Adequate equivalence relations and Pontryagin products

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Abstract

Let A be an abelian variety over a field k. We consider $CH_0(A)$ as a ring under Pontryagin product and relate powers of the ideal $I \subseteq CH_0(A)$ of degree zero elements to powers of the algebraic equivalence relation. We also consider a filtration $F^0 \supseteq F^1 \supseteq \ldots$ on the Chow groups of varieties of the form $T \times_k A$ (defined using Pontryagin products on $A \times_k A$ considered as an A-scheme via projection on the first factor) and prove that F^r coincides with the r-fold product $(F^1)^{*r}$ as adequate equivalence relations on the category of all such varieties.

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1 Introduction

Let k be a field and $\mathcal{V}_k$ the category of smooth projective varieties over k. We open with a well-known conjecture attributed to Bloch and Beilinson:

Conjecture 1.1. *For every object X of $\mathcal{V}_k$ there exists a descending filtration $F^\cdot$ on $CH^j(X;\mathbb{Q}) = CH^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $j \geq 0$ such that:*

1. *$F^0CH^j(X;\mathbb{Q}) = CH^j(X;\mathbb{Q})$ and $F^1CH^j(X;\mathbb{Q}) = CH^j(X;\mathbb{Q})_{hom}$ (cycles homologically equivalent to zero) for some fixed Weil cohomology theory.*

2. *$F^\cdot$ is preserved under intersection product, i.e. $F^rCH^i(X;\mathbb{Q}) \cdot F^sCH^j(X;\mathbb{Q}) \subseteq F^{r+s}CH^{i+j}(X;\mathbb{Q})$.*

3. *$F^\cdot$ is respected by f_* and f^* for morphisms $f : X \longrightarrow Y$.*

4. *Assuming that the Künneth components of the diagonal are algebraic, the rth graded piece $Gr^r_F CH^j(X;\mathbb{Q})$ depends only on the motive of X modulo homological equivalence.*

5. $F^rCH^j(X;\mathbb{Q}) = 0$ *for* $r >> 0$.

It is well-known that homological equivalence is an *adequate equivalence relation.* A precise definition is given in 2.10; roughly speaking, an adequate equivalence relation E is an assignment, to each smooth projective variety X, of a subgroup $ECH^*(X) \subseteq CH^*(X)$ preserved under pullback, pushforward, and intersection with arbitrary cycles. If E and E' are two adequate equivalence relations, one may define their sum and intersection in the obvious manner; these are also adequate equivalence relations. More interesting, though, is Hiroshi Saito's definition [15] of the product $E * E'$ of two adequate equivalence relations: for each X, $(E * E')CH^*(X)$ is the subgroup generated by cycles of the form $p_*(\alpha \cdot \beta)$, where T is some smooth projective variety and $p : X \times_k T \longrightarrow X$ is the projection map, $\alpha \in ECH^*(X \times_k T)$ and $\beta \in E'CH^*(X \times_k T)$. It is known [15] that this product structure is associative, commutative, and distributes in the expected manner over the sum discussed above. It is also the case that $E * E'$ is adequate; moreover, $(E * E')CH^*(X) \subseteq ECH^*(X) \cap E'CH^*(X)$.

Associativity of $*$ enables us to define the powers E^{*r} of an adequate equivalence relation. Assuming that the filtration of Conjecture 1.1 exists, it is clear from the second and third conditions that for each $r \geq 1$, F^r is also an adequate equivalence relation. A striking result of Jannsen ([9], Theorem 4.1) asserts that it must then be the case that $F^r = (F^1)^{*r}$ (as adequate equivalence relations on $\mathcal{V}_k$.)

This result of Jannsen provided the inspiration for this paper. Certain classes of smooth projective varieties (among them curves, surfaces, and abelian varieties) are known to have *Chow-Künneth decompositions*. Specifically, if X is one of the varieties listed above and $d = \dim X$, then the class of the diagonal $[\Delta_X] \in CH^d(X \times_k X;\mathbb{Q})$ has a decomposition:

$$[\Delta_X] = \sum_{i=0}^{2d} \pi_i$$

where $\pi_i \circ \pi_j = 0$ if $i \neq j$, and $\pi_i \circ \pi_i = \pi_i$ for each i. (Here, $\circ$ refers to composition of correspondences: if $\alpha \in CH^*(X \times_k Y)$ and $\beta \in CH^*(Y \times_k Z)$ and X, Y, Z are all smooth projective varieties, we define $\beta \circ \alpha = p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\beta)$, where the p_{ij} refer to projections of $X \times_k Y \times_k Z$ on the appropriate factors.) Moreover, the class of π_i modulo homological equivalence should coincide with the appropriate Künneth component of the class of Δ_X in $H^{2i}(X \times_k X)$.

Given $z \in CH^*(X;\mathbb{Q})$, we write $\pi_j(x)$ as shorthand for $p_{2*}(p_1^*x \cdot \pi_j)$. Then one might define a filtration on $CH^*(X;\mathbb{Q})$ by

$$F^r CH^p(X;\mathbb{Q}) = \sum_{j=0}^{2p-r} \pi_j(CH^p(X;\mathbb{Q}))$$

(This filtration ostensibly depends on Chow-Künneth decomposition)
Of course, one cannot hope interpret the F^r as adequate equivalence relations, if only because Chow-Künneth decompositions are not known to exist for arbitrary smooth projective varieties. Nevertheless, one might take some subcategory of $\mathcal{V}_k$, all of whose objects are known to have Chow-Künneth decompositions, and then ask, first, whether the F^r are equivalence relations which are adequate (in a sense made precise in the text) *with respect to this subcategory*, and second, whether the formula $F^r = (F^1)^{*r}$ holds for this filtration.

The filtration proposed above is supported by a conjecture of Murre [13] cited below. Jannsen [9] has proved that the two conjectures are in fact equivalent.

Conjecture 1.2. *(Murre)*
For every object X of $\mathcal{V}_k$, set $d = \dim X$. Then

1. *There exists a Chow-Künneth decomposition $[\Delta_X] = \sum_{i=0}^{2d} \pi_i$, where*

   $\pi_i \in CH^d(X \times_k X;\mathbb{Q})$.

2. *If $0 \leq i \leq j-1$ or $2j+1 \leq i \leq 2d$, $\pi_i(CH^j(X;\mathbb{Q})) = 0$.*

3. *Let $M^\cdot$ be the filtration on $CH^j(X;\mathbb{Q})$ defined by*

   $$M^\nu CH^j(X;\mathbb{Q}) = \cap_{k=2j-\nu+1}^{2j} Ker\ \pi_k.$$

   Then $M^\cdot$ is independent of ambiguity in the choice of projectors π_i.

