (left) transversal. For $g \in G$, define $t_g \in T$ and $n_g \in N$ as the unique elements such that $g = t_g n_g$. For any function $\beta : G/N \to G/N$, define $\tilde{\beta} : T \to T$ by $\beta(t_g N) = \tilde{\beta}(t_g)N$.

The following are the main negative results on strong admissibility:

Theorem 2.5. *1. [14, Theorem 5] A finite group whose* 2*-Sylow subgroup is nontrivial and cyclic is inadmissible.*

2. [11, Theorem 3] If a finite group G has a nontrivial, cyclic 3*-Sylow subgroup S, and H is a normal subgroup of G for which $G/H \cong S$, then G is not strongly admissible.*

3. [11, Corollary 4] A finite group of odd order with a nontrivial, cyclic 3*-Sylow subgroup is not strongly admissible.*

4. [12, Theorem 10] The dihedral and quaternion groups of order 8 *are not strongly admissible.*

The classification of strongly admissible abelian groups has been completed by Evans and will be indispensable to our work.

Theorem 2.6. *[10, Theorem 15] A finite abelian group is strongly admissible if and only if neither its* 2*-Sylow subgroup nor its* 3*-Sylow subgroup is nontrivial and cyclic.*

Considering that the dihedral and quaternion groups of order 8 are not strongly admissible, the following may seem somewhat surprising.

Theorem 2.7. *([6], [12, Theorems 11, 12]) Every noncyclic group of order* 16 *is strongly admissible.*

We will also need a result about complete mappings (or equivalently orthomorphisms), whose statement for complete mappings appears in [14].

Theorem 2.8. *[14, Corollary 2] Let N be a normal subgroup of a finite group G. If N and G/N admit orthomorphisms, then G admits orthomorphisms.*

The proof of this theorem proceeds exactly as one would expect. Fix a left transversal T for N in G, and choose orthomorphisms $\alpha : N \to N$ and $\beta : G/N \to G/N$. Having written $g \in G$ as $n_g t_g$ for $n_g \in N$ and $t_g \in T$, define $\gamma : G \to G$ by $\gamma(n_g t_g) = \alpha(n_g)\tilde{\beta}(t_g)$; then a routine verification shows that γ is an orthomorphism. Unfortunately, one cannot use these same methods to prove an analogue of Theorem 2.8 for complete mappings, the obstruction being that elements of N do not in general commute with elements of T. For the present, therefore, we are restricted to the following.

Proposition 2.9. *(cf. [9, Theorem 3], [17, Lemma 2.8]) Let N be a subgroup of the center of a finite group G. If N and G/N are strongly admissible, then G is strongly admissible.*

Finally, we note an important property of strong admissibility for all groups that can be proven by the obvious construction.

Proposition 2.10. *A direct product of strongly admissible groups is strongly admissible.*

2.3 Janko's Theorem

The following result appears in a paper of Janko [18], who describes it as a strengthening of a result of Berkovich.

Theorem 2.11. *([18, Proposition 1.12]) Let G be a 2-group having no normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose further that G is neither abelian nor of maximal class. Then one of the following holds:*

1. *G has a normal abelian subgroup $A \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ with $C_G(A)$ abelian of type $(2^n, 2)$, $n \geq 2$. In that case, $G/C_G(A)$ is isomorphic to a subgroup of $Aut(A) \cong D_8$. Moreover, if $G/C_G(A) \cong D_8$, then $C_G(A) = A$.*

2. *G has a normal abelian subgroup $W \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ with $C_G(W)$ metacyclic. In that case, the subgroup of $C_G(W)$ generated by the elements of order at most 4 is precisely W and $G/C_G(W)$ is isomorphic to a subgroup of $Aut(W)$.*

We will only make use of the existence part of Theorem 2.11. In the first case, let $x = (1, 0)$ and $y = (0, 1)$. Then for $g \in G$, $gxg^{-1} = x^i y^j$. It follows that $gx^2g^{-1} = x^{2i}y^{2j} = x^2$, so $x^2 \in Z(G)$ and $gyg^{-1} \in \{y, x^2y\}$. Likewise, in the second case, let $x = (1, 0)$ and $y = (0, 1)$. The elements of order 2 in W are x^2, y^2, and x^2y^2; so, replacing x by y or xy if necessary, we may assume without loss of generality that $x^2 \in Z(G)$. Now suppose that for $g \in G$, $gxg^{-1} = x^i y^j$. Then $x^2 = gx^2g^{-1} = x^{2i}y^{2j}$, so i is odd and j is even. It follows that $\langle x, y^2 \rangle$ is a normal subgroup of G, isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.

Definition 2.12. *A subgroup H of a 2-group G is* well-positioned *if H is normal in G, and H is one of the following types:*

- *Type I: $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$; in particular*
  $H = \langle x, y \mid x^4 = y^2 = 1,\ [x, y] = 1 \rangle$ *with* $x^2 \in Z(G)$.

- *Type II: $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; in particular*
  $H = \langle x, y, z \mid x^2 = y^2 = z^2 = 1,\ [x, y] = [x, z] = [y, z] = 1 \rangle$ *with* $x \in Z(G)$ *and* $\langle x, y \rangle$ *normal in* G.

Observe that if a normal subgroup H of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we may choose generators so that $(1, 0, 0) \in Z(G)$. Moreover, since H must contain a normal subgroup of order 4, we may further assume that $\langle (1, 0, 0), (0, 1, 0) \rangle$ is normal in G.

We may summarize the situation thus:

Corollary 2.13. *Let G be a 2-group that is neither abelian nor of maximal class. Then either G has a normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or there exists a normal subgroup $N = \langle a, b \rangle$ of G, isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$, with $|a| = 4$, $|b| = 2$, and $a^2 \in Z(G)$.*

In Section 3, we will need to consider particular factorizations of the quotient map $G \to G/N$, where N is a well-positioned subgroup of G. These are described in the following definition.

Definition 2.14. *Let H be a well-positioned subgroup of a 2-group G. The* canonical refinement *of the quotient map $\pi : G \to G/H$ is its factorization as a composition $\pi = \pi^{(2)} \circ \pi^{(1)} \circ \pi^{(0)}$ of homomorphisms:*

$$G_0 = G \xrightarrow{\pi^{(0)}} G_1 = G/\langle x^2 \rangle \xrightarrow{\pi^{(1)}} G_2 = G/\langle x^2, y \rangle \xrightarrow{\pi^{(2)}} G_3 = G/H$$

if H is of Type I or

$$G_0 = G \xrightarrow{\pi^{(0)}} G_1 = G/\langle x \rangle \xrightarrow{\pi^{(1)}} G_2 = G/\langle x, y \rangle \xrightarrow{\pi^{(2)}} G_3 = G/H$$

if H is of Type II.

Lemma 2.15. *(Moving lemma) Let H be a well-positioned subgroup of a group G of order 2^n, $n \geq 5$. If $G \notin \mathcal{C}_1$ and G/H is cyclic, then there exists a well-positioned subgroup H' of G such that G/H' is a noncyclic abelian group.*

Proof.

The hypothesis precludes G itself being cyclic. Suppose that H is of Type I, i.e. $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, generated by x, y, with $|x| = 4$, $|y| = 2$, and $x^2 \in Z(G)$. Let $\pi : G \to G/H$ be the quotient map and choose $a \in G$ such that $\pi(a)$ generates G/H. Then G is generated by a, x, and y. Moreover, normality of H in G implies that $axa^{-1} = x^i y^j$ where $i \in \{\pm 1\}$ and $j \in \{0, 1\}$, and $aya^{-1} = x^{2k} y$, where $k \in \{0, 1\}$. Direct computation shows that $a^2 x a^{-2} \in \{x, x^{-1}\}$ and a^2 commutes with y; thus, $a^4 \in Z(G)$. In particular, since $n \geq 5$, $a^{2^{n-3}} \in Z(G)$. Observe also that $|a| = 2^{n-2}$ or $|a| = 2^{n-3}$. Suppose first that $|a| = 2^{n-2}$. Then $a^{2^{n-3}}$ is an element of $\operatorname{Ker} \pi = H$ of order 2, so $a^{2^{n-3}} \in \{x^2, y, x^2 y\}$ but $a^{2^{n-4}} \notin H$. Then $H' = \langle a^{2^{n-4}}, x^2, y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ is a well-positioned subgroup of G, and $G/H' \cong \mathbb{Z}_{2^{n-4}} \times \mathbb{Z}_2$. If $|a| = 2^{n-3}$, then $a^{2^{n-4}}$ is an element of order 2 that is not in H; moreover, since $n \geq 5$, $a^{2^{n-4}}$ commutes with y. In this case, $H' = \langle a^{2^{n-4}}, x^2, y \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and normal in G; moreover, G/H' is not cyclic.

Suppose next that H is of Type II, i.e. $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by elements x, y, and z, each of order 2, with $x \in Z(G)$ and $\langle x, y\rangle$ normal in G. As before, let $\pi : G \to G/H$ denote the quotient map and $a \in G$ an element such that $\pi(a)$ generates G/H. Then $aya^{-1} = x^i y$ for some $i \in \{0,1\}$ and $aza^{-1} = x^j y^k z$ for some $j, k \in \{0,1\}$. It follows that $a^4 \in G$ and a has order 2^{n-2} or 2^{n-3}. If $|a| = 2^{n-2}$, then $a^{2^{n-3}}$ is an element of order 2 in $Z(G)$. If $a^{2^{n-3}} = x$, take $H' = \langle a^{2^{n-4}}, y\rangle$; otherwise, take $H' = \langle a^{2^{n-4}}, x\rangle$. In either case, $H' \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ is well-positioned in G and G/H' is noncyclic. Finally, if $|a| = 2^{n-3}$, then $H' = \langle a^{2^{n-4}}, x, y\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is easily seen to be a well-positioned subgroup of G with G/H' not cyclic. □

2.4 Stitching

Let N be a normal subgroup of a group G and T a left transversal for N. We now introduce the notion of *stitching*, which allows us to prove some version of Proposition 2.9 when N is not contained in the center of G. The main idea is to fix a strong complete mapping β of G/N and then to choose appropriate orthomorphisms $\alpha_t : N \to N$, $t \in T$ such that the maps $n_g t_g \mapsto \alpha_{t_g}(n_g)\tilde{\beta}(t_g)$ are bijections. The key difference between this construction and that of Theorem 2.8 is that the choice of map on N here varies with g, rather than being uniform across all choices.

Definition 2.16. *Let N be a normal subgroup of a group G. An N-*stitching *for G is a pair $(T, \{\alpha_t\}_{t\in T})$ where T is a left transversal for G, and for $t \in T$, $\alpha_t : N \to N$ is an orthomorphism such that $n \mapsto nt\alpha_t(n)t^{-1} = \delta_t(n)$ is bijective.*

The next result explains the relevance of N-stitchings to questions about strong admissibility. It is equivalent to Theorem 8 in [12], although the statement presented here is more suited to our needs.

Lemma 2.17. *(Stitching Lemma) Let N be a normal subgroup of a finite group G. If G has an N-stitching and G/N is strongly admissible, then G is strongly admissible.*

Proof.
Let $(T, \{\alpha_t\}_{t\in T})$ be an N-stitching and $\beta : G/N \to G/N$ a strong complete mapping. From bijectivity of the maps β and α_t, $t \in T$, it follows that Γ is bijective. To show that Γ is an orthomorphism, it suffices to show that the map $g \mapsto g^{-1}\Gamma(g)$ is injective. To this end, suppose $g_1, g_2 \in G$ are such that $g_1^{-1}\Gamma(g_1) = g_2^{-1}\Gamma(g_2)$, i.e.

$$t_{g_1}^{-1} n_{g_1}^{-1} \alpha_{t_{g_1}}(n_{g_1})\tilde{\beta}(t_{g_1}) = t_{g_2}^{-1} n_{g_2}^{-1} \alpha_{t_{g_2}}(n_{g_2})\tilde{\beta}(t_{g_2}). \tag{1}$$

Reducing both sides of this equation modulo N yields $t_{g_1}^{-1} N\beta(t_{g_1}N) = t_{g_2}^{-1} N\beta(t_{g_2}N)$. Since β is an orthomorphism, we have $t_{g_1}N = t_{g_2}N$, so $t_{g_1} = t_{g_2}$ and thus (1) reduces to

$$n_{g_1}^{-1}\alpha_{t_{g_1}}(n_{g_1}) = n_{g_2}^{-1}\alpha_{t_{g_1}}(n_{g_2}). \tag{2}$$

Now because $\alpha_{t_{g_1}}$ is an orthomorphism, it follows that $n_{g_1} = n_{g_2}$.

Finally, we show that Γ is a strong complete mapping. Suppose $g_1, g_2 \in G$ are such that $g_1\Gamma(g_1) = g_2\Gamma(g_2)$, i.e.

$$n_{g_1} t_{g_1} \alpha_{t_{g_1}}(n_{g_1}) \tilde{\beta}(t_{g_1}) = n_{g_2} t_{g_2} \alpha_{t_{g_2}}(n_{g_2}) \tilde{\beta}(t_{g_2}). \tag{3}$$

As before, reduction of this equation modulo N yields $t_{g_1}N\beta(t_{g_1}N) = t_{g_2}N\beta(t_{g_2}N)$. Because β is a complete mapping, we therefore have $t_{g_1} = t_{g_2}$; cancelling on the right then yields:

$$n_{g_1} t_{g_1} \alpha_{t_{g_1}}(n_{g_1}) = n_{g_2} t_{g_1} \alpha_{t_{g_1}}(n_{g_2}). \tag{4}$$

Now multiplying both sides on the right by $t_{g_1}^{-1}$ yields $\delta_{t_{g_1}}(n_{g_1}) = \delta_{t_{g_1}}(n_{g_2})$, from which we conclude that $n_{g_1} = n_{g_2}$. $\square$

The balance of this section consists of applications of the Stitching Lemma which will be needed for the proof of our main result.

Theorem 2.18. *Suppose $\overline{G} \in \mathcal{C}_1$ is a 2-group of order at least 8 and $\pi : G \to \overline{G}$ a 2-covering. Then G is strongly admissible.*

Proof.
Suppose $|\overline{G}| = 2^r$. By Theorem 2.1, $\overline{G}$ is generated by elements $\overline{a}$ and $\overline{b}$ satisfying $\overline{a}^{2^{r-1}} = 1$ and $\overline{b}\overline{a}\overline{b}^{-1} = \overline{a}^{\varepsilon + \delta 2^{r-2}}$ where $\varepsilon \in \{-1, 1\}$ and $\delta \in \{0, 1\}$ (in addition to other relations). Furthermore, Ker π is a cyclic subgroup of G generated by some $z \in Z(G)$, $|z| = 2$. Choose $a, b \in G$ such that $\pi(a) = \overline{a}$ and $\pi(b) = \overline{b}$; then G is generated by a,b, and z. Since $G \in \mathcal{C}_2$, $|a| = |\overline{a}| = 2^{r-1}$, and so $z \notin \langle a \rangle$. Furthermore, because $N' = \pi^{-1}(\langle \overline{a} \rangle) = \langle a, z \rangle$ is normal in G, it follows that $bab^{-1} = z^i a^j$ for some $i \in \{0, 1\}$ and odd integer j, $1 \leq j \leq 2^{r-1} - 1$. This implies $ba^2b^{-1} = a^{2j}$, and because $\overline{b}\overline{a}^2\overline{b}^{-1} = \overline{a}^{\pm 2}$, we must have $j = \pm 2$. It follows that $N = \langle a^2, z \rangle \cong \mathbb{Z}_{2^{r-2}} \times \mathbb{Z}_2$ is normal in G and that $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider the transversal $T = \{1, a, b, ba\}$ for N in G. We will construct an N-stitching $(N, \{\theta_t\}_{t \in T})$, taking all the maps θ_t to be the (same) map θ defined below.