4. *$M^1 CH^j(X;\mathbb{Q}) = CH^j(X;\mathbb{Q})_{hom}$, the subgroup of cycles homologically equivalent to zero (for some Weil cohomology theory).*

Assuming this conjecture, it follows from the first and second statements that

$$M^r CH^p(X;\mathbb{Q}) = \sum_{j=0}^{2p-r} \pi_j(CH^p(X;\mathbb{Q}))$$

which is exactly the filtration proposed above. The first two assertions of the conjecture also imply (cf. [13], 1.4) that $M^{j+1}CH^j(X;\mathbb{Q}) = 0$, $M^1 CH^j(X;\mathbb{Q}) \subseteq CH^j(X;\mathbb{Q})_{hom}$ and $M^1 CH^d(X;\mathbb{Q}) = \text{Ker}\ (\deg\ : CH^d(X;\mathbb{Q}) \to \mathbb{Q})$.

It is not surprising that there are close relationships among the various conjectures and conjectural filtrations described above. Let A be an abelian variety of dimension d. By interpreting $A \times_k A$ as an abelian A-scheme via projection on the first factor, Deninger and Murre [5] have constructed an explicit Chow-Künneth decomposition $[\Delta_A] = \sum_{i=0}^{2d} \pi_i$. Murre ([14], Corollary 2.5.2) has proved that the projectors π_i appearing in this decomposition act as zero on $CH^j(A;\mathbb{Q})$ if $i < j$ or $i > j+d$, which is part of the second statement of Murre's conjecture. Moreover, the remainder of the second statement is equivalent to Beauville's conjecture for A, which asserts that the groups $CH^j_s(A;\mathbb{Q}) = \{x \in CH^j(A;\mathbb{Q}) : \mathbf{n}^* x = n^{2j-s}x\}$ vanish when $s < 0$. At present, Beauville's conjecture is known to hold for (all) abelian varieties over a finite field [11] and for supersingular abelian varieties over fields of positive characteristic [6]. For abelian varieties over an arbitrary field, it is known to hold in the cases $j = 0, 1, d-2, d-1, d$; thus Beauville's conjecture is known for all abelian varieties of dimension ≤ 4. Finally, the third assertion of Murre's conjecture is also satisfied: while the projectors π_i may not themselves be unique, the corresponding motives (A, π_i) are unique up to isomorphism by results of Guletskii-Pedrini [7]. In any case, if we assume Beauville's conjecture for A, then there is a filtration $F^{\cdot}$ on $CH^*(A;\mathbb{Q})$ such that for zero-dimensional cycles, the first step is given by

$$F^1 CH^d(A;\mathbb{Q}) = \text{Ker}\ (\deg\ : CH^d(A;\mathbb{Q}) \to \mathbb{Q}) = CH^d(A;\mathbb{Q})_{alg},$$

the subgroup of cycles algebraically equivalent to zero.

After providing some preliminaries, we investigate the validity of the formula $F^r = (F^1)^{*r}$ in the context of abelian varieties. Let A be an abelian variety of dimension d over an algebraically closed field k, and L the (adequate) relation of algebraic equivalence. Observing that $CH_0(A)$ is a ring under Pontryagin product, let I be the kernel of the degree map $\deg\ : CH_0(A) \longrightarrow \mathbb{Z}$; it follows immediately that I is an ideal of $CH_0(A)$ and that $I = CH_0(A)_{alg}$. Let I^{*r} denote the rth power of I with respect to this structure. Our main result in the first section is that, under the assumption of Beauville's conjecture, we have the formula $L^{*r} CH_0(A) = I^{*r}$, the $*$ on the left representing (as before) the rth power of the algebraic equivalence relation. We stress that this formula holds integrally; that is, without tensoring Chow groups with $\mathbb{Q}$. Using a result of Beauville ([2], p. 649) to identify $I^{*r} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\sum_{j=0}^{2d-r} \pi_j(CH^d(A;\mathbb{Q}))$, it follows readily that $F^r CH^d(A) = (F^1)^{*r} CH^d(A)$ for the filtration described above, providing some evidence for the Bloch-Beilinson conjecture. While we are not yet able to prove an analogous result for cycles of positive dimension, the techniques used in the proof may be modified to prove that $L^{*n} = 0$ when $n > d$ if, again, we assume Beauville's conjecture. This is of interest in light of a result of

Voevodsky [17] that cycles algebraically equivalent to zero on a smooth projective variety X are nilpotent in the ring of correspondences from X to X.

The second part of the paper studies a similar formula, but in a relative setting. As above, fix an abelian variety A over a field k and consider the full subcategory $\mathcal{V}_k/A$ of $\mathcal{V}_k$ consisting of objects of the form $T \times_k A$ where T is a smooth projective variety. One may regard any such variety $T \times_k A$ as an abelian T-scheme via projection on the first factor. We then use the abovementioned Chow-Künneth decomposition of Deninger-Murre to define a filtration $F^{\cdot}$ on the groups $CH^*(T \times_k A; \mathbb{Q})$. Our result is that for each $r \geq 0$ we have $F^r = (F^1)^{*r}$ as adequate relations on $\mathcal{V}_k/A$.

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2 Preliminaries

2.1 Cycles and the Pontryagin Product

Let k be a field and X a scheme of finite type over k. We denote by $Z_i(X)$ the group of i-dimensional cycles on X, that is, the free abelian group generated by the set of dimension i subvarieties of X. We denote by $CH_i(X)$ the Chow group of i-dimensional cycles; that is, $Z_i(X)$ modulo the subgroup of cycles rationally equivalent to zero, and set $Z_*(X) = \oplus_i Z_i(X)$, $CH_*(X) = \oplus_i CH_i(X)$. The class of a subvariety $V \subseteq X$ in $CH_*(X)$ is denoted $[V]$. If X is equidimensional, we denote by $Z^j(X)$ (resp. $CH^j(X)$) the Chow group of codimension j cycles (resp. codimension j cycles modulo rational equivalence) on X; clearly, $Z^j(X) = Z_{d-j}(X)$ and $CH^j(X) = CH_{d-j}(X)$ where $d = \dim X$. It is well-known [8] that the graded group $CH^*(X) = \oplus_i CH^i(X)$ may be endowed with the structure of commutative graded ring under *intersection product*. Following convention, we will denote the intersection of two cycles $\alpha, \beta \in CH^*(X)$ by $\alpha \cdot \beta$. If $\alpha \in CH^*(X)$ and $\gamma \in CH^*(Y)$ are cycles, we denote by $\alpha \times \gamma \in CH^*(X \times_k Y)$ the cycle $p_1^*(\alpha) \cdot p_2^*(\gamma)$, where $p_1 : X \times_k Y \longrightarrow X$ and $p_2 : X \times_k Y \longrightarrow Y$ are the projection maps. If X, Y, and Z are smooth and projective, and $\alpha \in CH^*(X \times_k Y)$, $\beta \in CH^*(Y \times_k Z)$, we define their composition $\beta \circ \alpha = (p_{13})_*(p_{12}^* \alpha \cdot p_{23}^* \beta) \in CH^*(X \times_k Z)$. (Here p_{ij} is the projection of $X \times_k Y \times_k Z$ on the (i, j)th factor). If R is any ring, we write $CH^*(X; R)$ as shorthand for $CH^*(X) \otimes_{\mathbb{Z}} R$.

Now suppose A is an abelian variety and $\mu : A \times_k A \longrightarrow A$ is the morphism giving the group law on A. One may then define a product structure on $CH_*(A)$, namely the *Pontryagin product*, as follows:

$$* : CH_r(A) \otimes CH_s(A) \longrightarrow CH_{r+s}(A)$$

$$\alpha \times \beta \mapsto \alpha * \beta := \mu_*(\alpha \times \beta)$$

Clearly $CH_0(A)$ is a subring of $CH_*(A)$ for this ring structure. In the sequel, we will often use formal sums (in cycle groups) and addition of points on the abelian variety in the same expression; in an attempt to dispel potential confusion arising from this, we will denote the former by the ordinary summation symbol $\sum$ and the latter by $\overset{A}{\sum}$.

Now suppose k is algebraically closed. Consider the degree map $\deg : CH_0(A) \longrightarrow \mathbb{Z}$, and let $I = \mathrm{Ker} \deg$. Then I is generated by cycles of the form $[a] - [0]$ (where $a \in A$ is a closed point), and is an ideal of $CH_0(A)$ with respect to Pontryagin product. For any $n > 0$, we denote by I^{*n} the nth Pontryagin power of the ideal I, and define $I^{*0} = CH_0(A)$.

An elementary argument gives the following, cf. [4].

Lemma 2.1. *Let* $alb : I \longrightarrow A$ *be the Albanese map of* A, *i.e.* $alb(\sum[P_i]) = \overset{A}{\sum} P_i$. *Then there is an exact sequence*

$$0 \longrightarrow I^{*2} \longrightarrow I \xrightarrow{alb} A \longrightarrow 0$$

Other important properties of the ideal I are summarized below:

Proposition 2.2.

1. *(Bloch, [3] Lemma 1.4)*

   I *is divisible.*

2. *(Roitman, [12])*

   I^{*2} *is uniquely divisible.*

3. *(Bloch, [4], Theorem 0.1)*

   $I^{*(g+1)} = 0$.

Since I^{*n} is generated by products from I, it follows immediately from the above that I^{*n} is uniquely divisible when $n \geq 2$.

The following lemma will be necessary in the proof of our main result. The first assertion is elementary and follows from the definitions; the second is standard and may be proved by induction.

Lemma 2.3. *Let A be an abelian variety and $a \in A$ a closed point. Let $\tau_a : A \longrightarrow A$ denote the translation map $x \mapsto x + a$*

1. *For any $z \in CH_*(A)$, $(\tau_a)_* z = [a] * z$, $(\tau_a)^* z = [-a] * z$.*
2. *For any integer $n \geq 1$ and $a_1, \ldots, a_n \in A$,*

$$([a_1] - [0]) * \ldots * ([a_n] - [0]) = \sum_{e_1,\ldots e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} [\sum_{e_i=0}^{A} a_i]$$

An important tool in studying the Chow groups of an abelian variety is the *Fourier Transform.* We will not give the details of the construction here – these may be found in [1], [2], [5] – but rather list some important properties we will need. These will be stated in somewhat greater generality (i.e. for abelian schemes over a smooth quasiprojective base) than the present context, as we will adopt this perspective in the latter half of the paper.

Let k be a field, S a smooth quasiprojective algebraic k-scheme, and A an abelian scheme of fiber dimension $g = g_A$ over S. Let $\hat{A}$ denote the dual abelian scheme and $\ell \in CH^1(A \times_S \hat{A})$ the class of the Poincar'e bundle. For convenience, let $p : A \times_S \hat{A}$ denote projection on the first factor and $q : A \times_S \hat{A}$ projection on the second. Finally, denote by σ the involution $a \mapsto -a$ on A and $\hat{\sigma}$ the analogous involution on $\hat{A}$.

Proposition 2.4. *With notation as above, let $s : A \times_S \hat{A} \longrightarrow \hat{A} \times_S A$ denote the exchange of factors. There exist correspondences $F = e^{\ell} = \sum_{i=0}^{\infty} \frac{\ell^i}{i} \in CH^*(A \times_S \hat{A}; \mathbb{Q})$ and $\hat{F} = s^* F \in CH^*(\hat{A} \times_S A; \mathbb{Q})$ giving rise to homomorphisms ("Fourier transforms"):*

$$\mathcal{F} : CH^*(A; \mathbb{Q}) \longrightarrow CH^*(\hat{A}; \mathbb{Q}) \text{ and } \hat{\mathcal{F}} : CH^*(\hat{A}; \mathbb{Q}) \longrightarrow CH^*(A; \mathbb{Q})$$

defined by

$$\mathcal{F}(x) = q_*(p^* x \cdot F) \quad \text{and } \hat{\mathcal{F}}(y) = p_*(q^* y \cdot \hat{F})$$

such that

- $\hat{\mathcal{F}}(\mathcal{F}(x)) = (-1)^g \sigma_* x$ *for all* $x \in CH^*(A;\mathbb{Q})$ *and* $\mathcal{F}(\hat{\mathcal{F}}(y)) = (-1)^g \hat{\sigma}_* y$ *for all* $y \in CH^*(\hat{A};\mathbb{Q})$.
- $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha)\cdot\mathcal{F}(\beta)$ *for all* $\alpha,\beta \in CH^*(A;\mathbb{Q})$.
- $\mathcal{F}(\alpha\cdot\beta) = (-1)^g(\mathcal{F}(\alpha) * \mathcal{F}(\beta))$ *for all* $\alpha,\beta \in CH^*(A;\mathbb{Q})$.

We remark that first formula above may be used to obtain analogues of the second and third formulae for $\hat{\mathcal{F}}$.

Now let $\mathbf{n} : A \times A$ denote multiplication by n on A. For each $s \in \mathbb{Z}$, define:

$$CH^p_s(A;\mathbb{Q}) := \{x \in CH^p(A;\mathbb{Q}) : \mathbf{n}^* x = n^{2p-s} x\}$$

An important property of the Fourier transform is

Proposition 2.5. *([2] Prop. 2, [5] Lemma 2.18)*

$$\mathcal{F}(CH^p_s(A);\mathbb{Q}) = CH^{g-p+s}_s(\hat{A};\mathbb{Q})$$

Using Fourier theory, one can prove the following theorem on diagonalizability of the multiplication by n morphism on A.