Let $S = \{1, \ldots, 2^{r-3}\}$ and define $\theta : N \to N$ by

$$\theta(a^{2i}) = \begin{cases} a^{4i-2} & i \in S \\ a^{4i}z & i \notin S \end{cases}$$

$$\theta(a^{2i}z) = \begin{cases} a^{4i} & i \in S \\ a^{4i-2}z & i \notin S \end{cases}$$

It is easy to check that θ is an orthomorphism of N, and clearly conjugation by a acts trivially on N. If $ba^2b^{-1} = a^2$, then the map $\delta_b : N \to N$ defined by $\delta_b(n) = nb\theta(n)b^{-1}$ is simply $n \mapsto n\theta(n)$, which is easily seen to be bijective.

If $ba^2b^{-1} = a^{-2}$, then

$$\delta_b(a^{2i}) = \begin{cases} a^{-2i+2} & i \in S \\ a^{-2i}z & i \notin S \end{cases}$$

$$\delta_b(a^{2i}z) = \begin{cases} a^{-2i}z & i \in S \\ a^{-2i+2} & i \notin S \end{cases}$$

which is still bijective. Since conjugation by a acts trivially on N, $\delta_{ba} = \delta_b$, and we conclude that $(T, \{\theta\}_{t\in T})$ is an N-stitching. Finally, by Theorem 2.6, $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is strongly admissible; thus, by Lemma 2.17, G is strongly admissible. □

Remark. The map θ is the same one used by Evans [9, Lemma 1] to prove strong admissibility of $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$. This map was itself based on a construction of Carlitz [8].

The last three results of this section use a related, but slightly different version of stitching, in which N is a normal subgroup of a 2-group G, but no transversal for N in G is given. Since conjugation by $g \in G$ induces an automorphism on N whose order is a power of 2, this forces us to consider all automorphisms of N of such order. We then have the following analogue of Lemma 2.17.

Lemma 2.19. *Let N be a normal subgroup of a 2-group G. Suppose that for each $\varphi \in Aut(N)$ of 2-power order, there exists an orthomorphism α_φ of N such that the map $n \mapsto n\varphi(\alpha_\varphi(n)) = \delta_\varphi(n)$ is bijective. If G/N is strongly admissible, then G is strongly admissible.*

Proposition 2.20. *($\mathbb{Z}_4 \times \mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-stitching) Let G be a 2-group and N a well-positioned subgroup of G. If G/N is strongly admissible, then so is G.*

Proof.

First suppose N is of Type I. Identify N with $\mathbb{Z}_4 \times \mathbb{Z}_2$ by identifying the generators x and y with $(1, 0)$ and $(0, 1)$, respectively. For convenience of presentation, we write xy in place of (x, y). The automorphisms of N, defined on the generators of N are listed in Table 1. Table 2 shows how to define the maps α_{φ_i}, $1 \le i \le 8$. A routine verification shows that these maps are orthomorphisms, and that the associated maps δ_{φ_i} of Lemma 2.19 are bijective.

If N is of Type II, identify N with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by identifying the generators x, y, and z with, respectively, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Again, convenience of presentation, we write xyz in place of (x, y, z). As before, the automorphisms of N are listed in Table 3. Table 4 shows how to define the maps α_{ψ_i}, $1 \le i \le 8$. A routine verification shows that these maps are indeed orthomorphisms, and that the associated maps δ_{ψ_i} of Lemma 2.19 are bijective. □

Table 1: Automorphisms of $\mathbb{Z}_4 \times \mathbb{Z}_2$ induced by conjugation

Automorphism	$\varphi(10)$	$\varphi(01)$
φ_1	10	01
φ_2	30	01
φ_3	11	01
φ_4	31	01
φ_5	10	21
φ_6	30	21
φ_7	11	21
φ_8	31	21

Table 2: Stitching data for $N = \mathbb{Z}_4 \times \mathbb{Z}_2$

i	$\varphi_i(00)$	$\varphi_i(10)$	$\varphi_i(20)$	$\varphi_i(30)$	$\varphi_i(01)$	$\varphi_i(11)$	$\varphi_i(21)$	$\varphi_i(31)$
1	00	20	11	31	10	30	01	21
2,5,7	00	20	01	21	10	31	11	30
3	00	20	01	21	31	10	30	11
4,8	00	20	21	01	30	31	11	10
6	00	30	01	31	10	21	11	20

Table 3: Automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ induced by conjugation

Automorphism	$\psi(010)$	$\psi(001)$
ψ_1	010	001
ψ_2	010	011
ψ_3	010	101
ψ_4	010	111
ψ_5	110	001
ψ_6	110	011
ψ_7	110	101
ψ_8	110	111

Table 4: Stitching data for $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

i	$\psi_i(000)$	$\psi_i(001)$	$\psi_i(010)$	$\psi_i(011)$	$\psi_i(100)$	$\psi_i(101)$	$\psi_i(110)$	$\psi_i(111)$
1,4,8	000	010	100	110	011	001	111	101
2,7	000	010	101	111	001	011	100	110
3	000	010	110	100	001	011	111	101
5	000	010	111	101	011	001	100	110
6	000	010	100	110	101	111	001	011

Proposition 2.21. *(Stitching for groups of order* 16*) Let G be a* 2*-group containing a normal subgroup N of order* 16 *that is neither cyclic nor isomorphic to Q_{16}. If G/N is strongly admissible, then so is G.*

Proof.
Table 5 lists the isomorphism classes of groups of order 16. We construct stitchings on all of these except for the first (the cyclic group $\mathbb{Z}_{16}$) and the ninth (the generalized quaternion group Q_{16}). When $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we may select generators in an appropriate manner to ensure that $(1, 0, 0, 0) \in Z(G)$, and that

$$N \supseteq 1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \supseteq 1 \times 1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \supseteq 1 \times 1 \times 1 \times \mathbb{Z}_2 \supseteq 1$$

is a filtration of N by subgroups invariant under the conjugation action of G. This reduces the number of automorphisms to be considered from 20160 to 64. The presentations of the various subgroups N by generators and relations are given in Table 5, and encodings of their elements by integers a, $0 \leq a \leq 15$ are given in Table 6. The relevant automorphisms φ_i and the maps α_{φ_i} found by `Python` and used to define the stitchings, are listed in the (very long) table following the appendix. $\square$

3 Results

Let $\mathcal{A}$ be the class consisting of all 2-groups in $\mathcal{C}_1$, all noncyclic abelian 2-groups, and all noncyclic groups of order 16.

Definition 3.1. *A* coherent resolution *of a* 2*-group G is a sequence:*

$$\mathcal{R} : G = G_0 \xrightarrow{\pi_0} G_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{r-1}} G_r$$

in which $r \geq 0$; $G_0, \ldots, G_r$ are noncyclic 2*-groups; and the π_i, $0 \leq i \leq r-1$, are surjective homomorphisms such that:*

1. *For $0 \leq i \leq r-1$, Ker π_i is a well-positioned subgroup of G_i*

2. $G_r \in \mathcal{A}$

3. $G_{r-1} \notin \mathcal{A}$.

We refer to r as the length *of $\mathcal{R}$.*

Remark.
It follows easily that $G_i \notin \mathcal{A}$ for i, $0 \leq i \leq r-2$. Note also that $|G_r| \geq 4$, and that if $|G_r| = 4$, then $G_r \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 3.2. *Every noncyclic 2-group has a coherent resolution.*

Proof.
Every noncyclic group of order 4 or 8 is either abelian or in $\mathcal{C}_1$. Thus, if a noncyclic group G is either abelian or of order at most 16, then it has a coherent resolution of length 0.

Assume henceforth that G is neither abelian nor in $\mathcal{C}_1$, and has order 2^n, $n \geq 5$. We will prove the assertion by induction on n. By Corollary 2.13, G has a well-positioned subgroup H of order 8, and by Lemma 2.15, we may assume that G/H is not cyclic. If $n = 5$, then G/H has order 4, and as such must be isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $n = 6$, then G/H has order 8, and as such is either abelian or in $\mathcal{C}_1$. If $n = 7$, then G/H is a noncyclic group of order 16. In all three cases, the quotient map $G \to G/H$ constitutes a coherent resolution of length 1. If $n \geq 8$, then by induction G/H has a coherent resolution $\mathcal{R}'$. Splicing the quotient map $G \to G/H$ with $\mathcal{R}'$, we obtain a coherent resolution of G. $\square$

For a noncyclic 2-group G, we define $\rho(G)$ to be the length of the longest coherent resolution of G.

Definition 3.3. *Suppose*

$$\mathcal{R} : G_0 \xrightarrow{\pi_0} G_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{r-1}} G_r$$

is a coherent resolution of length $r \geq 1$ of some 2-group G. For i, $0 \leq i \leq r-1$, denote by

$$G_i \overset{\pi_i^{(0)}}{\to} G_{i,1} \overset{\pi_i^{(1)}}{\to} G_{i,2} \overset{\pi_i^{(2)}}{\to} G_{i+1}$$

the canonical refinement (see Definition 2.14) of the map π_i.
For $0 \leq i \leq r$, define $L_{3i} = G_i$; for $0 \leq i \leq r-1$, define $L_{3i+1} = G_{i,1}$, $L_{3i+2} = G_{i,2}$, and $\sigma_{3i+j} = \pi_i^{(j)}$ for $j = 0, 1, 2$.
If L_{3r} is abelian or of order 16, let $d(\mathcal{R}) = 3$. In all other cases, let

$$d(\mathcal{R}) = \begin{cases} 1 & \text{if } L_{3r-2} \in \mathcal{C}_1 \\ 2 & \text{if } L_{3r-2} \notin \mathcal{C}_1 \text{ and } L_{3r-1} \in \mathcal{C}_1 \\ 3 & \text{otherwise.} \end{cases}$$

The truncated expansion *of $\mathcal{R}$ is the sequence:*

$$L_0 \overset{\sigma_0}{\to} L_1 \overset{\sigma_1}{\to} \cdots \overset{\sigma_{3(r-1)+d(\mathcal{R})-1}}{\to} L_{3(r-1)+d(\mathcal{R})}.$$

For $0 \leq i < j \leq 3r-d$, let $K_{i,j}$ denote the kernel of the composition $\kappa_{i,j} = \sigma_{j-1} \circ \ldots \circ \sigma_i : L_i \to L_j$. Note that from the definition of canonical refinement,

$$K_{3i,3i+2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ and } K_{3i+1,3i+3} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{5}$$

for $0 \leq i \leq r-2$ and also for $i = r-1$ when these are defined.

Remark.

The parameter $d(\mathcal{R})$ is defined so as to arrange that $L_{3r-1+d(\mathcal{R})} \in \mathcal{A}$. If, moreover, $L_{3(r-1)+d(\mathcal{R})} \in \mathcal{C}_1$, then $L_i \notin \mathcal{C}_1$ for all $i < 3(r-1)+d(\mathcal{R})$; in other words, $L_{3(r-1)+d(\mathcal{R})}$ is the first group (starting from the left) from $\mathcal{C}_1$ that appears in the refinement of $\mathcal{R}$.

Theorem 3.4. *Every 2-group with $c(G) \geq 3$ is strongly admissible.*

Proof.

Let G be a group of order 2^n with $c(G) \geq 3$. The only group of order 16 satisfying this condition is $\mathbb{Z}_2^4$, which is strongly admissible by Theorem 2.6; thus, we may assume $|G| \geq 32$. It suffices to show that any of the G_i, $0 \leq i \leq r-1$, is strongly admissible.

Fix a coherent resolution

$$\mathcal{R} : G_0 \overset{\pi_0}{\to} G_1 \overset{\pi_1}{\to} G_2 \overset{\pi_2}{\to} \cdots \overset{\pi_{r-1}}{\to} G_r.$$

for G. Because $c(G) \geq 3$, $G \notin \mathcal{C}_1$. If $r = 0$, this forces G to be a noncyclic abelian group, which is strongly admissible by Theorem 2.6. Now suppose $r \geq 1$ and let

$$\mathcal{R}' : G = L_0 \overset{\sigma_0}{\to} L_1 \overset{\sigma_1}{\to} \cdots \overset{\sigma_{3r-4}}{\to} L_{3(r-1)} \overset{\sigma_{3r-3}}{\to} \cdots \to L_{3(r-1)+d-1} \overset{\sigma_{3(r-1)+d-1}}{\longrightarrow} L_{3(r-1)+d},$$

where $1 \leq d = d(\mathcal{R}) \leq 3$, be the truncated expansion of $\mathcal{R}$. If G_r is either abelian or noncyclic of order 16, then $d = 3$ and $L_{3(r-1)+3} = G_r$ is strongly admissible by Theorem 2.6 or Theorem 2.7. We are thus reduced to the case that $G_r \in \mathcal{C}_1$, $|G_r| \neq 16$. The construction then implies that $L_{3(r-1)+d}$ is nonabelian and in $\mathcal{C}_1$.

If $|G| = 32$, then $|G_1| = 4$; since G_1 is noncyclic, we must have $G_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is strongly admissible. It follows from Proposition 2.20 that G is strongly admissible. We assume henceforth $|G| \geq 64$.

If $d = 1$, then since $L_{3r-2} \in \mathcal{C}_1$ but $L_{3r-3} \notin \mathcal{C}_1$, it follows that $\sigma_{3(r-1)} : L_{3r-3} \to L_{3r-2}$ is a 2-covering. By Theorem 2.18, $L_{3r-3} = G_{r-1}$ is strongly admissible.