Theorem 2.6. *(Beauville,[2]; Deninger-Murre [5], Theorem 2.19) With notation as before, let d be the dimension of S and g the fiber dimension of A over S. Let n be an integer $\neq 0, \pm 1$. Then there is an isomorphism*

$$CH^p(A;\mathbb{Q}) \cong \bigoplus_{s=p''}^{p'} CH^p_s(A;\mathbb{Q})$$

where $p' = \min(2p, p+d)$ *and* $p'' = \max(p-g, 2(p-g))$.
Furthermore, $CH^p_s(A;\mathbb{Q}) = 0$ *if* $s < 2p-2g$ *or* $s > 2p$.

The last statement is not stated explicitly in either of the sources [5], [11] but follows easily from Proposition 2.5.

The next result provides an important link between the above eigenspaces and the ideal $I \subset CH_0(A)$. The result was proven by Beauville [2] for abelian varieties over $\mathbb{C}$, but the proof works over an arbitrary algebraically closed field:

Proposition 2.7. *Let A be an abelian variety over an algebraically closed field and $J = I$ the image of I under the natural map $q : CH^g(A) \longrightarrow CH^g(A;\mathbb{Q})$. Then $J^{*r} = \bigoplus_{s\geq r} CH^g_s(A;\mathbb{Q})$.*

Next, we recall some functorial properties of the eigenspaces. These are proved for abelian varieties over $\mathbb{C}$ in [1]; the same proofs apply to the general situation.

Proposition 2.8.

1. *Let A be an abelian scheme of fiber dimension g over a scheme S (as in Proposition 2.8). If $x \in CH^p_s(A;\mathbb{Q})$ and $y \in CH^q_t(A;\mathbb{Q})$, then $x \cdot y \in CH^{p+q}_{s+t}(A;\mathbb{Q})$ and $x * y \in CH^{p+q-g}_{s+t}(A;\mathbb{Q})$.*

2. *Let $f : A \longrightarrow B$ be a homomorphism of abelian schemes over S. Then $f^*(CH^p_s(B;\mathbb{Q})) \subseteq CH^p_s(A;\mathbb{Q})$ and $f_*(CH^p_s(A;\mathbb{Q})) \subseteq CH^{p+c}_s(B;\mathbb{Q})$, where $c = g_B - g_A$.*

We close this section with an important conjecture due to Beauville ([1], p.255):

Conjecture 2.9. *Let A be an abelian variety. Then for any $p \geq 0$ and $s < 0$, $CH^p_s(A;\mathbb{Q}) = 0$.*

At present, Künnemann has proven that the conjecture holds for any abelian variety when k is an algebraic extension of a finite field ([11], Theorem 7.1); Fakhruddin ([6], Proposition 1) has verified the conjecture for supersingular abelian varieties over any field. It is also known ([1], p. 255; see also [14]) that $CH^p_s(A;\mathbb{Q}) = 0$ if $p = 0, 1, g-2, g-1$, or g.

2.2 Adequate Equivalence Relations

Let k be a field and $\mathcal{V}_k$ the category of smooth projective varieties over k. The following definition is due (at least in the case $\mathcal{C} = \mathcal{V}_k$) to Samuel:

Definition 2.10. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{V}_k$. An* adequate equivalence relation *on $\mathcal{C}$ is an assignment, to every object X of $\mathcal{C}$ of a graded subgroup $EZ^*(X) \subseteq Z^*(X)$ with the following properties:*

1. *If $\alpha, \beta \in Z^*(X)$, then there exists a cycle $\alpha' \in Z^*(X)$ such that α' and β intersect properly, and $\alpha - \alpha' \in EZ^*(X)$.*

2. *Let $\alpha \in Z^*(X)$ and $\beta \in Z^*(X \times_k Y)$ such that the intersection $\beta \cap (\alpha \times_k Y)$ is defined. If $\alpha \in EZ^*(X)$, then $(p_2)_*(\beta \cap p_1^*(\alpha)) \in EZ^*(Y)$, where p_1 and p_2 are the respective projections of $X \times Y$ onto the first and second factors.*

Essentially, the first condition implies that some sort of moving lemma holds for E-equivalence, and the second condition guarantees preservation of E-equivalence under the action of correspondences. If we do not specify the subcategory $\mathcal{C}$, we will assume without further comment that $\mathcal{C} = \mathcal{V}_k$.

Rational equivalence, algebraic equivalence, homological equivalence (with respect to some Weil cohomology theory, cf. [10]) and numerical equivalence are all examples of adequate equivalence relations. If E and E' are two equivalence relations, we say that E is *finer* than E' if $EZ^*(X) \subseteq E'Z^*(X)$ for all $X \in \mathcal{V}_k$. The following theorem summarizes some well-known relationships among the equivalence relations mentioned above.

Theorem 2.11.

- *Rational equivalence is strictly finer than algebraic equivalence, which is strictly finer than homological equivalence, which in turn is strictly finer than numerical equivalence. With $\mathbb{Q}$-coefficients, Grothendieck's standard conjectures predict that numerical equivalence and homological equivalence (with respect to any Weil cohomology theory) coincide [10].*
- *(Samuel, [16]) Rational equivalence is the finest adequate equivalence relation.*
- *With $\mathbb{Q}$-coefficients, numerical equivalence is the coarsest non-trivial adequate equivalence relation.*

We will make use of another important (adequate) equivalence relation, called ℓ-cubical equivalence, was defined by Samuel in [16]:

Definition 2.12. *Let k be an algebraically closed field and $\ell \geq 0$ an integer. Two cycles $\alpha_1, \alpha_2 \in Z^j(X)$ are called ℓ-cubically equivalent if there exist curves $C_1, \ldots, C_\ell$, a cycle $z \in Z^j(C_1 \times_k \ldots \times_k C_\ell \times_k X)$ and closed points $p_i^0, p_i^1 \in C_i$ $(i = 1, \ldots, \ell)$ such that $s(p_1^{e_1}, \ldots, p_\ell^{e_\ell})^*(Z) \in Z^j(X)$ (pullback of Z via the inclusion $s(p_1^{e_1}, \ldots, p_\ell^{e_\ell}) : X \hookrightarrow C_1 \times_k \ldots \times_k C_\ell \times_k X$ induced by the closed point $p = (p_1^{e_1}, \ldots, p_\ell^{e_\ell}) \in C_1 \times_k \ldots \times_k C_\ell$) exists for all $e_1, \ldots, e_\ell \in \{0, 1\}$ and such that*

$$\alpha_1 - \alpha_2 = \sum_{e_1,\ldots,e_\ell \in \{0,1\}} (-1)^{e_1+\ldots e_\ell} s(p_1^{e_1}, \ldots, p_\ell^{e_\ell})^*(Z).$$

As noted in [9], p. 229, a Bertini-type argument implies that the same equivalence relation is obtained if one replaces the "parameter varieties" C_i above by arbitrary smooth projective varieties, or by abelian varieties; alternatively, one may take $C_1, \ldots, C_\ell$ to be the same curve.