If $d = 2$, then $\sigma_{3r-2} : L_{3r-2} \to L_{3r-1}$ is a 2-covering. Again, by Theorem 2.18, L_{3r-2} is strongly admissible. If $c(G) = c(L_0) \geq 4$, then $c(L_1) \geq 3$ and hence $r \geq 2$. Now let K be the kernel of the composition $\tau = \sigma_{3r-3} \circ \pi_{r-2} : L_{3r-6} \to L_{3r-2}$. Then $K_{3r-6,3r-2}$ is a group of order 16 containing the subgroup $K_{3r-6,3r-4} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so $K_{3r-6,3r-2}$ cannot be isomorphic either to $\mathbb{Z}_{16}$ or to Q_{16}. By Proposition 2.21, $L_{3r-6} = G_{r-2}$ is strongly admissible.
The above argument also applies if $c(G) = 3$ and $r = 2$, so assume henceforth that $c(G) = 3$ and $r = 1$. Let

$$G = L_0 \overset{\pi^{(0)}}{\to} L_1 \overset{\pi^{(1)}}{\to} L_2 \overset{\pi^{(2)}}{\to} L_3 = G_1.$$

denote the canonical refinement of the morphism $\pi_0 : G_0 \to G_1$. Since $d = 2$, L_2 is either abelian, of order 16, or in $\mathcal{C}_1$. If L_2 is abelian, then $L_3 = G_1$ is abelian and noncyclic, hence strongly admissible. If $|L_2| = 16$, then $|G_1| = 8$; as such, there exists $z \in Z(G_1)$ such that $\widetilde{G_1} = G_1/\langle z \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the kernel of the composition $G \to G_1 \to \widetilde{G_1}$ is of order 16 and (as it contains Ker π_0 as a subgroup) cannot be isomorphic either to $\mathbb{Z}_{16}$ or to Q_{16}. Thus, we conclude by Proposition 2.21 that G is strongly admissible. We are thereby reduced to the case $|G| \geq 128$.
Last, suppose L_2 is nonabelian and in $\mathcal{C}_1$. Then L_2 is generated by elements $\overline{a}, \overline{b}$ subject to relations $\overline{a}^{2^{n-3}} = 1$, $\overline{b}^2 = 1$, and $\overline{b}\overline{a}\overline{b}^{-1} = \overline{a}^{\delta + \varepsilon 2^{n-4}}$, where $\delta \in \{1, -1\}$ and $\varepsilon \in \{0, 1\}$. Pick elements $a, b \in G$ such that $(\pi^{(1)} \circ \pi^{(0)})(a) = \overline{a}$ and $(\pi^{(1)} \circ \pi^{(0)})(b) = \overline{b}$; let $K = K_{0,2}$. Because $c(G) = 3$ and $|\overline{a}| = 2^{n-3}$, we must also have $|a| = 2^{n-3}$. Moreover, the map $\pi_2 : L_2 \to L_3$ is the quotient of the nonabelian group $L_2 \in \mathcal{C}_1$ by its unique central element of order 2, so Ker $\pi_2 = \{1, \overline{a}^{2^{n-4}}\}$. Thus, $a^{2^{n-4}}$ is an element of G of order 2 that lies in $N =$ Ker π_0 but not in K. Because $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, this precludes the case $N \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, so we must have $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $N = \langle x, y, a^{2^{n-4}} \rangle$, where (according to the definition of canonical refinement), $x \in Z(G)$ and $K = \langle x, y \rangle$ is a normal subgroup of G. From normality of K in G, one easily sees that a^4 commutes with x and y. We also have the relation $bab^{-1} = a^{\delta + \varepsilon 2^{n-4}} x^i y^j$ for some $i, j \in \{0, 1\}$; direct computation then shows that $ba^4b^{-1} = a^{4\delta}$. In particular, our assumption $|G| \geq 128$, i.e. $n \geq 7$, implies $a^{2^{n-4}} \in Z(G)$. Now consider the subgroup $K' = \langle x, a^{2^{n-4}} \rangle$ of N. Then K', being a subgroup of $Z(G)$, is normal in G, and we may consider another decomposition of the map π by successive quotients:

$$G \overset{\alpha}{\to} G/K' \overset{\beta}{\to} G/N = G_1.$$

Now $\alpha(a)$ is an element of order 2^{n-4} in G/K', but G/K' contains no elements of order 2^{n-3}. It follows that $\beta : G/K' \to G_1$ is a 2-covering, so G/K' is strongly admissible by Theorem 2.18. Because $K' \leq Z(G)$ and $K' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, strong admissibility of G follows from Proposition 2.9.

The final case to consider is $d = 3$. Recall that $G_r \in \mathcal{C}_1$ and $|G_r| \neq 16$. If $|G_r| = 8$, then

G_r is isomorphic to either D_8 or Q_8; in either case, $\widetilde{G_r} = G_r/Z(G_r) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is strongly admissible. The kernel of the composite map $G_{r-1} \overset{\pi_{r-1}}{\to} G_r \to \widetilde{G_r}$ has order 16 and (since it contains Ker π_{r-1} as a subgroup) is neither cyclic nor isomorphic to Q_{16}. We may therefore apply Proposition 2.21 to conclude that G_{r-1} is strongly admissible.

If $|G_r| = |L_{3r}| \geq 32$, then $|L_{3r-1}| \geq 64$. In that case, $\sigma_{3r-1} : L_{3r-1} \to L_{3r}$ is a 2-covering, so L_{3r-1} is strongly admissible. Applying the "hat" construction of Lemma 2.3 twice, we obtain a commutative diagram of maps:

$$\begin{tikzcd}
L_{3r-3} \arrow[r, "\sigma_{3r-3}"] & L_{3r-2} \arrow[r, "\sigma_{3r-2}"] & L_{3r-1} \arrow[r, "\sigma_{3r-1}"] \arrow[d, "q_{L_{3r-1}}"] & L_{3r} \arrow[d, "q_{L_{3r}}"] \\
 & & \widehat{L}_{3r-1} \arrow[r, "\widehat{\sigma}_{3r-1}"] \arrow[d, "q_{\widehat{L}_{3r-1}}"] & \widehat{L}_{3r} \arrow[d, "q_{\widehat{L}_{3r}}"] \\
 & & \widehat{\widehat{L}}_{3r-1} \arrow[r, "\widehat{\widehat{\sigma}}_{3r-1}"] & \widehat{\widehat{L}}_{3r}
\end{tikzcd}$$

Define $\psi = q_{\widehat{L}_{3r-1}} \circ q_{L_{3r-1}} \circ \sigma_{3r-2} \circ \sigma_{3r-3} : L_{3r-3} \to \widehat{\widehat{L}}_{3r-1}$. Then ψ is surjective and $K =$ Ker ψ is an order 16 subgroup of L_{3r-3} that contains the subgroup $K_{3r-3,3r-1} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. As before, K is isomorphic neither to $\mathbb{Z}_{16}$ nor to Q_{16}. Then $\widehat{\widehat{\sigma}}_{3r-1} : \widehat{\widehat{L}}_{3r-1} \to \widehat{\widehat{L}}_{3r}$ is a 2-covering with $\widehat{\widehat{L}}_{3r} \in \mathcal{C}_1$; therefore, $\widehat{\widehat{L}}_{3r-1}$ is strongly admissible. It ensues that $L_{3r-3} = G_{r-1}$ is strongly admissible by Proposition 2.21. $\square$

Remark. The presence of a large cyclic subgroup renders an inductive strategy impossible to execute for groups in $\mathcal{C}_1$. Explicit constructions of strong complete mappings for such groups have thus far proved elusive. The omission of groups in $\mathcal{C}_2$ from Theorem 3.4 is more perplexing. Indeed, if $\pi : G \to \overline{G}$ is a 2-covering with $\overline{G} \in \mathcal{C}_1$ then $G \in \mathcal{C}_2$ is strongly admissible by Theorem 2.18. In that case, we were able to construct a strong complete mapping explicitly because G has a normal subgroup with respect to which the conjugation action of G is relatively uncomplicated. This does not seem to be the case for all groups in $\mathcal{C}_2$, the classification of which may be found in [5, Theorem 74.2].

Our methods are nonetheless sufficient to prove the following:

Theorem 3.5. *Suppose G_1 and G_2 are nontrivial 2-groups. Then $G_1 \times G_2$ is strongly admissible.*

Proof.
Let $|G_1| = 2^r$ and $|G_2| = 2^s$, with $r \geq s \geq 1$. We will prove the theorem by simultaneous induction on s and r.

Suppose first $s = 1$, i.e. $G_2 \cong \mathbb{Z}_2$. If G_1 is abelian, the result follows from Theorem 2.6, so we may assume that G_1 is a nonabelian group of order at least 8. If $G_1 \in \mathcal{C}_1$, then

$G = G_1 \times G_2$ is admissible by Theorem 2.18. If $G_1 \notin \mathcal{C}_1$, then $r \geq 4$ and by Corollary 2.13, G_1 contains a well-positioned subgroup K. Then $N = K \times 1$ is a normal subgroup of G and by induction, $(G/K) \times \mathbb{Z}_2$ is strongly admissible. It follows from Proposition 2.20 that G is strongly admissible.

Now suppose $s \geq 2$. Then there exist $x \in Z(G_1)$ and $y \in Z(G_2)$, with $|x| = |y| = 2$, so $H = \langle (x, 1), (1, y) \rangle$ is a subgroup of $Z(G)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and as such is strongly admissible. By induction, G/H is strongly admissible, so G is strongly admissible by Proposition 2.9. □

Appendix: Groups of order 16

The classification of groups of order 16 (up to isomorphism) is summarized in Table 5, along with descriptions of each group in terms of generators and relations. For ease of reference, the numbering of the isomorphism classes follows [13], borrowing the group's identification number from that used the `GAP` software. In the present context, we will not need to work with class 1 (the cyclic group of order 16) or class 9 (the generalized quaternion group of order 16).

We say that a group G has *type* $(k_1, \ldots, k_n)$ if $|G| = \prod_{i=1}^{n} k_i$ and there are generators $a_1, \ldots, a_n \in G$ such that every element of G is uniquely expressible as $a_1^{i_1} \cdots a_n^{i_n}$, where $0 \leq i_j \leq k_j$ for each j, $1 \leq j \leq n$.
For ease of reference, we fix a labeling of elements in a group of order 16 by elements in $\{0, \ldots, 15\}$ according to its type, as described in the following table. We abbreviate the ordered pair (a, b) by ab, (a, b, c) by abc, etc.

Table 5: Isomorphism classes of groups of order 16

Name	GAP ID	Type	Gen / Rel	$\|Aut(G)\|$
$\mathbb{Z}_{16}$	1	(16)	$a^{16} = 1$	8
$\mathbb{Z}_4 \times \mathbb{Z}_4$	2	$(4, 4)$	$a^4 = b^4 = 1,\ ab = ba$	96
SmallGroup(16,3)	3	$(4, 2, 2)$	$a^4 = b^2 = c^2 = 1,$ $ab = ba, bc = cb, cac^{-1} = ab$	32
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ (nontrivial)	4	$(4, 4)$	$a^4 = b^4 = 1,\ bab^{-1} = a^3$	32
$\mathbb{Z}_8 \times \mathbb{Z}_2$	5	$(8, 2)$	$a^8 = b^2 = 1,\ ab = ba$	16
M_{16}	6	$(8, 2)$	$a^8 = b^2 = 1,\ bab^{-1} = a^5$	16
D_{16}	7	$(8, 2)$	$a^8 = b^2 = 1,\ bab^{-1} = a^{-1}$	32
S_{16}	8	$(8, 2)$	$a^8 = b^2 = 1,\ bab^{-1} = a^3$	16
Q_{16}	9	$(8, 2)$	$a^4 = b^2 = abab$	32
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	10	$(4, 2, 2)$	$a^4 = b^2 = c^2 = 1,$ $ab = ba, ac = ca, bc = cb$	192
$D_8 \times \mathbb{Z}_2$	11	$(4, 2, 2)$	$a^4 = b^2 = c^2 = 1$ $bab^{-1} = a^{-1}, ac = ca, bc = cb$	64
$Q_8 \times \mathbb{Z}_2$	12	$(4, 2, 2)$	$a^2 = b^2 = abab$ $c^2 = 1, ac = ca, bc = cb$	192
SmallGroup(16,13)	13	$(4, 2, 2)$	$a^4 = b^2 = 1,\ a^2 = c^2$ $bab^{-1} = a^{-1},\ ac = ca,\ bc = cb$	48
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	14	$(2, 2, 2, 2)$	$a^2 = b^2 = c^2 = d^2 = 1, ab = ba, ac = ca,$ $bc = ca, ad = da, bd = db, cd = dc$	20160

Table 6: Dictionary of labelings

Type	0	1	2	3	4	5	6	7
$(4, 4)$	00	01	02	03	10	11	12	13
$(8, 2)$	00	10	20	30	40	50	60	70
$(4, 2, 2)$	000	001	010	011	100	101	110	111
$(2, 2, 2, 2)$	0000	0001	0010	0011	0100	0101	0110	0111

Type	8	9	10	11	12	13	14	15
$(4, 4)$	20	21	22	23	30	31	32	33
$(8, 2)$	01	11	21	31	41	51	61	71
$(4, 2, 2)$	200	201	210	211	300	301	310	311
$(2, 2, 2, 2)$	1000	1001	1010	1011	1100	1101	1110	1111

Given a particular type $(k_1, \ldots, k_n)$, we fix a bijection between the elements of any group of that type (represented by tuples $(i_1, \ldots, i_n)$, as above) and the set $\{0, 1, \ldots, 15\}$; these bijections are outlined in Table 6.

Since we will be considering N-stitchings of 2-groups G when N is a normal subgroup of order 16, we will need to describe those automorphisms of N that could be induced by conjugation by an element of G. As G is a 2-group, we may restrict attention to those automorphisms of N whose order is a power of 2. When $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (`GAP` ID 14), we know that N contains a sequence of subgroups $N \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \{1\}$, each invariant under the conjugation action of G on itself. By selecting generators appropriately, we may assume that $N_3 = 1 \times 1 \times 1 \times \mathbb{Z}_2$, $N_2 = 1 \times 1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $N_1 = 1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This allows us to describe an automorphism as a matrix over $\mathbb{Z}_2$ of the following form:

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

As an automorphism of N is determined by its action on the generators $a_1, \ldots, a_n$ of G, we may describe an automorphism φ by an n-tuple $(t^{(1)} \ldots, t^{(n)})$, in which $t^{(i)} = (r_{i1}, \ldots, r_{in})$, where $\varphi(a_i) = a_1^{r_{i1}} \cdots a_n^{r_{in}}$. For convenience of presentation in tables, we omit internal parentheses and represent φ by an n^2-tuple. In the case of $\mathbb{Z}_2^4$, we represent the automorphism associated to the matrix (6) by the 6-tuple (a, b, c, d, e, f).

The following (very long) table lists all groups N of order 16 considered in Proposition 2.21, together with their relevant automorphisms. For each automorphism φ of N, an orthomorphism $\alpha_{N,\varphi} : N \to N$ such that $n \mapsto n\varphi(\alpha_{N,\varphi}(n))$ satisfies the hypotheses of Proposition 2.19, is given. All maps send 0 to itself, so that datum is omitted from the table. The maps were found by a `Python` program that randomly generated a permutation σ of $\{1, \ldots, 15\}$ and then checked to see if σ could be used to define any of the maps $\alpha_{N,\varphi}$, stopping when all such maps were found.

ID	Automorphism	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	(0, 1, 1, 0)	11	1	14	5	7	9	3	12	2	15	13	4	10	8	6
2	(2, 1, 1, 0)	11	7	12	8	3	15	4	10	1	13	6	2	9	5	14
2	(0, 3, 1, 0)	10	9	4	12	6	1	3	14	11	13	5	15	7	2	8
2	(2, 3, 1, 0)	14	9	1	7	11	2	10	12	3	15	6	13	5	8	4
2	(0, 1, 3, 0)	4	7	1	3	14	2	15	9	13	12	10	6	11	5	8
2	(2, 1, 3, 0)	10	6	5	3	2	14	4	15	11	1	12	8	7	13	9
2	(0, 3, 3, 0)	6	3	13	15	2	8	5	11	12	14	1	4	9	7	10

2	(2, 3, 3, 0)	15	9	1	8	6	5	3	13	2	12	14	11	10	4	7
2	(1, 0, 0, 1)	8	1	9	11	3	4	15	5	13	6	12	2	14	7	10
2	(1, 0, 0, 3)	4	10	9	13	2	5	14	7	11	6	15	1	3	8	12
2	(1, 0, 3, 1)	8	3	4	13	2	14	6	15	11	12	7	10	1	9	5
2	(1, 0, 1, 3)	5	8	14	11	7	2	9	1	6	15	3	10	4	13	12
2	(1, 0, 2, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
2	(1, 0, 2, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
2	(1, 0, 1, 1)	8	3	4	13	2	14	6	15	11	12	7	10	1	9	5
2	(1, 0, 3, 3)	15	8	14	7	12	10	4	13	6	3	9	2	5	1	11
2	(1, 3, 0, 1)	12	9	15	2	7	5	11	14	3	13	8	1	10	6	4
2	(1, 1, 0, 3)	8	1	7	13	6	11	15	5	3	4	14	2	9	12	10
2	(3, 1, 2, 3)	8	10	2	6	11	12	1	13	5	7	15	3	14	9	4
2	(3, 3, 2, 1)	8	11	5	3	9	4	1	13	12	2	7	6	10	15	14
2	(1, 2, 0, 1)	9	3	10	8	2	15	6	14	4	13	5	1	7	12	11
2	(1, 2, 0, 3)	12	11	4	10	2	9	3	6	8	14	1	7	5	15	13
2	(3, 2, 1, 3)	9	13	6	5	1	8	14	10	3	7	12	15	11	2	4
2	(3, 2, 3, 1)	8	3	4	13	2	14	6	15	11	12	7	10	1	9	5
2	(1, 2, 2, 1)	2	13	8	1	10	4	11	14	12	6	3	15	7	5	9
2	(1, 2, 2, 3)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
2	(3, 2, 3, 3)	14	4	9	6	12	1	10	13	5	3	8	15	11	2	7
2	(3, 2, 1, 1)	3	13	5	12	8	15	14	11	10	6	4	1	7	2	9
2	(1, 1, 0, 1)	4	9	15	14	6	1	11	5	8	12	3	10	2	7	13
2	(1, 3, 0, 3)	14	4	2	8	10	7	13	15	5	1	3	11	6	12	9
2	(3, 3, 2, 3)	3	9	8	1	10	2	11	7	15	4	14	13	12	6	5
2	(3, 1, 2, 1)	8	10	2	6	11	12	1	13	5	7	15	3	14	9	4
2	(0, 1, 1, 2)	3	6	8	1	15	5	14	7	12	11	13	4	2	10	9
2	(2, 1, 1, 2)	11	12	7	10	13	9	3	5	8	1	6	14	2	15	4
2	(0, 3, 1, 2)	10	6	5	3	2	14	4	15	11	1	12	8	7	13	9
2	(2, 3, 1, 2)	4	7	1	3	14	2	15	9	13	12	10	6	11	5	8
2	(0, 1, 3, 2)	2	4	15	11	13	5	14	12	3	7	9	1	6	8	10
2	(2, 1, 3, 2)	8	7	9	5	13	2	1	11	15	14	6	3	10	12	4
2	(0, 3, 3, 2)	11	13	4	6	8	3	15	14	7	9	2	5	1	10	12
2	(2, 3, 3, 2)	5	12	6	14	2	8	4	13	1	3	10	7	15	9	11
2	(3, 0, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
2	(3, 0, 0, 3)	10	1	5	3	15	14	11	13	6	4	8	7	9	12	2
2	(3, 0, 1, 1)	13	3	4	12	15	9	14	6	11	5	10	2	1	7	8
2	(3, 0, 3, 3)	10	1	11	3	9	7	14	13	6	12	5	8	15	4	2
2	(3, 0, 2, 1)	9	8	2	10	7	13	12	4	6	5	14	1	11	15	3
2	(3, 0, 2, 3)	14	12	10	13	4	7	8	15	3	5	9	2	1	6	11
2	(3, 0, 3, 1)	13	3	4	12	15	9	14	6	11	5	10	2	1	7	8
2	(3, 0, 1, 3)	12	4	1	14	2	13	11	9	8	15	5	3	6	10	7
2	(3, 1, 0, 1)	14	8	2	5	11	4	12	13	7	1	15	3	9	6	10
2	(3, 3, 0, 3)	11	9	2	12	1	8	4	15	13	7	5	14	6	3	10

2	(1, 3, 2, 3)	14	7	11	2	6	15	9	4	13	1	10	3	8	12	5
2	(1, 1, 2, 1)	8	11	1	5	4	12	15	7	14	6	13	3	10	2	9
2	(3, 2, 0, 1)	15	9	14	1	11	7	10	12	5	8	3	6	2	13	4
2	(3, 2, 0, 3)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
2	(1, 2, 3, 3)	7	8	12	5	14	10	1	4	11	13	3	15	2	9	6
2	(1, 2, 1, 1)	8	3	4	13	2	14	6	15	11	12	7	10	1	9	5
2	(3, 2, 2, 1)	10	6	5	3	2	14	4	15	11	1	12	8	7	13	9
2	(3, 2, 2, 3)	2	7	5	3	14	10	15	4	12	1	9	6	11	13	8
2	(1, 2, 1, 3)	8	3	5	12	15	10	1	4	6	13	9	11	2	7	14
2	(1, 2, 3, 1)	7	15	9	3	13	4	11	1	8	6	14	10	2	5	12
2	(3, 3, 0, 1)	14	12	2	15	13	8	10	9	11	5	7	6	1	3	4
2	(3, 1, 0, 3)	3	7	6	12	15	2	1	14	13	11	10	9	8	5	4
2	(1, 1, 2, 3)	14	7	11	2	6	15	9	4	13	1	10	3	8	12	5
2	(1, 3, 2, 1)	8	10	2	6	11	12	1	13	5	7	15	3	14	9	4
3	(1, 0, 0, 0, 1, 0, 0, 0, 1)	14	5	8	6	1	15	9	7	10	2	12	13	11	4	3
3	(1, 0, 0, 0, 1, 0, 0, 1, 1)	7	11	15	10	3	1	8	9	12	2	4	14	6	13	5
3	(1, 0, 0, 0, 1, 0, 2, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
3	(1, 0, 0, 0, 1, 0, 2, 1, 1)	15	11	5	3	6	12	8	9	13	2	7	14	10	1	4
3	(1, 0, 1, 0, 1, 0, 0, 0, 1)	6	8	7	12	14	3	9	4	10	5	2	13	15	1	11
3	(1, 1, 1, 0, 1, 0, 0, 1, 1)	4	9	15	10	7	1	14	2	6	11	13	8	5	3	12
3	(3, 1, 1, 0, 1, 0, 2, 1, 1)	4	12	1	8	9	5	13	3	7	15	2	11	10	6	14
3	(3, 0, 1, 0, 1, 0, 2, 0, 1)	13	3	6	7	11	8	14	4	2	15	1	9	5	12	10
3	(1, 1, 0, 0, 1, 0, 0, 0, 1)	11	15	6	2	14	5	9	7	1	8	12	13	4	10	3
3	(1, 1, 0, 0, 1, 0, 0, 1, 1)	6	1	14	5	13	8	9	2	12	3	4	7	15	10	11
3	(1, 1, 0, 0, 1, 0, 2, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
3	(1, 1, 0, 0, 1, 0, 2, 1, 1)	6	11	7	2	13	3	4	10	15	1	14	8	12	9	5
3	(1, 1, 1, 0, 1, 0, 0, 0, 1)	9	11	7	15	3	1	10	5	8	4	12	6	14	2	13
3	(1, 0, 1, 0, 1, 0, 0, 1, 1)	10	8	2	6	13	3	9	14	7	5	12	15	4	1	11
3	(3, 0, 1, 0, 1, 0, 2, 1, 1)	4	9	15	10	7	1	14	2	6	11	13	8	5	3	12
3	(3, 1, 1, 0, 1, 0, 2, 0, 1)	14	8	4	3	1	9	11	12	2	6	10	5	15	13	7
3	(3, 0, 0, 0, 1, 0, 0, 0, 1)	2	3	1	9	14	8	15	5	12	4	13	11	7	10	6
3	(3, 0, 0, 0, 1, 0, 0, 1, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
3	(3, 0, 0, 0, 1, 0, 2, 0, 1)	14	5	8	6	1	15	9	7	10	2	12	13	11	4	3
3	(3, 0, 0, 0, 1, 0, 2, 1, 1)	15	11	5	3	6	12	8	9	13	2	7	14	10	1	4
3	(3, 1, 1, 0, 1, 0, 0, 0, 1)	8	1	9	5	2	4	3	13	12	14	15	11	6	10	7
3	(3, 0, 1, 0, 1, 0, 0, 1, 1)	9	11	7	15	3	1	10	5	8	4	12	6	14	2	13
3	(1, 0, 1, 0, 1, 0, 2, 1, 1)	11	5	4	1	7	15	3	9	2	12	13	8	14	6	10
3	(1, 1, 1, 0, 1, 0, 2, 0, 1)	4	9	15	10	7	1	14	2	6	11	13	8	5	3	12
3	(3, 1, 0, 0, 1, 0, 0, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
3	(3, 1, 0, 0, 1, 0, 0, 1, 1)	6	14	7	1	8	15	9	10	2	4	3	11	12	13	5
3	(3, 1, 0, 0, 1, 0, 2, 0, 1)	13	8	5	15	12	7	4	10	6	2	14	1	11	9	3
3	(3, 1, 0, 0, 1, 0, 2, 1, 1)	5	8	13	1	6	9	14	12	2	4	10	11	15	3	7
3	(3, 0, 1, 0, 1, 0, 0, 0, 1)	8	10	2	6	11	12	1	13	5	7	15	3	14	9	4

3	(3, 1, 1, 0, 1, 0, 0, 1, 1)	10	7	5	3	2	15	4	14	12	11	1	8	9	6	13
3	(1, 1, 1, 0, 1, 0, 2, 1, 1)	4	9	15	10	7	1	14	2	6	11	13	8	5	3	12
3	(1, 0, 1, 0, 1, 0, 2, 0, 1)	9	11	2	6	3	1	4	15	13	14	12	10	7	5	8
4	(1, 0, 0, 1)	4	10	5	15	2	12	8	1	7	11	6	14	9	13	3
4	(1, 0, 0, 3)	6	9	11	13	2	7	1	12	15	5	10	14	8	4	3
4	(1, 0, 3, 1)	11	6	4	7	12	15	2	13	10	5	3	14	1	8	9
4	(1, 0, 3, 3)	2	5	6	14	12	11	8	1	7	4	3	15	9	10	13
4	(1, 0, 2, 1)	14	8	6	15	4	10	9	5	11	13	3	2	1	7	12
4	(1, 0, 2, 3)	3	13	8	7	6	12	14	15	5	2	4	9	11	10	1
4	(1, 0, 1, 1)	6	1	5	9	13	8	14	10	12	11	15	3	7	2	4
4	(1, 0, 1, 3)	15	3	14	8	4	11	5	1	12	2	13	9	7	10	6
4	(1, 2, 0, 1)	6	1	14	5	13	8	9	2	12	3	4	7	15	10	11
4	(1, 2, 0, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
4	(1, 2, 3, 3)	9	15	8	7	6	2	10	12	3	13	4	14	11	5	1
4	(1, 2, 3, 1)	2	10	13	9	8	3	14	12	11	4	6	15	1	7	5
4	(1, 2, 2, 1)	9	15	8	10	11	4	6	2	5	13	14	7	1	3	12
4	(1, 2, 2, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
4	(1, 2, 1, 3)	6	1	5	9	13	8	14	10	12	11	15	3	7	2	4
4	(1, 2, 1, 1)	6	13	12	5	9	10	14	11	7	2	1	3	15	8	4
4	(3, 0, 0, 1)	6	1	14	5	13	8	9	2	12	3	4	7	15	10	11
4	(3, 0, 0, 3)	11	14	12	6	8	13	1	5	2	9	7	4	3	15	10
4	(3, 0, 1, 1)	15	12	7	14	13	3	5	1	6	9	8	11	4	2	10
4	(3, 0, 1, 3)	15	3	14	8	4	11	5	1	12	2	13	9	7	10	6
4	(3, 0, 2, 1)	2	4	8	11	13	1	14	10	15	6	7	9	12	3	5
4	(3, 0, 2, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
4	(3, 0, 3, 1)	6	1	5	9	13	8	14	10	12	11	15	3	7	2	4
4	(3, 0, 3, 3)	2	7	9	12	3	10	8	15	11	6	14	5	4	13	1
4	(3, 2, 0, 1)	7	5	6	15	9	3	12	13	10	8	1	4	11	2	14
4	(3, 2, 0, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
4	(3, 2, 1, 3)	15	6	10	5	11	9	13	4	12	8	3	1	2	7	14
4	(3, 2, 1, 1)	15	3	14	8	4	11	5	1	12	2	13	9	7	10	6
4	(3, 2, 2, 1)	14	8	6	15	4	10	9	5	11	13	3	2	1	7	12
4	(3, 2, 2, 3)	3	10	4	8	12	2	9	15	6	5	1	13	11	7	14
4	(3, 2, 3, 3)	6	1	5	9	13	8	14	10	12	11	15	3	7	2	4
4	(3, 2, 3, 1)	15	9	12	11	10	1	5	4	8	3	13	6	14	2	7
5	(1, 0, 0, 1)	14	3	11	8	1	15	4	10	7	13	9	6	12	5	2
5	(1, 0, 4, 1)	9	11	14	10	7	3	6	4	13	1	5	15	2	12	8
5	(1, 1, 0, 1)	12	10	9	7	3	11	6	2	13	15	4	14	1	5	8
5	(5, 1, 4, 1)	3	8	14	13	9	7	4	11	6	2	15	10	12	5	1
5	(3, 0, 0, 1)	12	15	13	2	11	5	4	10	1	3	7	8	14	9	6
5	(3, 0, 4, 1)	15	7	13	11	10	2	6	14	4	12	3	5	1	9	8
5	(3, 1, 0, 1)	4	1	13	8	12	11	3	6	2	5	9	14	10	15	7
5	(7, 1, 4, 1)	3	6	1	8	15	14	13	9	4	7	2	5	12	11	10