Let $F_\ell Z^*(X)$ denote the group of cycles ℓ-cubically equivalent to zero. It is clear from the definition that $F_1 Z^*(X)$ coincides with the subgroup of cycles algebraically equivalent to zero, which we henceforth denote $LZ^*(X)$

In light of the fact that rational equivalence is the finest adequate equivalence relation, it is often convenient to adopt the following notation: given an adequate equivalence relation E, let $ECH^*(X)$ denote the image of $EZ^*(X)$ under the quotient map $Z^*(X) \longrightarrow CH^*(X)$. Then giving an adequate equivalence relation E is equivalent to specifying subgroups $ECH^*(X)$ preserved under pushforwards and pullbacks and such that $\alpha \in CH^*(X),\ \beta \in ECH^*(X) \Longrightarrow \alpha \cdot \beta \in ECH^*(X)$. (cf. [9], Lemma 1.3) Equivalently, one could stipulate simply that the subgroups $ECH^*(X)$ be preserved under composition of correspondences.

Hiroshi Saito [15] has defined the following notion of *product* of equivalence relations. In view of the above remarks, we give all our definitions modulo rational equivalence.

Definition 2.13. *Let E and E' be adequate equivalence relations. We define $E * E'$ as follows: $\alpha \in (E * E')CH^*(X))$ if α is a sum of cycles of the form $p_*(\alpha_1 \cdot \alpha_2)$, where T is a smooth projective variety, $\alpha_1 \in ECH^*(X \times_k T)$, $\alpha_2 \in E'CH^*(X \times_k T)$ and $p : X \times_k T \longrightarrow X$ represents projection on the first factor.*

Proposition 2.14. *[15] $E * E'$ is an adequate equivalence relation finer than both E and E'.*

This product operation is evidently associative (and commutative); hence we may speak of the nth power E^{*n} of E for any $n \geq 1$; by convention E^{*0} is the trivial relation, that is, $E^{*0}CH^*(X) = CH^*(X)$ for all X. An important observation proceeding straight from the definition and linking two of the examples above is:

Proposition 2.15. *The ℓ-cubical equivalence relation is the ℓth power of the algebraic equivalence relation, i.e. $F_\ell = L^{*\ell}$*

3 Zero-cycles on an abelian variety

Let A be an abelian variety over an algebraically closed field k. It is well-known that $I = \text{Ker } (\deg\ : CH_0(A) \to \mathbb{Z})$ coincides with the subgroup of zero-dimensional cycles algebraically equivalent to zero.
Our main result is:

Theorem 3.1. *For any $n \geq 0$,*

$$I^{*n} \subseteq L^{*n}CH_0(A)$$

If Conjecture 2.9 (Beauville's Conjecture) is true for abelian varieties over k, then

$$I^{*n} = L^{*n}CH_0(A)$$

In particular, $L^{*2}CH_0(A) = I^{*2} = Ker\ (alb : I \to A)$, and if $n > g = \dim A$, then $L^{*n}CH_0(A) = 0$.

For emphasis, we note that the $*$ on the left represents the Pontryagin power of the ideal I, while the $*$ on the right represents the power of L(=algebraic equivalence) as an (adequate) equivalence relation. Note also that, in contrast to [9], we work with integral, not rational coefficients.

Proof.

When $n = 0$, the statement is trivial, and when $n = 1$, the assertion is that I is equal to the group of cycles algebraically equivalent to zero; this is well-known ([8], 19.3.5). We assume henceforth that $n = 2$. In light of Proposition 2.15, it suffices to prove that $I^{*n} = F_nCH_0(A)$. Note that I^{*n} is generated by elements of the form $c = ([a_1] - [0]) * \ldots * ([a_n] - [0])$.

Let $Z = \{z = (y, x_1, \ldots, x_n) \in A \times_k A^n : y = x_1 + \ldots + x_n\} \subseteq A \times_k A^n$, and define points $p_i^0 = a_i \in A$ and $p_i^1 = 0 \in A$ for $i = 1, \ldots, n$. Direct computation then shows that (in the notation of Definition 2.12):

$$s(p_1^{e_1}, \ldots, p_n^{e_n})^*(Z) = \sum_{i:e_i=0}^{A} a_i$$

(the sum on the right is the group law on the abelian variety)

By Lemma 2.3, the class of the cycle

$$\sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1+\ldots e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*(Z) = \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1+\ldots e_n} \sum_{i:e_i=0}^{A} a_i$$

modulo rational equivalence is equal to $([a_1] - [0]) * \ldots * ([a_n] - [0]) = c$. This suffices to show $c \in F_nCH_0(A)$.

Conversely, suppose $c \in F_nCH_0(A)$. By the remark following Definition 2.12, we may assume that the "parameter varieties" are all abelian varieties. Thus, we are reduced to the situation in which there are abelian varieties $A_1, \ldots, A_n$, a subvariety

$Z \subseteq A \times_k A_1 \times_k \ldots \times_k A_n$ and points $p_i^0, p_i^1 \in A_i$ for $i = 1, \ldots, n$ such that

$$c = \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*(Z)$$

Without loss of generality, we may assume that $p_i^1 = 0 \in A_i$ for all $i = 1, \ldots, n$.
Recall that $s(p_1^{e_1}, \ldots, p_n^{e_n})$ is the natural embedding $A \hookrightarrow A \times_k A_1 \times_k \ldots \times_k A_n$ using the point $(p_1^{e_1}, \ldots, p_n^{e_n}) \in A_1 \times_k \ldots \times_k A_n$. Letting $\tau(p_1^{e_1}, \ldots, p_n^{e_n}) : A_1 \times_k \ldots \times_k A_n \longrightarrow A_1 \times_k \ldots \times_k A_n$ denote the translation map $z \mapsto z + (p_1^{e_1}, \ldots, p_n^{e_n})$, we have

$$s(p_1^{e_1}, \ldots, p_n^{e_n}) = (id_A \times \tau(p_1^{e_1}, \ldots, p_n^{e_n})) \circ s(0, \ldots, 0)$$

For $i = 1, \ldots, n$, define

$$b_i = (0, 0, \ldots, -p_i^0, \ldots, 0) \in A \times_k A_1 \times_k \ldots \times_k A_n$$

where p_i^0 appears in the factor corresponding to A_i.
For convenience, define

$$\beta_i = [b_i] - [(0, 0, \ldots, 0)] \in CH_0(A \times_k A_1 \times_k \ldots \times_k A_n)$$

Then

$$c = \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} s(p_1^{e_1}, \ldots, p_n^{e_n})^*[Z]$$

$$= s(0, \ldots, 0)^* \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} \tau(p_1^{e_1}, \ldots, p_n^{e_n})^*[Z]$$

By the first formula of Lemma 2.3,

$$= s(0, \ldots, 0)^* \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} ([(-p_1^{e_1}, \ldots, -p_n^{e_n})] * [Z])$$

$$= s(0, \ldots, 0)^* \sum_{e_1,\ldots,e_n \in \{0,1\}} (-1)^{e_1 + \ldots e_n} (\sum_{e_i = 0}^{A} [b_i] * [Z])$$

By the second formula of Lemma 2.3,

$$= s(0, \ldots, 0)^* (\beta_1 * \ldots * \beta_n * [Z])$$

Following the notation of Proposition 2.7, let q denote any of the maps $CH^*(\cdot) \to CH^*(\cdot; \mathbb{Q})$ obtained by tensoring with $\mathbb{Q}$. Since each of the zero-cycles β_i has degree

0, $q(\beta_i) \in \oplus_{s\geq 1} CH^*_s(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q})$ by Proposition 2.7. From Beauville's conjecture, $[Z] \in \oplus_{s\geq 0} CH^g_s(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q})$. Thus, by Proposition 2.8,

$$q(\beta_1 * \ldots * \beta_n * [Z]) = q(\beta_1) * \ldots * q(\beta_n) * q([Z]) \in \oplus_{s\geq n} CH^g_s(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q}).$$

Next, applying the second assertion of Proposition 2.8, we conclude that $q(c) \in \oplus_{s\geq n} CH^g_s(A;\mathbb{Q})$; the latter may be identified with J^{*n} by means of Proposition 2.7. For every $n \geq 1$, $q : I \to J$ restricts to a map $q_n : I^{*n} \to J^{*n}$. However, by Roitman's Theorem (Theorem 2.1, part 2) I^{*n} is uniquely divisible for $n \geq 2$, so q_n is an isomorphism and $c \in I^{*n}$ as desired.

Corollary 3.2. *Let C be a smooth projective curve over an algebraically closed field k, and suppose Beauville's conjecture holds for abelian varieties over k. Then $LCH_0(C) = Ker\ (deg\ : CH_0(C) \longrightarrow \mathbb{Z})$ and $L^{*n}CH_0(C) = 0$ for $n \geq 2$.*

Proof.
The first assertion is classical. For the second, let J be the Jacobian of C and $\iota : C \hookrightarrow J$ the associated map. Functoriality of the Albanese map yields a commutative diagram:

$$\begin{array}{ccc} LCH_0(C) & \xrightarrow{alb_C} & \mathrm{Alb}(C)(k) = J(k) \\ \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle =} \\ LCH_0(J) & \xrightarrow{alb_J} & \mathrm{Alb}(J)(k) = J(k) \end{array}$$

Since L^{*n} is adequate, $\iota_*(L^{*n}CH_0(C)) \subseteq L^{*n}CH_0(J) \subseteq L^{*2}CH_0(J) = I^{*2} = (\mathrm{Ker}\ alb_J)$ by Theorem 3.1. By commutativity of the diagram, $alb_C(L^{*n}CH_0(C)) = 0$. However, alb_C is an isomorphism, so $L^{*n}CH_0(C) = 0$.

If we allow ourselves $\mathbb{Q}$-coefficients, the method employed in the second half of the proof of Theorem 3.1 may be modified to prove a more general statement on the "nilpotence" of algebraic equivalence.

Proposition 3.3. *Let A be an abelian variety of dimension g over an algebraically closed field k. If Beauville's conjecture holds, then $L^{*n}CH^*(A;\mathbb{Q}) = 0$ for $n > g$.*

Proof.
As before, we identify L^{*n} with F_n. An argument analogous to that used in the second half of the proof of Theorem 3.1 shows that $F_nCH^*(A;\mathbb{Q})$ is generated by elements of the form

$$c = s(0, 0, \ldots, 0)^*(\beta_1 * \ldots * \beta_n * \zeta)$$

where $A_1, \ldots, A_n$ are "parameter" abelian varieties and $\beta_1, \ldots, \beta_n$ are zero-cycles of degree 0 on $A \times_k A_1 \times_k \ldots \times_k A_n$, hence members of $\oplus_{s \geq 1} CH^*(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q})$. By Beauville's conjecture, $\zeta \in \oplus_{s \geq 0} CH^*_s(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q})$. Now, $\beta_1 * \ldots * \beta_n * \zeta \in \oplus_{s \geq n} CH^*_s(A \times_k A_1 \times_k \ldots \times_k A_n; \mathbb{Q})$; hence $s(0, 0, \ldots, 0)^*(\beta_1 * \ldots * \beta_n * \zeta) \in \oplus_{s \geq n} CH^*_s(A; \mathbb{Q})$. By Theorem 2.6, we have $CH^p_s(A; \mathbb{Q}) = 0$ if $s > p$, so when $n > g$, $c = 0$ as desired.

Remark.
It is possible to strengthen Proposition 3.3 so that its conclusion holds integrally. The Fourier transform $\mathcal{F} : CH^*(A; \mathbb{Q}) \longrightarrow CH^*(\hat{A}; \mathbb{Q})$ involves composition with the correspondence $\sum_{i=0}^{\infty} \frac{\ell^i}{i!} = \sum_{i=0}^{2g} \frac{\ell^i}{i!}$. If we choose N to be a multiple of $(2g)!$, we may define (integrally) maps $\mathcal{F}_N = N\mathcal{F} : CH^*(A) \longrightarrow CH^*(\hat{A})$ and $\hat{\mathcal{F}}_N = N\hat{\mathcal{F}} : CH^*(\hat{A}) \longrightarrow CH^*(A)$ with properties analogous to those of Proposition 2.4 Using these properties, together with the fact that the group $LCH^*(A)$ is divisible (cf. [8], Example 19.1.2), the techniques in the proof of Proposition 3.3 may be adapted to show that $L^{*n}CH^*(A) = 0$ for $n > g$.

4 Filtrations in the relative setting

In this section, we investigate a version of the formula $F^r = (F^1)^{*r}$ in a relative setting. We first recall the following theorem giving a Künneth decomposition of the class of the diagonal of an abelian variety. In the interest of keeping the exposition self-contained, we will refrain from explicit mention of Chow motives and instead refer the reader to [5] and [11] for details.

Theorem 4.1. *(Deninger-Murre, Theorem 3.1; Künnemann), Theorem 3.1.1)*
Let S be a smooth quasiprojective scheme over a base field k and B/S an abelian scheme of fiber dimension g. Let Δ_B be the diagonal of B; that is, the graph of the identity morphism $B \longrightarrow B$. There is a unique decomposition:

$$[\Delta_B] = \sum_{i=0}^{2g} \pi_i \text{ in } CH^g(B \times_S B)$$

such that $(1 \times \mathbf{n})^* \pi_i = n^i \pi_i$ for each i and all $n \in \mathbb{Z}$. Furthermore, $\pi_i \circ \pi_j = 0$ for $i \neq j$, and for all i, $\pi_i \circ \pi_i = \pi_i$. Also, $s^*(\pi_i) = \pi_{2g-i}$, where $s : B \times_S B \longrightarrow B \times_S B$ is the exchange of factors.