5	(5, 0, 0, 1)	6	3	8	12	11	15	1	2	4	13	7	10	9	5	14
5	(5, 0, 4, 1)	14	3	11	8	1	15	4	10	7	13	9	6	12	5	2
5	(5, 1, 0, 1)	5	10	2	1	8	13	9	6	11	3	7	15	14	12	4
5	(1, 1, 4, 1)	3	8	10	9	2	5	15	11	4	14	12	6	1	7	13
5	(7, 0, 0, 1)	2	9	11	8	1	12	4	10	15	13	5	7	6	3	14
5	(7, 0, 4, 1)	10	7	4	9	1	8	13	14	11	5	2	15	12	6	3
5	(7, 1, 0, 1)	5	13	4	12	7	11	8	14	3	15	10	2	1	9	6
5	(3, 1, 4, 1)	3	6	1	8	15	14	13	9	4	7	2	5	12	11	10
6	(1, 0, 0, 1)	3	14	2	11	8	1	13	5	10	4	15	9	6	12	7
6	(1, 0, 4, 1)	10	7	4	9	1	8	13	14	11	5	2	15	12	6	3
6	(1, 1, 0, 1)	15	13	10	14	7	3	6	4	1	9	8	5	2	12	11
6	(5, 1, 4, 1)	5	11	8	15	12	4	1	13	10	2	7	6	3	9	14
6	(3, 0, 0, 1)	4	8	2	1	14	13	12	10	3	6	15	9	11	5	7
6	(3, 0, 4, 1)	10	6	9	3	8	1	15	2	4	12	7	5	14	11	13
6	(7, 1, 0, 1)	4	3	10	9	1	8	15	7	6	13	12	14	11	2	5
6	(3, 1, 4, 1)	5	14	4	7	10	9	15	6	3	11	13	1	8	12	2
6	(5, 0, 0, 1)	10	7	4	9	1	8	13	14	11	5	2	15	12	6	3
6	(5, 0, 4, 1)	13	11	2	15	3	8	4	1	7	5	10	14	9	6	12
6	(5, 1, 0, 1)	5	11	8	15	12	4	1	13	10	2	7	6	3	9	14
6	(1, 1, 4, 1)	6	10	13	3	9	1	14	5	11	8	15	2	4	7	12
6	(7, 0, 0, 1)	15	9	5	7	13	8	3	11	6	1	4	10	14	2	12
6	(7, 0, 4, 1)	3	7	14	8	10	13	5	6	1	4	15	11	2	9	12
6	(3, 1, 0, 1)	15	14	12	11	3	1	6	13	10	4	2	5	9	8	7
6	(7, 1, 4, 1)	5	10	14	2	4	8	11	3	7	15	12	1	6	13	9
7	(1, 0, 0, 1)	8	11	4	12	10	5	2	14	7	13	15	6	1	3	9
7	(1, 0, 7, 1)	12	7	2	8	1	13	9	3	11	15	5	10	4	6	14
7	(1, 0, 6, 1)	13	9	12	6	4	11	5	15	1	14	8	2	7	3	10
7	(1, 0, 5, 1)	5	11	14	7	2	8	13	1	4	6	3	10	12	15	9
7	(1, 0, 4, 1)	15	13	1	14	9	5	3	7	12	8	4	11	10	6	2
7	(1, 0, 3, 1)	13	3	12	14	1	11	2	10	6	4	9	5	8	15	7
7	(1, 0, 2, 1)	5	12	8	6	13	10	2	9	11	1	4	15	7	3	14
7	(1, 0, 1, 1)	8	13	11	14	2	7	5	9	4	1	15	6	10	12	3
7	(3, 0, 0, 1)	7	14	9	1	10	13	2	12	6	8	3	11	4	15	5
7	(3, 0, 5, 1)	12	10	8	14	2	4	6	15	13	1	5	9	11	7	3
7	(3, 0, 2, 1)	4	10	5	1	12	11	9	15	3	14	7	13	2	8	6
7	(3, 0, 7, 1)	4	13	15	2	6	11	9	7	12	8	3	5	14	10	1
7	(3, 0, 4, 1)	11	15	2	12	8	3	1	13	5	9	4	14	7	10	6
7	(3, 0, 1, 1)	4	10	8	6	2	12	14	9	13	7	1	11	15	5	3
7	(3, 0, 6, 1)	10	14	2	11	15	3	1	5	8	4	13	7	9	6	12
7	(3, 0, 3, 1)	10	4	14	12	2	8	6	1	13	9	5	7	15	11	3
7	(5, 0, 0, 1)	12	7	15	3	10	4	9	13	8	14	2	5	11	6	1
7	(5, 0, 3, 1)	2	12	8	6	4	14	10	11	13	3	7	9	15	5	1
7	(5, 0, 6, 1)	10	3	9	6	13	5	14	12	11	7	8	15	1	4	2

7	(5, 0, 1, 1)	8	14	2	6	10	12	4	7	1	15	13	11	9	3	5
7	(5, 0, 4, 1)	8	4	15	3	10	7	13	11	14	12	2	1	9	6	5
7	(5, 0, 7, 1)	4	10	8	6	2	12	14	9	13	7	1	11	15	5	3
7	(5, 0, 2, 1)	14	5	10	2	11	3	15	9	6	1	7	8	12	4	13
7	(5, 0, 5, 1)	11	9	2	6	13	15	4	12	14	5	10	1	7	8	3
7	(7, 0, 0, 1)	5	9	13	8	7	3	15	10	14	11	2	6	12	1	4
7	(7, 0, 1, 1)	2	7	5	15	12	10	9	13	3	1	6	4	14	8	11
7	(7, 0, 2, 1)	12	4	2	8	15	14	5	13	3	1	6	11	9	7	10
7	(7, 0, 3, 1)	12	7	15	14	6	13	1	3	11	4	2	9	5	10	8
7	(7, 0, 4, 1)	13	9	12	6	4	11	5	15	1	14	8	2	7	3	10
7	(7, 0, 5, 1)	15	9	14	8	7	5	2	3	11	13	1	4	12	10	6
7	(7, 0, 6, 1)	11	8	4	9	13	1	6	12	15	3	7	14	2	5	10
7	(7, 0, 7, 1)	5	3	6	12	9	13	8	10	15	11	14	1	7	4	2
8	(1, 0, 0, 1)	15	3	14	7	13	4	12	5	8	6	1	9	11	10	2
8	(1, 0, 6, 1)	9	1	13	2	8	7	12	15	4	6	5	10	14	3	11
8	(1, 0, 4, 1)	12	3	15	7	14	4	13	6	1	9	8	10	2	5	11
8	(1, 0, 2, 1)	14	5	13	2	9	3	10	12	15	4	6	1	8	11	7
8	(3, 0, 0, 1)	4	11	1	5	9	12	14	6	8	2	15	3	10	7	13
8	(3, 0, 2, 1)	4	10	9	11	3	7	12	14	5	15	1	8	6	13	2
8	(3, 0, 4, 1)	4	14	1	5	15	9	8	7	13	2	10	6	11	3	12
8	(3, 0, 6, 1)	3	12	8	1	13	9	6	5	11	4	15	7	14	2	10
8	(5, 0, 0, 1)	15	1	6	11	13	10	4	14	12	3	2	7	9	8	5
8	(5, 0, 6, 1)	5	8	6	2	13	10	12	3	7	15	4	1	11	9	14
8	(5, 0, 4, 1)	7	3	13	12	1	9	11	15	6	4	8	10	14	5	2
8	(5, 0, 2, 1)	4	8	15	2	12	14	1	11	13	5	10	9	3	7	6
8	(7, 0, 0, 1)	15	6	4	13	12	9	5	7	8	2	1	10	14	11	3
8	(7, 0, 2, 1)	4	6	12	10	7	14	11	1	8	15	13	3	2	9	5
8	(7, 0, 4, 1)	7	10	9	1	15	11	3	13	12	8	6	5	14	2	4
8	(7, 0, 6, 1)	4	8	15	2	12	14	1	11	13	5	10	9	3	7	6
10	(1, 0, 0, 0, 0, 1, 0, 1, 0)	7	9	15	3	13	10	4	1	14	8	5	6	12	11	2
10	(1, 0, 0, 2, 0, 1, 0, 1, 0)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(1, 0, 0, 0, 1, 0, 0, 0, 1)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(1, 0, 0, 0, 1, 0, 0, 1, 1)	13	7	14	6	4	10	15	1	3	12	8	11	2	5	9
10	(1, 0, 0, 0, 1, 0, 2, 0, 1)	8	4	14	9	13	10	5	2	6	1	7	3	11	15	12
10	(1, 0, 0, 0, 1, 0, 2, 1, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(1, 0, 0, 0, 1, 1, 0, 0, 1)	11	7	10	8	3	2	15	5	6	12	9	13	14	1	4
10	(1, 0, 0, 2, 1, 1, 2, 0, 1)	6	15	9	13	8	7	3	12	10	2	4	14	11	5	1
10	(1, 0, 0, 0, 0, 1, 2, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(1, 0, 0, 2, 0, 1, 2, 1, 0)	2	14	8	9	7	15	11	6	1	5	13	3	12	4	10
10	(1, 0, 0, 2, 1, 0, 0, 0, 1)	9	6	15	10	2	5	12	13	4	8	1	3	11	7	14
10	(1, 0, 0, 2, 1, 0, 2, 1, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
10	(1, 0, 0, 2, 1, 0, 2, 0, 1)	7	14	11	8	6	13	1	2	12	5	9	3	4	15	10
10	(1, 0, 0, 2, 1, 0, 0, 1, 1)	9	4	8	1	2	10	6	15	12	3	5	14	7	13	11

10	(1, 0, 0, 2, 1, 1, 0, 0, 1)	6	1	12	5	11	15	13	10	4	2	14	7	9	8	3
10	(1, 0, 0, 0, 1, 1, 2, 0, 1)	13	6	9	3	7	14	8	11	4	15	2	10	12	5	1
10	(1, 1, 0, 0, 0, 1, 0, 1, 0)	7	14	11	8	6	13	1	2	12	5	9	3	4	15	10
10	(1, 1, 0, 2, 0, 1, 0, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(1, 0, 1, 0, 1, 0, 0, 0, 1)	8	4	7	9	3	14	13	10	2	5	6	15	12	1	11
10	(1, 1, 1, 0, 1, 0, 0, 1, 1)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(3, 0, 1, 0, 1, 0, 2, 0, 1)	8	4	14	9	13	10	5	2	6	1	7	3	11	15	12
10	(3, 1, 1, 0, 1, 0, 2, 1, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(1, 0, 1, 0, 1, 1, 0, 0, 1)	14	4	11	5	6	15	12	10	13	7	1	3	9	8	2
10	(3, 0, 1, 2, 1, 1, 2, 0, 1)	11	9	4	1	12	5	6	14	13	2	7	10	15	3	8
10	(3, 1, 0, 0, 0, 1, 2, 1, 0)	13	11	6	12	3	7	8	2	4	9	15	14	10	5	1
10	(3, 1, 0, 2, 0, 1, 2, 1, 0)	4	5	2	1	12	13	3	10	15	14	8	11	7	6	9
10	(1, 0, 1, 2, 1, 0, 0, 0, 1)	15	3	14	7	8	9	13	12	11	1	4	2	5	10	6
10	(3, 1, 1, 2, 1, 0, 2, 1, 1)	10	4	14	5	15	2	8	13	7	9	3	11	1	12	6
10	(3, 0, 1, 2, 1, 0, 2, 0, 1)	2	4	14	12	8	10	6	1	3	13	7	11	15	5	9
10	(1, 1, 1, 2, 1, 0, 0, 1, 1)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(1, 0, 1, 2, 1, 1, 0, 0, 1)	11	5	14	15	3	8	10	4	6	2	9	13	1	7	12
10	(3, 0, 1, 0, 1, 1, 2, 0, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(1, 0, 1, 0, 0, 1, 0, 1, 0)	13	6	9	3	7	14	8	11	4	15	2	10	12	5	1
10	(3, 0, 1, 2, 0, 1, 0, 1, 0)	7	10	1	13	8	2	11	6	3	5	12	9	14	15	4
10	(1, 1, 0, 0, 1, 0, 0, 0, 1)	5	12	8	13	2	11	4	15	3	7	10	14	9	6	1
10	(1, 1, 0, 0, 1, 0, 0, 1, 1)	2	10	7	9	14	8	3	1	6	4	12	13	15	11	5
10	(1, 1, 0, 0, 1, 0, 2, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(1, 1, 0, 0, 1, 0, 2, 1, 1)	4	3	11	7	14	1	9	15	13	6	2	10	8	12	5
10	(1, 1, 1, 0, 1, 1, 0, 0, 1)	11	10	1	2	14	5	8	7	4	12	15	13	9	3	6
10	(3, 1, 1, 2, 1, 1, 2, 0, 1)	14	5	2	8	13	15	3	11	12	7	9	10	6	4	1
10	(1, 0, 1, 0, 0, 1, 2, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(3, 0, 1, 2, 0, 1, 2, 1, 0)	3	8	6	5	14	1	15	11	13	7	12	2	4	10	9
10	(3, 1, 0, 2, 1, 0, 0, 0, 1)	10	13	14	9	6	12	11	4	7	8	2	3	5	15	1
10	(3, 1, 0, 2, 1, 0, 2, 1, 1)	12	9	15	7	11	1	8	10	3	14	5	13	4	6	2
10	(3, 1, 0, 2, 1, 0, 2, 0, 1)	7	13	1	11	15	14	3	12	4	9	10	5	6	8	2
10	(3, 1, 0, 2, 1, 0, 0, 1, 1)	13	6	9	3	7	14	8	11	4	15	2	10	12	5	1
10	(3, 1, 1, 2, 1, 1, 0, 0, 1)	6	1	14	3	13	12	10	9	7	8	15	2	4	5	11
10	(1, 1, 1, 0, 1, 1, 2, 0, 1)	13	7	10	8	4	14	2	15	3	9	5	11	6	12	1
10	(1, 1, 1, 0, 0, 1, 0, 1, 0)	13	12	6	7	14	8	5	2	1	11	15	3	4	9	10
10	(3, 1, 1, 2, 0, 1, 0, 1, 0)	11	6	1	3	8	13	2	14	10	4	12	5	9	15	7
10	(1, 1, 1, 0, 1, 0, 0, 0, 1)	8	4	7	9	3	14	13	10	2	5	6	15	12	1	11
10	(1, 0, 1, 0, 1, 0, 0, 1, 1)	9	13	8	11	1	3	6	12	10	4	14	2	15	7	5
10	(3, 1, 1, 0, 1, 0, 2, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(3, 0, 1, 0, 1, 0, 2, 1, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(1, 1, 0, 0, 1, 1, 0, 0, 1)	7	14	11	8	6	13	1	2	12	5	9	3	4	15	10
10	(1, 1, 0, 2, 1, 1, 2, 0, 1)	2	5	7	3	1	15	13	9	11	12	14	10	8	6	4
10	(3, 1, 1, 0, 0, 1, 2, 1, 0)	5	12	6	14	11	1	8	4	2	3	10	9	15	13	7

10	(1, 1, 1, 2, 0, 1, 2, 1, 0)	9	1	15	14	10	13	5	12	8	7	4	2	11	3	6
10	(3, 1, 1, 2, 1, 0, 0, 0, 1)	8	4	12	11	6	2	15	5	13	1	9	10	7	3	14
10	(1, 0, 1, 2, 1, 0, 2, 1, 1)	9	6	10	2	14	4	13	5	12	11	7	15	3	1	8
10	(1, 1, 1, 2, 1, 0, 2, 0, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(3, 0, 1, 2, 1, 0, 0, 1, 1)	2	5	7	3	1	15	13	9	11	12	14	10	8	6	4
10	(3, 1, 0, 2, 1, 1, 0, 0, 1)	8	4	12	11	6	2	15	5	13	1	9	10	7	3	14
10	(3, 1, 0, 0, 1, 1, 2, 0, 1)	8	1	9	3	11	7	15	5	13	4	12	14	6	10	2
10	(3, 0, 0, 0, 0, 1, 0, 1, 0)	4	12	8	6	2	3	13	11	15	14	10	5	9	1	7
10	(3, 0, 0, 2, 0, 1, 0, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(3, 0, 0, 0, 1, 0, 0, 0, 1)	8	9	4	1	15	5	10	7	13	6	3	2	11	12	14
10	(3, 0, 0, 0, 1, 0, 0, 1, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(3, 0, 0, 0, 1, 0, 2, 0, 1)	2	10	7	9	14	8	3	1	6	4	12	13	15	11	5
10	(3, 0, 0, 0, 1, 0, 2, 1, 1)	3	12	11	1	8	10	9	2	5	13	4	15	6	7	14
10	(3, 0, 0, 0, 1, 1, 0, 0, 1)	7	12	10	11	6	1	13	5	8	2	15	14	9	3	4
10	(3, 0, 0, 2, 1, 1, 2, 0, 1)	5	13	9	6	4	15	2	3	14	12	8	1	11	10	7
10	(3, 0, 0, 0, 0, 1, 2, 1, 0)	3	15	2	14	11	1	10	6	13	9	12	8	5	7	4
10	(3, 0, 0, 2, 0, 1, 2, 1, 0)	4	12	8	6	2	3	13	11	15	14	10	5	9	1	7
10	(3, 0, 0, 2, 1, 0, 0, 0, 1)	3	12	5	14	13	9	6	4	10	15	2	7	1	11	8
10	(3, 0, 0, 2, 1, 0, 2, 1, 1)	12	11	4	14	3	1	5	13	8	9	15	2	6	10	7
10	(3, 0, 0, 2, 1, 0, 2, 0, 1)	8	15	5	14	2	7	1	12	10	13	9	4	6	3	11
10	(3, 0, 0, 2, 1, 0, 0, 1, 1)	7	14	11	6	2	5	12	9	13	15	1	3	4	8	10
10	(3, 0, 0, 2, 1, 1, 0, 0, 1)	5	13	2	14	10	11	9	1	7	6	3	15	8	12	4
10	(3, 0, 0, 0, 1, 1, 2, 0, 1)	11	7	10	8	3	2	15	5	6	12	9	13	14	1	4
10	(3, 1, 0, 0, 0, 1, 0, 1, 0)	6	4	8	7	2	3	15	9	12	14	1	5	11	10	13
10	(3, 1, 0, 2, 0, 1, 0, 1, 0)	2	3	1	8	10	15	13	14	12	6	4	9	11	5	7
10	(3, 0, 1, 0, 1, 0, 0, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(3, 1, 1, 0, 1, 0, 0, 1, 1)	7	12	10	8	1	14	6	11	3	13	4	9	15	5	2
10	(1, 0, 1, 0, 1, 0, 2, 0, 1)	14	7	5	12	15	9	6	4	11	1	2	10	8	13	3
10	(1, 1, 1, 0, 1, 0, 2, 1, 1)	13	1	4	10	15	14	11	3	6	8	5	9	12	7	2
10	(3, 0, 1, 0, 1, 1, 0, 0, 1)	5	15	11	7	10	13	1	2	6	12	9	8	4	3	14
10	(1, 0, 1, 2, 1, 1, 2, 0, 1)	7	8	11	13	1	9	6	5	2	15	4	10	14	12	3
10	(1, 1, 0, 0, 0, 1, 2, 1, 0)	3	15	2	14	11	1	10	6	13	9	12	8	5	7	4
10	(1, 1, 0, 2, 0, 1, 2, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(3, 0, 1, 2, 1, 0, 0, 0, 1)	11	1	4	13	8	14	12	6	5	7	10	2	15	9	3
10	(1, 1, 1, 2, 1, 0, 2, 1, 1)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(1, 0, 1, 2, 1, 0, 2, 0, 1)	6	4	8	7	2	3	15	9	12	14	1	5	11	10	13
10	(3, 1, 1, 2, 1, 0, 0, 1, 1)	13	6	9	3	7	14	8	11	4	15	2	10	12	5	1
10	(3, 0, 1, 2, 1, 1, 0, 0, 1)	4	5	14	10	12	7	11	2	6	1	3	8	15	13	9
10	(1, 0, 1, 0, 1, 1, 2, 0, 1)	10	7	12	13	6	2	9	5	3	8	15	4	11	1	14
10	(3, 0, 1, 0, 0, 1, 0, 1, 0)	11	15	6	2	14	5	9	7	1	8	12	13	4	10	3
10	(1, 0, 1, 2, 0, 1, 0, 1, 0)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
10	(3, 1, 0, 0, 1, 0, 0, 0, 1)	8	4	7	9	3	14	13	10	2	5	6	15	12	1	11
10	(3, 1, 0, 0, 1, 0, 0, 1, 1)	2	10	7	9	14	8	3	1	6	4	12	13	15	11	5

10	(3, 1, 0, 0, 1, 0, 2, 0, 1)	13	15	12	5	7	9	11	14	10	3	1	4	6	8	2
10	(3, 1, 0, 0, 1, 0, 2, 1, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(3, 1, 1, 0, 1, 1, 0, 0, 1)	5	15	11	7	10	13	1	2	6	12	9	8	4	3	14
10	(1, 1, 1, 2, 1, 1, 2, 0, 1)	2	14	4	9	12	8	5	7	13	1	3	6	11	15	10
10	(3, 0, 1, 0, 0, 1, 2, 1, 0)	10	7	4	3	9	12	15	1	11	6	5	2	8	13	14
10	(1, 0, 1, 2, 0, 1, 2, 1, 0)	12	10	7	2	4	9	14	13	11	6	1	15	3	5	8
10	(1, 1, 0, 2, 1, 0, 0, 0, 1)	8	4	12	11	6	2	15	5	13	1	9	10	7	3	14
10	(1, 1, 0, 2, 1, 0, 2, 1, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(1, 1, 0, 2, 1, 0, 2, 0, 1)	7	14	11	8	6	13	1	2	12	5	9	3	4	15	10
10	(1, 1, 0, 2, 1, 0, 0, 1, 1)	15	11	6	14	9	5	2	7	1	8	12	13	3	10	4
10	(1, 1, 1, 2, 1, 1, 0, 0, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(3, 1, 1, 0, 1, 1, 2, 0, 1)	13	7	10	8	4	14	2	15	3	9	5	11	6	12	1
10	(3, 1, 1, 0, 0, 1, 0, 1, 0)	8	7	1	3	14	5	13	9	4	12	15	10	2	6	11
10	(1, 1, 1, 2, 0, 1, 0, 1, 0)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(3, 1, 1, 0, 1, 0, 0, 0, 1)	6	1	7	2	13	3	12	10	15	11	14	8	4	9	5
10	(3, 0, 1, 0, 1, 0, 0, 1, 1)	8	1	9	3	11	7	15	5	13	4	12	14	6	10	2
10	(1, 1, 1, 0, 1, 0, 2, 0, 1)	13	8	6	1	7	10	15	3	14	11	5	2	4	9	12
10	(1, 0, 1, 0, 1, 0, 2, 1, 1)	2	5	7	3	1	15	13	9	11	12	14	10	8	6	4
10	(3, 1, 0, 0, 1, 1, 0, 0, 1)	5	13	9	15	4	11	2	1	10	6	12	14	3	8	7
10	(3, 1, 0, 2, 1, 1, 2, 0, 1)	2	14	4	9	12	8	5	7	13	1	3	6	11	15	10
10	(1, 1, 1, 0, 0, 1, 2, 1, 0)	9	1	6	14	4	13	3	10	15	7	12	5	11	2	8
10	(3, 1, 1, 2, 0, 1, 2, 1, 0)	8	13	2	10	6	4	12	15	1	14	7	9	11	3	5
10	(1, 1, 1, 2, 1, 0, 0, 0, 1)	10	12	2	6	11	5	15	4	3	13	14	9	1	7	8
10	(3, 0, 1, 2, 1, 0, 2, 1, 1)	2	9	6	14	1	7	5	15	13	12	3	11	4	8	10
10	(3, 1, 1, 2, 1, 0, 2, 0, 1)	15	11	9	12	7	10	8	4	2	5	6	13	14	3	1
10	(1, 0, 1, 2, 1, 0, 0, 1, 1)	2	5	7	3	1	15	13	9	11	12	14	10	8	6	4
10	(1, 1, 0, 2, 1, 1, 0, 0, 1)	11	5	6	10	7	15	2	3	1	14	4	13	9	8	12
10	(1, 1, 0, 0, 1, 1, 2, 0, 1)	7	14	11	8	6	13	1	2	12	5	9	3	4	15	10
11	(1, 0, 0, 0, 1, 0, 0, 0, 1)	12	5	1	15	6	10	9	2	14	7	3	13	4	8	11
11	(1, 0, 0, 0, 1, 0, 2, 0, 1)	15	6	2	13	8	14	4	12	11	5	1	7	10	3	9
11	(1, 0, 0, 0, 1, 1, 0, 0, 1)	10	7	15	13	4	5	9	2	1	14	6	11	3	12	8
11	(1, 0, 0, 2, 1, 1, 2, 0, 1)	13	8	11	3	6	1	2	10	15	12	7	9	4	5	14
11	(1, 0, 0, 3, 1, 0, 0, 0, 1)	4	11	8	2	1	9	13	10	14	6	5	15	12	3	7
11	(1, 0, 0, 3, 1, 0, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(1, 0, 0, 3, 1, 1, 0, 0, 1)	13	8	11	3	6	1	2	10	15	12	7	9	4	5	14
11	(1, 0, 0, 1, 1, 1, 2, 0, 1)	7	15	5	13	2	14	12	11	4	8	10	6	1	9	3
11	(1, 0, 0, 2, 1, 0, 0, 0, 1)	8	14	4	2	10	11	13	9	1	7	5	15	6	12	3
11	(1, 0, 0, 2, 1, 0, 2, 0, 1)	5	15	1	2	12	10	13	9	14	4	8	11	6	3	7
11	(1, 0, 0, 2, 1, 1, 0, 0, 1)	8	3	11	2	14	1	5	4	12	15	7	6	10	13	9
11	(1, 0, 0, 0, 1, 1, 2, 0, 1)	6	1	14	3	13	12	10	9	7	8	15	2	4	5	11
11	(1, 0, 0, 1, 1, 0, 0, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(1, 0, 0, 1, 1, 0, 2, 0, 1)	4	6	5	8	14	15	10	7	11	2	1	13	3	9	12
11	(1, 0, 0, 1, 1, 1, 0, 0, 1)	14	4	15	6	8	12	10	11	5	1	2	13	3	9	7

11	(1, 0, 0, 3, 1, 1, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(1, 0, 1, 0, 1, 0, 0, 0, 1)	8	14	4	2	10	11	13	9	1	7	5	15	6	12	3
11	(3, 0, 1, 0, 1, 0, 2, 0, 1)	13	11	4	12	15	2	1	10	7	9	6	3	8	5	14
11	(1, 0, 1, 0, 1, 1, 0, 0, 1)	8	14	6	10	2	12	4	3	11	7	15	13	5	9	1
11	(3, 0, 1, 2, 1, 1, 2, 0, 1)	8	13	9	7	2	4	11	3	12	15	5	10	1	6	14
11	(1, 0, 1, 3, 1, 1, 0, 0, 1)	6	4	11	1	15	2	12	7	8	3	5	14	9	13	10
11	(3, 0, 1, 1, 1, 1, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(1, 0, 1, 3, 1, 0, 0, 0, 1)	11	12	5	15	9	4	6	7	1	14	2	3	8	13	10
11	(3, 0, 1, 3, 1, 0, 2, 0, 1)	15	11	12	6	9	13	10	2	5	7	8	4	3	1	14
11	(1, 0, 1, 2, 1, 0, 0, 0, 1)	8	14	6	10	2	12	4	3	11	7	15	13	5	9	1
11	(3, 0, 1, 2, 1, 0, 2, 0, 1)	6	9	1	7	2	14	13	12	8	15	5	10	4	11	3
11	(1, 0, 1, 2, 1, 1, 0, 0, 1)	8	3	11	2	14	1	5	4	12	15	7	6	10	13	9
11	(3, 0, 1, 0, 1, 1, 2, 0, 1)	14	10	15	5	1	12	4	3	11	7	13	9	2	8	6
11	(1, 0, 1, 1, 1, 1, 0, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(3, 0, 1, 3, 1, 1, 2, 0, 1)	14	11	8	6	10	7	2	12	3	15	5	4	9	13	1
11	(1, 0, 1, 1, 1, 0, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
11	(3, 0, 1, 1, 1, 0, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(3, 0, 0, 0, 1, 0, 0, 0, 1)	12	5	1	15	6	10	9	2	14	7	3	13	4	8	11
11	(3, 0, 0, 0, 1, 0, 2, 0, 1)	14	10	15	5	1	12	4	3	11	7	13	9	2	8	6
11	(3, 0, 0, 0, 1, 1, 0, 0, 1)	6	1	4	13	15	10	2	14	8	12	9	7	5	3	11
11	(3, 0, 0, 2, 1, 1, 2, 0, 1)	2	5	13	11	8	12	10	1	7	6	3	14	9	15	4
11	(3, 0, 0, 1, 1, 0, 0, 0, 1)	4	11	8	2	1	9	13	10	14	6	5	15	12	3	7
11	(3, 0, 0, 1, 1, 0, 2, 0, 1)	12	15	4	5	1	13	3	14	11	2	8	10	7	9	6
11	(3, 0, 0, 1, 1, 1, 0, 0, 1)	4	6	9	11	14	7	8	10	15	2	13	5	1	3	12
11	(3, 0, 0, 3, 1, 1, 2, 0, 1)	2	13	9	1	12	4	8	14	5	7	3	10	6	15	11
11	(3, 0, 0, 2, 1, 0, 0, 0, 1)	5	11	12	9	1	14	4	3	7	13	10	6	15	8	2
11	(3, 0, 0, 2, 1, 0, 2, 0, 1)	8	13	9	7	2	4	11	3	12	15	5	10	1	6	14
11	(3, 0, 0, 2, 1, 1, 0, 0, 1)	8	3	11	2	14	1	5	4	12	15	7	6	10	13	9
11	(3, 0, 0, 0, 1, 1, 2, 0, 1)	13	8	11	3	6	1	2	10	15	12	7	9	4	5	14
11	(3, 0, 0, 3, 1, 0, 0, 0, 1)	15	6	2	13	8	14	4	12	11	5	1	7	10	3	9
11	(3, 0, 0, 3, 1, 0, 2, 0, 1)	6	15	11	10	4	2	14	5	3	8	12	7	9	13	1
11	(3, 0, 0, 3, 1, 1, 0, 0, 1)	13	8	11	3	6	1	2	10	15	12	7	9	4	5	14
11	(3, 0, 0, 1, 1, 1, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(3, 0, 1, 0, 1, 0, 0, 0, 1)	2	14	12	13	15	3	11	6	4	8	10	7	5	1	9
11	(1, 0, 1, 0, 1, 0, 2, 0, 1)	11	14	4	6	10	5	15	13	7	1	2	9	12	8	3
11	(3, 0, 1, 0, 1, 1, 0, 0, 1)	6	1	4	13	15	10	2	14	8	12	9	7	5	3	11
11	(1, 0, 1, 2, 1, 1, 2, 0, 1)	10	4	11	9	1	5	14	2	6	15	7	3	12	8	13
11	(3, 0, 1, 1, 1, 1, 0, 0, 1)	9	4	12	15	7	5	13	14	6	11	2	1	8	10	3
11	(1, 0, 1, 3, 1, 1, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(3, 0, 1, 1, 1, 0, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
11	(1, 0, 1, 1, 1, 0, 2, 0, 1)	6	14	10	7	3	9	15	13	11	1	5	8	12	4	2
11	(3, 0, 1, 2, 1, 0, 0, 0, 1)	3	7	5	12	14	15	13	11	8	6	4	2	9	1	10
11	(1, 0, 1, 2, 1, 0, 2, 0, 1)	13	11	12	7	15	9	2	3	4	8	5	10	1	6	14