In fact, Künnemann has given the following explicit formula for π_i ([11], p. 200):

$$\pi_i = \frac{1}{(2g-i)!}(\log[\Delta])^{*(2g-i)}$$

where

$$\log([\Delta]) = \sum_{m=1}^{\infty}(-1)^m \frac{([\Delta]-[\Gamma_e])^{*m}}{m},$$

Γ_e is the graph of the map $B \longrightarrow B$ sending everything to the identity section of B, and $*$ represents Pontryagin product on $B \times_S B$, considered as an abelian B-scheme via projection on the first factor. Only finitely many of the terms in the series defining $\log([\Delta])$ are nonzero (cf. [11], Theorem 1.4.1), so this expression is well-defined.

It follows readily from the definitions that $CH^g(B \times_S B)$ is a (noncommutative) ring under composition of correspondences; the above theorem asserts that the unit element for this ring structure may be decomposed as a sum of mutually orthogonal idempotents ("projectors"), each of which is an eigenvector for the maps $1 \times \mathbf{n}$.

Now let k be any field and A a (fixed) abelian variety over k; set $g = \dim A$. Let $\mathcal{V}_k/A$ denote the full subcategory of $\mathcal{V}_k$ consisting of objects of the form $T \times_k A$ where T is any smooth projective variety over k; morphisms are of the form $f \times 1 : S \times_k A \longrightarrow T \times_k A$, where $f : S \longrightarrow T$ is a morphism in $\mathcal{V}_k$.. Viewing $T \times_k A$ as an abelian T-scheme via projection on the first factor, Theorem 2.6 gives a decomposition (in which some of the eigenspaces may be zero):

$$CH^p(T \times_k A; \mathbb{Q}) \cong \bigoplus_{s=max(p-g,2p-2g)}^{2p} CH^p_s(T \times_k A; \mathbb{Q})$$

For emphasis, we note:

$$CH^p_s(T \times_k A; \mathbb{Q}) = \{\alpha \in CH^p(T \times_k A; \mathbb{Q}) : (1 \times \mathbf{n})^* \alpha = n^{2p-s}\alpha\}$$

The following statement relates the eigenspaces to composition (as correspondences) with the projectors defined above.

Proposition 4.2. *For any i, $0 \le i \le 2g$ and any p,*

$$\pi_i \circ CH^p(T \times_k A; \mathbb{Q}) = CH^p_{2p-i}(T \times_k A; \mathbb{Q})$$

where π_i is interpreted as a correspondence from A to A and elements of $CH^p(T \times_k A; \mathbb{Q})$ are interpreted as correspondences from T to A.

Proof.
Let p_{ij} denote the projection from map from $T \times_k A \times_k A$ onto the (i,j)th factor. Then for $\alpha \in CH^p(T \times_k A; \mathbb{Q})$,

$$(1 \times \mathbf{n})^*(\pi_i \circ \alpha) = (1 \times \mathbf{n})^*(p_{13*}(p_{12}^*\alpha \cdot p_{23}^*\pi_i))$$

$$= p_{13*}((1 \times 1 \times \mathbf{n})^*(p_{12}^*\alpha \cdot p_{23}^*\pi_i))$$

$$= p_{13*}(p_{12}^*\alpha \cdot p_{23}^*(1 \times \mathbf{n})^*\pi_i)$$

$$= n^i(\pi_i \circ \alpha)$$

Thus, $\pi_i \circ CH^p(T \times_k A; \mathbb{Q}) \subseteq CH^p_{2p-i}(T \times_k A; \mathbb{Q})$.

For the other inclusion, observe that

$$CH^p_{2p-i}(T \times_k A; \mathbb{Q}) = [\Delta] \circ CH^p_{2p-i}(T \times_k A; \mathbb{Q}) = \sum_{j=0}^{2g} \pi_j \circ CH^p_{2p-i}(T \times_k A; \mathbb{Q})$$

From the inclusion just proved, $\pi_j \circ CH^p(T \times_k A; \mathbb{Q}) \subseteq CH^p_{2p-j}(T \times_k A; \mathbb{Q})$; hence

$$\sum_{j=0}^{2g} \pi_j \circ CH^p_{2p-i}(T \times_k A; \mathbb{Q}) = \pi_i \circ CH^p_{2p-i}(T \times_k A; \mathbb{Q}) = \pi_i \circ CH^p(T \times_k A; \mathbb{Q})$$

as desired.

Our main result is:

Theorem 4.3. *For each integer $r \geq 0$ and each $T \in V_k$, define a filtration $F^\cdot$ on $CH^*(T \times_k A; \mathbb{Q})$ by viewing $T \times_k A$ as a T-scheme and setting*

$$F^r CH^p(T \times_k A; \mathbb{Q}) = \bigoplus_{s \leq 2p-r} CH^p_s(T \times_k A; \mathbb{Q}).$$

Then:

1. *For all $a, b \geq 0$, $F^a \cdot F^b \subseteq F^{a+b}$. Furthermore, for any $r \geq 0$, F^r is an adequate equivalence relation on $\mathcal{V}_k/A$.*

2. *F^r coincides with $(F^1)^{*r}$ (as adequate equivalence relations on $\mathcal{V}_k/A$).*

Proof.
We prove first that F^r is preserved under pullbacks and pushforwards; then we show $F^a \cdot F^b \subseteq F^{a+b}$ for all $a, b \geq 0$. This will suffice to show that F^r is adequate.
Let $f : T \times_k A \longrightarrow S \times_k A$ be a morphism in $\mathcal{V}_k/A$. Set $d_T = \dim T$, $d_S = \dim S$. Then for $\alpha \in CH^p_s(S \times_k A; \mathbb{Q})$, we have

$$(1_T \times \mathbf{n})^* f^*(\alpha) = f^*(1_S \times \mathbf{n})^*(\alpha) = n^{2p-s} f^*(\alpha)$$

Thus,

$$f^*(F^r CH^p(S \times_k A; \mathbb{Q})) = f^*(\bigoplus_{s \leq 2p-r} CH^p_s(S \times_k A; \mathbb{Q}))$$

$$\subseteq \bigoplus_{s \leq 2p-r} CH^p_s(T \times_k A; \mathbb{Q}) = F^r CH^p(T \times_k A; \mathbb{Q})$$

Furthermore, for $\beta \in CH^p_t(T \times_k A; \mathbb{Q})$, we have

$$(1_S \times \mathbf{n})^* f_*(\beta) = f_*(1_T \times \mathbf{n})^* \beta = n^{2p-t} f_*(\beta) \in CH^{p+d_S-d_T}_{t+2(d_S-d_T)}(S \times_k A; \mathbb{Q})$$