11	(3, 0, 1, 2, 1, 1, 0, 0, 1)	9	15	7	10	14	4	3	1	8	5	12	6	11	13	2
11	(1, 0, 1, 0, 1, 1, 2, 0, 1)	8	3	11	2	14	1	5	4	12	15	7	6	10	13	9
11	(3, 0, 1, 3, 1, 1, 0, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
11	(1, 0, 1, 1, 1, 1, 2, 0, 1)	6	15	11	10	4	2	14	5	3	8	12	7	9	13	1
11	(3, 0, 1, 3, 1, 0, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
11	(1, 0, 1, 3, 1, 0, 2, 0, 1)	11	5	14	8	3	13	6	1	10	4	15	9	2	12	7
12	(0, 1, 0, 1, 0, 0, 0, 0, 1)	11	5	15	6	12	10	9	3	1	7	4	13	14	8	2
12	(0, 1, 0, 1, 0, 0, 2, 0, 1)	8	10	2	14	6	9	1	13	5	15	7	11	3	4	12
12	(0, 1, 1, 1, 0, 0, 0, 0, 1)	4	10	7	3	15	8	14	6	13	11	1	9	2	5	12
12	(2, 1, 1, 1, 0, 0, 2, 0, 1)	8	10	2	14	6	9	1	13	5	15	7	11	3	4	12
12	(2, 1, 0, 1, 0, 0, 0, 0, 1)	2	11	12	14	3	8	10	9	13	7	15	4	6	1	5
12	(2, 1, 0, 1, 0, 0, 2, 0, 1)	13	11	6	2	10	5	9	7	12	8	3	15	1	4	14
12	(2, 1, 1, 1, 0, 0, 0, 0, 1)	4	10	8	15	9	7	13	6	5	3	14	11	2	12	1
12	(0, 1, 1, 1, 0, 0, 2, 0, 1)	14	11	4	1	9	13	5	6	8	12	3	7	15	2	10
12	(0, 1, 0, 1, 0, 1, 0, 0, 1)	11	5	15	6	12	10	9	3	1	7	4	13	14	8	2
12	(0, 1, 0, 3, 0, 1, 2, 0, 1)	8	10	2	14	6	9	1	13	5	15	7	11	3	4	12
12	(0, 1, 1, 1, 0, 1, 0, 0, 1)	2	12	4	5	13	11	3	10	15	14	9	1	6	7	8
12	(2, 1, 1, 3, 0, 1, 2, 0, 1)	14	12	1	13	9	5	8	4	11	7	10	15	3	6	2
12	(2, 1, 0, 1, 0, 1, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(2, 1, 0, 3, 0, 1, 2, 0, 1)	6	10	1	2	12	5	9	11	13	7	4	14	8	15	3
12	(2, 1, 1, 1, 0, 1, 0, 0, 1)	15	13	9	10	12	7	5	3	4	2	6	8	14	1	11
12	(0, 1, 1, 3, 0, 1, 2, 0, 1)	2	7	4	6	13	15	5	14	12	11	8	10	1	9	3
12	(1, 0, 0, 0, 1, 0, 0, 0, 1)	13	3	9	10	2	5	8	12	1	4	14	15	7	11	6
12	(1, 0, 0, 0, 1, 0, 2, 0, 1)	5	6	11	15	7	13	8	1	12	4	9	2	10	3	14
12	(1, 0, 0, 0, 1, 1, 0, 0, 1)	4	3	12	15	7	11	8	1	5	2	13	6	14	10	9
12	(1, 0, 0, 2, 1, 1, 2, 0, 1)	5	1	10	2	8	12	9	11	14	15	4	13	7	6	3
12	(1, 0, 0, 3, 1, 0, 0, 0, 1)	12	11	1	8	4	9	15	6	2	7	13	14	10	5	3
12	(1, 0, 0, 3, 1, 0, 2, 0, 1)	7	10	15	13	6	1	2	4	11	9	5	3	8	12	14
12	(1, 0, 0, 3, 1, 1, 0, 0, 1)	7	12	8	3	14	9	10	2	13	11	15	1	4	6	5
12	(1, 0, 0, 1, 1, 1, 2, 0, 1)	3	1	7	10	9	11	5	15	12	4	2	13	14	6	8
12	(1, 0, 0, 2, 1, 0, 0, 0, 1)	9	6	15	10	2	5	12	13	4	8	1	3	11	7	14
12	(1, 0, 0, 2, 1, 0, 2, 0, 1)	13	12	10	15	11	1	3	5	8	7	9	14	2	6	4
12	(1, 0, 0, 2, 1, 1, 0, 0, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
12	(1, 0, 0, 0, 1, 1, 2, 0, 1)	6	1	14	3	13	12	10	9	7	8	15	2	4	5	11
12	(1, 0, 0, 1, 1, 0, 0, 0, 1)	9	5	6	14	2	12	4	11	7	3	13	8	1	15	10
12	(1, 0, 0, 1, 1, 0, 2, 0, 1)	12	11	1	8	4	9	15	6	2	7	13	14	10	5	3
12	(1, 0, 0, 1, 1, 1, 0, 0, 1)	13	5	15	14	11	12	2	3	6	1	10	9	4	8	7
12	(1, 0, 0, 3, 1, 1, 2, 0, 1)	10	8	13	5	2	14	3	4	15	1	12	9	7	11	6
12	(1, 0, 1, 0, 1, 0, 0, 0, 1)	8	5	7	9	2	14	1	3	11	15	13	6	12	10	4
12	(3, 0, 1, 0, 1, 0, 2, 0, 1)	3	1	7	10	9	11	5	15	12	4	2	13	14	6	8
12	(1, 0, 1, 0, 1, 1, 0, 0, 1)	8	1	5	9	15	11	12	6	14	2	7	13	10	4	3
12	(3, 0, 1, 2, 1, 1, 2, 0, 1)	4	1	12	6	10	2	15	11	8	14	5	9	3	7	13
12	(1, 0, 1, 3, 1, 1, 0, 0, 1)	10	6	9	3	13	4	8	11	12	15	5	2	1	7	14

12	(3, 0, 1, 1, 1, 1, 2, 0, 1)	2	6	12	9	11	15	5	10	8	4	14	3	1	13	7
12	(1, 0, 1, 3, 1, 0, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(3, 0, 1, 3, 1, 0, 2, 0, 1)	3	1	7	10	9	11	5	15	12	4	2	13	14	6	8
12	(1, 0, 1, 2, 1, 0, 0, 0, 1)	4	6	12	2	9	15	5	14	11	1	10	3	8	13	7
12	(3, 0, 1, 2, 1, 0, 2, 0, 1)	15	3	12	11	9	13	10	4	2	8	6	14	5	7	1
12	(1, 0, 1, 2, 1, 1, 0, 0, 1)	9	14	13	15	7	4	11	1	8	12	6	3	10	5	2
12	(3, 0, 1, 0, 1, 1, 2, 0, 1)	12	15	4	5	1	13	3	14	11	2	8	10	7	9	6
12	(1, 0, 1, 1, 1, 1, 0, 0, 1)	10	12	14	1	11	13	15	2	8	5	7	3	9	4	6
12	(3, 0, 1, 3, 1, 1, 2, 0, 1)	3	14	8	12	15	5	11	13	6	4	10	9	2	7	1
12	(1, 0, 1, 1, 1, 0, 0, 0, 1)	10	12	14	1	11	13	15	2	8	5	7	3	9	4	6
12	(3, 0, 1, 1, 1, 0, 2, 0, 1)	3	1	7	10	9	11	5	15	12	4	2	13	14	6	8
12	(3, 1, 0, 0, 1, 0, 0, 0, 1)	12	13	1	6	9	3	4	14	10	11	15	5	2	8	7
12	(3, 1, 0, 0, 1, 0, 2, 0, 1)	3	6	4	7	13	15	5	9	2	12	14	10	1	11	8
12	(3, 1, 1, 0, 1, 1, 0, 0, 1)	6	1	14	3	13	12	10	9	7	8	15	2	4	5	11
12	(1, 1, 1, 2, 1, 1, 2, 0, 1)	11	8	14	5	6	15	9	7	4	13	3	10	1	2	12
12	(1, 1, 0, 2, 1, 0, 0, 0, 1)	5	12	8	13	2	11	4	15	3	7	10	14	9	6	1
12	(1, 1, 0, 2, 1, 0, 2, 0, 1)	6	7	9	2	15	1	4	12	10	3	13	8	5	11	14
12	(1, 1, 1, 2, 1, 1, 0, 0, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
12	(3, 1, 1, 0, 1, 1, 2, 0, 1)	14	7	1	12	3	10	13	9	15	6	8	5	2	11	4
12	(3, 1, 1, 0, 1, 0, 0, 0, 1)	6	1	14	3	13	12	10	9	7	8	15	2	4	5	11
12	(1, 1, 1, 0, 1, 0, 2, 0, 1)	6	7	9	2	15	1	4	12	10	3	13	8	5	11	14
12	(3, 1, 0, 0, 1, 1, 0, 0, 1)	5	12	11	3	6	10	8	2	4	9	13	1	15	7	14
12	(3, 1, 0, 2, 1, 1, 2, 0, 1)	3	6	8	1	15	5	14	7	12	11	13	4	2	10	9
12	(1, 1, 1, 2, 1, 0, 0, 0, 1)	14	3	8	1	13	4	9	6	11	7	15	5	2	10	12
12	(3, 1, 1, 2, 1, 0, 2, 0, 1)	12	8	13	3	11	4	2	9	14	1	15	5	6	10	7
12	(1, 1, 0, 2, 1, 1, 0, 0, 1)	11	7	12	8	3	15	4	10	1	13	6	2	9	5	14
12	(1, 1, 0, 0, 1, 1, 2, 0, 1)	4	8	7	14	2	11	12	3	15	13	10	5	1	6	9
12	(0, 1, 0, 3, 0, 0, 0, 0, 1)	11	5	15	6	12	10	9	3	1	7	4	13	14	8	2
12	(0, 1, 0, 3, 0, 0, 2, 0, 1)	2	11	12	5	10	4	15	14	13	7	1	8	6	3	9
12	(0, 1, 1, 3, 0, 0, 0, 0, 1)	5	6	11	7	2	4	8	1	12	15	10	14	3	13	9
12	(2, 1, 1, 3, 0, 0, 2, 0, 1)	2	11	12	5	10	4	15	14	13	7	1	8	6	3	9
12	(2, 1, 0, 3, 0, 0, 0, 0, 1)	4	1	12	13	15	9	11	10	7	6	3	2	8	5	14
12	(2, 1, 0, 3, 0, 0, 2, 0, 1)	15	11	8	1	7	3	5	9	14	6	13	4	10	2	12
12	(2, 1, 1, 3, 0, 0, 0, 0, 1)	15	13	11	9	12	5	10	14	8	6	1	3	7	2	4
12	(0, 1, 1, 3, 0, 0, 2, 0, 1)	14	11	4	1	9	13	5	6	8	12	3	7	15	2	10
12	(0, 1, 0, 3, 0, 1, 0, 0, 1)	11	5	15	6	12	10	9	3	1	7	4	13	14	8	2
12	(0, 1, 0, 1, 0, 1, 2, 0, 1)	15	3	4	13	14	8	5	11	12	6	1	9	10	2	7
12	(0, 1, 1, 3, 0, 1, 0, 0, 1)	4	10	6	5	14	15	3	11	7	13	1	2	9	12	8
12	(2, 1, 1, 1, 0, 1, 2, 0, 1)	14	10	15	11	6	7	1	4	12	8	2	9	3	13	5
12	(2, 1, 0, 3, 0, 1, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(2, 1, 0, 1, 0, 1, 2, 0, 1)	7	5	6	8	2	14	1	10	12	11	9	15	4	13	3
12	(2, 1, 1, 3, 0, 1, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(0, 1, 1, 1, 0, 1, 2, 0, 1)	2	1	14	11	15	7	10	12	6	3	13	4	9	8	5

12	(3, 0, 0, 0, 1, 0, 0, 0, 1)	3	7	10	2	13	8	5	4	14	11	6	15	1	9	12
12	(3, 0, 0, 0, 1, 0, 2, 0, 1)	10	9	4	1	15	14	11	7	13	3	6	2	12	8	5
12	(3, 0, 0, 0, 1, 1, 0, 0, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
12	(3, 0, 0, 2, 1, 1, 2, 0, 1)	7	12	8	3	14	9	10	2	13	11	15	1	4	6	5
12	(3, 0, 0, 1, 1, 0, 0, 0, 1)	5	8	7	11	6	14	1	2	15	3	4	9	12	13	10
12	(3, 0, 0, 1, 1, 0, 2, 0, 1)	5	10	1	9	7	13	6	14	2	15	12	11	4	8	3
12	(3, 0, 0, 1, 1, 1, 0, 0, 1)	3	11	13	9	1	2	4	15	8	12	14	6	10	5	7
12	(3, 0, 0, 3, 1, 1, 2, 0, 1)	3	9	14	1	15	10	4	12	6	11	13	5	2	8	7
12	(3, 0, 0, 2, 1, 0, 0, 0, 1)	9	6	15	10	2	5	12	13	4	8	1	3	11	7	14
12	(3, 0, 0, 2, 1, 0, 2, 0, 1)	6	8	12	5	9	10	4	1	7	2	14	15	3	11	13
12	(3, 0, 0, 2, 1, 1, 0, 0, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
12	(3, 0, 0, 0, 1, 1, 2, 0, 1)	12	13	14	5	10	4	15	3	11	1	7	2	9	8	6
12	(3, 0, 0, 3, 1, 0, 0, 0, 1)	10	12	14	1	11	13	15	2	8	5	7	3	9	4	6
12	(3, 0, 0, 3, 1, 0, 2, 0, 1)	12	15	4	5	1	13	3	14	11	2	8	10	7	9	6
12	(3, 0, 0, 3, 1, 1, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(3, 0, 0, 1, 1, 1, 2, 0, 1)	2	15	7	5	11	14	1	12	3	13	8	9	10	4	6
12	(3, 0, 1, 0, 1, 0, 0, 0, 1)	10	15	13	3	7	11	9	2	8	14	12	4	1	5	6
12	(1, 0, 1, 0, 1, 0, 2, 0, 1)	5	6	11	7	2	4	8	1	12	15	10	14	3	13	9
12	(3, 0, 1, 0, 1, 1, 0, 0, 1)	4	3	12	15	7	11	8	1	5	2	13	6	14	10	9
12	(1, 0, 1, 2, 1, 1, 2, 0, 1)	11	13	15	9	10	14	12	5	8	4	1	7	3	2	6
12	(3, 0, 1, 1, 1, 1, 0, 0, 1)	6	4	11	1	15	2	12	7	8	3	5	14	9	13	10
12	(1, 0, 1, 3, 1, 1, 2, 0, 1)	2	6	12	9	11	15	5	10	8	4	14	3	1	13	7
12	(3, 0, 1, 1, 1, 0, 0, 0, 1)	9	11	15	7	3	4	13	5	12	14	10	2	6	1	8
12	(1, 0, 1, 1, 1, 0, 2, 0, 1)	12	13	1	6	9	3	4	14	10	11	15	5	2	8	7
12	(3, 0, 1, 2, 1, 0, 0, 0, 1)	4	3	12	15	7	11	8	1	5	2	13	6	14	10	9
12	(1, 0, 1, 2, 1, 0, 2, 0, 1)	12	13	1	6	9	3	4	14	10	11	15	5	2	8	7
12	(3, 0, 1, 2, 1, 1, 0, 0, 1)	6	12	9	15	1	2	10	14	8	7	3	11	4	5	13
12	(1, 0, 1, 0, 1, 1, 2, 0, 1)	15	10	5	13	14	11	12	9	6	8	7	1	2	4	3
12	(3, 0, 1, 3, 1, 1, 0, 0, 1)	10	12	14	1	11	13	15	2	8	5	7	3	9	4	6
12	(1, 0, 1, 1, 1, 1, 2, 0, 1)	14	11	8	6	10	7	2	12	3	15	5	4	9	13	1
12	(3, 0, 1, 3, 1, 0, 0, 0, 1)	10	12	14	1	11	13	15	2	8	5	7	3	9	4	6
12	(1, 0, 1, 3, 1, 0, 2, 0, 1)	12	13	1	6	9	3	4	14	10	11	15	5	2	8	7
12	(1, 1, 0, 0, 1, 0, 0, 0, 1)	2	5	12	14	9	8	11	13	4	3	1	10	6	15	7
12	(1, 1, 0, 0, 1, 0, 2, 0, 1)	7	3	4	8	13	1	2	10	5	9	14	15	11	12	6
12	(1, 1, 1, 0, 1, 1, 0, 0, 1)	9	13	6	15	4	8	1	10	3	7	12	5	14	2	11
12	(3, 1, 1, 2, 1, 1, 2, 0, 1)	14	7	1	12	3	10	13	9	15	6	8	5	2	11	4
12	(3, 1, 0, 2, 1, 0, 0, 0, 1)	14	7	15	10	8	9	11	6	3	2	1	5	12	13	4
12	(3, 1, 0, 2, 1, 0, 2, 0, 1)	15	6	9	13	10	8	12	5	3	2	4	7	1	11	14
12	(3, 1, 1, 2, 1, 1, 0, 0, 1)	11	8	4	7	1	2	15	5	12	3	13	10	6	9	14
12	(1, 1, 1, 0, 1, 1, 2, 0, 1)	11	8	14	5	6	15	9	7	4	13	3	10	1	2	12
12	(1, 1, 1, 0, 1, 0, 0, 0, 1)	3	11	8	15	1	4	2	5	14	12	7	13	10	6	9
12	(3, 1, 1, 0, 1, 0, 2, 0, 1)	6	8	7	12	15	1	2	3	4	11	13	5	14	10	9
12	(1, 1, 0, 0, 1, 1, 0, 0, 1)	13	4	9	3	15	2	14	5	8	1	12	10	6	11	7

12	(1, 1, 0, 2, 1, 1, 2, 0, 1)	5	1	10	9	15	7	12	6	11	14	13	4	2	3	8
12	(3, 1, 1, 2, 1, 0, 0, 0, 1)	5	13	6	15	12	14	9	11	7	8	10	1	2	4	3
12	(1, 1, 1, 2, 1, 0, 2, 0, 1)	15	3	6	13	8	10	9	2	14	1	7	11	5	4	12
12	(3, 1, 0, 2, 1, 1, 0, 0, 1)	7	1	6	13	2	3	12	15	8	14	9	4	11	10	5
12	(3, 1, 0, 0, 1, 1, 2, 0, 1)	4	8	7	14	2	11	12	3	15	13	10	5	1	6	9
13	(2, 1, 1, 3, 0, 1, 0, 0, 1)	8	15	2	14	13	1	9	10	5	4	7	11	6	3	12
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13	(0, 1, 1, 3, 0, 1, 0, 0, 1)	14	10	5	6	1	2	13	9	15	7	8	11	12	3	4
13	(2, 1, 1, 1, 0, 1, 2, 0, 1)	3	5	14	13	9	7	8	4	15	2	1	10	6	11	12
13	(1, 0, 0, 0, 1, 0, 0, 0, 1)	3	12	8	1	4	9	13	6	5	2	14	7	10	15	11
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13	(1, 0, 0, 2, 1, 0, 0, 0, 1)	14	8	11	15	9	5	10	1	7	13	6	2	4	12	3
13	(1, 0, 0, 2, 1, 0, 2, 0, 1)	9	6	2	11	14	7	3	13	12	4	1	10	15	5	8
13	(1, 0, 0, 1, 1, 0, 0, 0, 1)	14	4	8	15	1	9	10	12	3	11	6	2	5	7	13
13	(1, 0, 0, 1, 1, 0, 2, 0, 1)	4	6	5	8	14	15	10	7	11	2	1	13	3	9	12
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13	(1, 1, 1, 0, 1, 0, 2, 0, 1)	12	11	5	8	13	7	4	2	6	9	15	14	3	1	10
13	(1, 1, 1, 2, 1, 0, 0, 0, 1)	12	11	15	14	2	10	1	3	7	8	4	9	5	13	6
13	(3, 1, 1, 2, 1, 0, 2, 0, 1)	9	12	5	14	4	2	8	11	10	7	6	13	15	1	3
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13	(3, 0, 0, 0, 1, 0, 2, 0, 1)	14	9	15	1	10	4	13	11	5	12	3	2	8	7	6
13	(3, 0, 0, 1, 1, 0, 0, 0, 1)	13	8	6	12	9	15	1	10	7	5	2	11	14	3	4
13	(3, 0, 0, 1, 1, 0, 2, 0, 1)	14	11	10	12	9	8	1	3	5	7	6	15	2	4	13
13	(3, 0, 0, 2, 1, 0, 0, 0, 1)	10	7	15	11	12	9	5	4	3	1	13	2	8	6	14
13	(3, 0, 0, 2, 1, 0, 2, 0, 1)	8	13	12	10	14	15	3	2	11	7	6	4	9	5	1
13	(3, 0, 0, 3, 1, 0, 0, 0, 1)	10	4	15	1	2	9	6	14	5	8	3	13	7	11	12
13	(3, 0, 0, 3, 1, 0, 2, 0, 1)	3	5	9	11	14	15	1	13	7	2	6	8	4	10	12
13	(1, 1, 1, 0, 1, 0, 0, 0, 1)	3	1	2	11	9	8	10	5	14	4	15	13	7	6	12
13	(3, 1, 1, 0, 1, 0, 2, 0, 1)	12	6	4	1	15	2	14	10	7	3	8	11	5	13	9
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14	(0, 0, 1, 0, 0, 0)	13	5	14	15	7	8	2	11	3	12	10	4	9	1	6
14	(0, 0, 1, 0, 0, 0)	13	9	10	3	11	7	1	5	6	2	15	14	8	4	12

14	(0, 0, 1, 0, 1, 0)	11	7	4	12	8	9	5	1	2	6	13	15	3	10	14
14	(0, 0, 1, 0, 1, 0)	5	7	12	9	2	8	11	14	3	1	10	4	15	13	6
14	(0, 0, 0, 1, 0, 0)	6	13	5	10	15	3	12	4	1	11	9	8	14	7	2
14	(0, 0, 0, 1, 1, 0)	2	8	12	9	11	7	3	15	1	6	14	10	4	5	13
14	(0, 0, 0, 1, 1, 0)	13	9	10	3	11	7	1	5	6	2	15	14	8	4	12
14	(0, 0, 0, 1, 0, 0)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(0, 0, 1, 1, 0, 0)	15	8	7	13	2	3	12	10	5	11	4	1	14	6	9
14	(0, 0, 1, 1, 1, 0)	2	8	12	9	11	7	3	15	1	6	14	10	4	5	13
14	(0, 0, 1, 1, 1, 0)	8	5	12	15	1	4	13	9	10	7	14	2	11	6	3
14	(0, 0, 1, 1, 0, 0)	4	5	7	12	9	13	10	14	11	3	8	6	2	15	1
14	(0, 1, 0, 0, 0, 0)	13	5	14	15	7	8	2	11	3	12	10	4	9	1	6
14	(0, 1, 1, 0, 0, 0)	5	1	4	8	15	9	14	6	2	7	3	13	11	12	10
14	(0, 1, 0, 0, 1, 0)	10	1	6	9	7	12	15	4	14	5	2	13	3	8	11
14	(0, 1, 1, 0, 1, 0)	2	8	12	9	11	7	3	15	1	6	14	10	4	5	13
14	(0, 1, 1, 0, 0, 0)	5	1	9	11	3	8	10	13	2	6	12	14	4	15	7
14	(0, 1, 0, 0, 0, 0)	15	3	9	2	10	13	4	5	1	14	7	11	8	12	6
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14	(0, 1, 0, 0, 1, 0)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(0, 1, 0, 1, 0, 0)	2	12	1	13	15	14	3	4	6	7	10	9	11	5	8
14	(0, 1, 1, 1, 1, 0)	8	9	5	10	2	3	15	4	6	7	1	14	12	13	11
14	(0, 1, 0, 1, 1, 0)	11	13	10	8	7	1	6	12	15	9	14	4	3	5	2
14	(0, 1, 1, 1, 0, 0)	13	11	6	3	14	8	5	12	1	7	10	15	2	4	9
14	(0, 1, 1, 1, 0, 0)	4	6	11	5	14	15	10	7	3	13	9	2	1	8	12
14	(0, 1, 0, 1, 1, 0)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(0, 1, 1, 1, 1, 0)	2	5	10	8	7	13	15	12	4	11	14	6	3	1	9
14	(0, 1, 0, 1, 0, 0)	12	9	4	11	7	3	14	6	13	2	8	10	1	15	5
14	(1, 0, 0, 0, 0, 1)	14	10	8	9	4	3	1	11	13	6	12	2	15	7	5
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14	(1, 0, 1, 0, 1, 1)	9	6	15	7	14	1	8	2	4	12	10	5	3	11	13
14	(1, 0, 1, 0, 1, 1)	2	8	14	3	12	10	1	6	13	11	4	9	15	5	7
14	(1, 0, 1, 0, 0, 1)	13	9	10	3	11	7	1	5	6	2	15	14	8	4	12
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14	(1, 0, 0, 0, 1, 1)	7	3	4	9	15	13	11	12	10	8	14	2	5	1	6
14	(1, 0, 0, 0, 1, 1)	7	14	6	15	12	11	5	2	10	4	3	13	9	1	8
14	(1, 1, 0, 1, 0, 1)	2	12	1	13	15	14	3	4	6	7	10	9	11	5	8
14	(1, 1, 1, 1, 1, 1)	5	12	9	13	7	14	4	15	8	1	6	10	2	11	3
14	(1, 1, 1, 1, 1, 1)	10	14	7	9	4	3	13	15	6	2	8	5	11	12	1
14	(1, 1, 0, 1, 0, 1)	13	11	6	3	14	8	5	12	1	7	10	15	2	4	9
14	(1, 1, 1, 1, 0, 1)	14	5	10	7	8	2	9	4	15	1	3	6	12	11	13
14	(1, 1, 0, 1, 1, 1)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(1, 1, 0, 1, 1, 1)	5	12	9	13	7	14	4	15	8	1	6	10	2	11	3
14	(1, 1, 1, 1, 0, 1)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(1, 1, 0, 0, 0, 1)	2	12	1	13	15	14	3	4	6	7	10	9	11	5	8

14	(1, 1, 1, 0, 0, 1)	15	9	7	3	13	11	2	1	10	6	4	14	12	8	5
14	(1, 1, 1, 0, 1, 1)	12	13	4	2	15	3	11	9	1	8	5	7	14	10	6
14	(1, 1, 0, 0, 1, 1)	4	13	1	5	9	8	12	14	10	3	15	11	7	6	2
14	(1, 1, 1, 0, 0, 1)	8	5	6	11	9	14	1	10	2	4	15	13	7	3	12
14	(1, 1, 0, 0, 0, 1)	13	11	6	3	14	8	5	12	1	7	10	15	2	4	9
14	(1, 1, 0, 0, 1, 1)	10	7	11	8	2	15	3	6	4	9	13	14	12	1	5
14	(1, 1, 1, 0, 1, 1)	2	8	12	9	11	7	3	15	1	6	14	10	4	5	13
14	(1, 0, 0, 1, 0, 1)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(1, 0, 0, 1, 1, 1)	8	4	1	9	14	5	15	7	13	11	12	6	3	2	10
14	(1, 0, 1, 1, 1, 1)	13	9	10	3	11	7	1	5	6	2	15	14	8	4	12
14	(1, 0, 1, 1, 0, 1)	9	7	10	3	14	4	13	12	5	11	6	15	2	8	1
14	(1, 0, 1, 1, 0, 1)	10	7	11	8	2	15	3	6	4	9	13	14	12	1	5
14	(1, 0, 1, 1, 1, 1)	13	11	6	3	14	8	5	12	1	7	10	15	2	4	9
14	(1, 0, 0, 1, 1, 1)	5	8	14	6	13	3	11	15	7	9	2	10	12	1	4
14	(1, 0, 0, 1, 0, 1)	13	9	10	3	11	7	1	5	6	2	15	14	8	4	12

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