Therefore

$$f_*(F^r CH^p(T \times_k A; \mathbb{Q})) = f_*(\bigoplus_{t \leq 2p-r} CH^p_t(T \times_k A; \mathbb{Q}))$$

$$\subseteq \bigoplus_{t \leq 2(p+d_S-d_T)-r} CH^p_s(S \times_k A; \mathbb{Q}) = F^r CH^{p+d_S-d_T}(S \times_k A; \mathbb{Q})$$

Finally, if $\alpha \in CH^p_s(S \times_k A)$ and $\beta \in CH^q_t(S \times_k A)$, then

$$(1 \times \mathbf{n})^*(\alpha \cdot \beta) = (1 \times \mathbf{n})^*(\alpha) \cdot (1 \times \mathbf{n})^* \beta = n^{2(p+q)-(s+t)}(\alpha \cdot \beta)$$

Thus,

$$F^a CH^p(S \times_k A; \mathbb{Q}) \cdot F^b CH^q(S \times_k A; \mathbb{Q}) = \bigoplus_{s \leq 2p-a} CH^p_s(S \times_k A; \mathbb{Q}) \cdot \bigoplus_{t \leq 2q-b} CH^q_t(S \times_k A; \mathbb{Q})$$

$$\subseteq \bigoplus_{u \leq 2(p+q)-(a+b)} CH^{p+q}_u(S \times_k A; \mathbb{Q}) = F^{a+b} CH^{p+q}_u(S \times_k A; \mathbb{Q}).$$

For the second assertion, we need the following:

Lemma 4.4. $\pi_i \in (F^1)^{*(2g-i)} CH^g(A \times_k A; \mathbb{Q})$.

By Künnemann's formula (following Theorem 4.1), formal properties of the logarithm imply that:

$$\pi_{2g-r} = \frac{1}{r!}\pi_{2g-1}^{*r}$$

in which $*$ represents Pontryagin product on $A \times_k A$, considered as an A-scheme via projection on the first factor.

Since $(1 \times \mathbf{n})^* \pi_i = n^i \pi_i$ by Theorem 4.1, it follows that $\pi_i \in CH^g_{2g-i}(A \times_k A; \mathbb{Q}) \subseteq F^{2g-i}CH^g_{2g-i}(A \times_k A; \mathbb{Q})$; in particular, $\pi_{2g-1} \in F^1 CH^g(A \times_k A; \mathbb{Q})$.

Since we are considering $A \times_k A$ as an A-scheme via projection on the first factor, the dual abelian scheme for this structure is $A \times_k \hat{A}$. Denote by

$$\mathcal{F} : CH^*(A \times_k A; \mathbb{Q}) \longrightarrow CH^*(A \times_k \hat{A}; \mathbb{Q}), \;\; \hat{\mathcal{F}} : CH^*(A \times_k \hat{A}; \mathbb{Q}) \longrightarrow CH^*(A \times_k A; \mathbb{Q})$$

the various Fourier transforms of 2.4 for this structure.

Let p_{ij} denote the various projections from $A \times_k A \times_k \hat{A}$ onto two factors. Let $F = e^{\ell}$ as in Propositon 2.4. Then

$$\mathcal{F}(\pi_{2g-1}) = p_{13*}(p_{12}^* \pi_{2g-1} \cdot (1 \times F)) \in CH^*(A \times_k \hat{A}; \mathbb{Q})$$

Since $\pi_{2g-1} \in F^1 CH^*(A \times_k A; \mathbb{Q})$ and F^1 is adequate, it follows from the above formula that $\mathcal{F}(\pi_{2g-1}) \in F^1(CH^*(A \times_k \hat{A}; \mathbb{Q}))$.
Thus for any $i \geq 1$, 2.4 implies:

$$\mathcal{F}(\pi_{2g-i}) = \mathcal{F}(\frac{1}{i!}\pi_{2g-1}^{*i}) = \frac{1}{i!}(\mathcal{F}(\pi_{2g-1}))^{\cdot i} \in (F^1)^{*i} CH^*(A \times_k \hat{A}; \mathbb{Q}))$$

by the definition of the product of equivalence relations.
Finally, because $(F^1)^{*i}$ is adequate (by Proposition 2.14), we have

$$\pi_{2g-i} = (-1)^g \sigma^* \hat{\mathcal{F}}(\mathcal{F}(\pi_{2g-i})) \in (F^1)^{*i} CH^*(A \times_k \hat{A}; \mathbb{Q})$$

which completes the proof of the Lemma.

Returning to the proof of Theorem 4.3, the inclusion $(F^1)^{*r} \subseteq F^r$ may be proved by induction on r, the case $r = 1$ being trivial. Evidently, $(F^1)^{*r} = (F^1)^{*(r-1)} * F^1$, which by the induction hypothesis equals $F^{r-1} * F^1$. Now if $\gamma \in (F^{r-1} * F^1)CH^*(S \times_k A)$, there exists a smooth projective variety T and elements $\alpha \in F^{r-1}CH^*(T \times_k S \times_k A)$, $\beta \in F^1 CH^*(T \times_k S \times_k A)$ such that $\gamma = p_*(\alpha \cdot \beta)$ where $p : T \times_k S \times_k A \longrightarrow S \times_k A$ is the projection map. From the first statement of Theorem 4.3, it is clear that

$\alpha \cdot \beta \in F^r(T \times_k S \times_k A)$, and, since F^r is adequate, it follows that $\gamma = p_*(\alpha \cdot \beta) \in F^r CH^*(S \times_k A)$.

For the reverse inclusion, suppose

$$\alpha \in F^r CH^p(S \times_k A; \mathbb{Q}) = \bigoplus_{s \leq 2p-r} CH^p_s(S \times_k A; \mathbb{Q}) = \bigoplus_{s \leq 2p-r} \pi_{2p-s} \circ CH^p(S \times_k A; \mathbb{Q}),$$

the last equality by Proposition 4.2.
Since $\pi_{2p-s} \in (F^1)^{*(2g-2p+s)} CH^g(A \times_k A; \mathbb{Q})$ by Lemma 4.4, it follows from adequacy of $(F^1)^{*(2g-2p+s)}$ that $\pi_{2p-s} \circ CH^p(S \times_k A; \mathbb{Q}) \subseteq (F^1)^{*(2g-2p+s)} CH^p(S \times_k A; \mathbb{Q})$.
Thus

$$\alpha \in \bigoplus_{s \leq 2p-r} (F^1)^{*(2g-2p+s)} CH^p(S \times_k A; \mathbb{Q}) = \bigoplus_{t \geq r} (F^1)^{*t} CH^p(S \times_k A; \mathbb{Q})$$

$$\subseteq (F^1)^{*r} CH^p(S \times_k A; \mathbb{Q})$$

as desired.

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