

Group actions and Messiaen's modes of limited transposition

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Introduction

The intimate relationship between mathematics and music has been recognized, investigated, and celebrated since ancient times. Pythagoras, who was not only a mathematician, but also a composer, music theorist, and instrumentalist, studied the connection between musical intervals and rational numbers [1]. On the other end of the continent, the Indian mathematician Piṅgala used combinatorics to analyze metrical patterns in Sanskrit poetry [2]. Before distinct academic disciplines began to take form in the nineteenth century, it was quite common for musicians to maintain a keen interest in mathematics and vice versa. Papadopoulos notes in particular the correspondence among the composer Rameau and the mathematicians Euler and d'Alembert. [1] Closer to our own time, the American serialist composer Milton Babbitt was noted for his use of set theory and combinatorics in musical analysis and composition ([3], [4]).

The French composer Olivier Messiaen (1908-1992) was also greatly influenced by mathematics, although in a different way. Messiaen likely did not have much training in higher mathematics [1], yet his work reveals a profound respect for the role of mathematics in music. One often finds him grappling with fairly sophisticated concepts (for example, the order of a permutation) in his work, but from outside the formalism of advanced mathematics. In 1944, Messiaen published *The Technique of my Musical Language*, an exposition of the theory behind his music. This work was intended as a service to his students and also to correct various misconceptions that had arisen concerning his own (musical) intentions. In the last of its three sections, Messiaen discusses tonal systems and introduces his now-famous “modes of limited transposition.” Along the way, he makes a curious claim about the number of such modes, which – if interpreted literally – is not true. Equally intriguing is another comment he makes about a generalization of his modes to a tonal system in which the octave is divided into quarter-tones rather than semitones.

Two questions immediately arise: what could Messiaen have meant in making his claim as stated, and how might mathematics be put to use to address the enumeration problem to which he drew attention? The objective of this article is to address these questions, along with suitable generalizations of them, using tools from elementary number theory. We begin with a quick survey of Messiaen's life and work to situate him within the history of Western classical music and to give some sense of how mathematics fit into his conception of music. This is followed by a very brief overview of concepts from music theory and a discussion of Messiaen's classification of modes of limited transposition. The last three sections are concerned with mathematics and address the aforementioned questions about enumeration at suitable levels of generality.

In preparing this article, we have tried to assume as little as possible about music theory. Readers conversant with music theory can easily skim large parts of the

third section. Likewise, those familiar with Möbius inversion could probably skip the mathematical preliminaries. The main mathematical results are found in the last two sections of the paper. For a sampling of applications of enumeration to problems in music theory beyond those addressed in this article, the reader is directed to the works [5], [6], [7], [8] and [9].

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Olivier Messiaen

The turn of the twentieth century was an era of considerable musical innovation. In the second half of the nineteenth century, the tonal system that had dominated European classical music since 1600 had been challenged and stretched to its limits. By the time Messiaen was born in Avignon, France in 1908, Romanticism had largely given way to new stylistic movements. Some of them, like Impressionism, retained elements of traditional European music – scales and chords, for instance – but employed them in highly innovative ways. Others, like Expressionism, discarded the tonal system in favor of a serialist approach.

Messiaen entered the Paris Conservatory at the age of eleven, eventually studying organ under Marcel Dupré and composition under Paul Dukas. He was appointed assistant organist at the Église de la Sainte-Trinité in Paris in 1929 and promoted to organist in 1931, a position he held until his death. It is not surprising, therefore, that a substantial proportion of his output consists of sacred music, or at least music inspired by religious themes. In 1966, Messiaen was appointed professor of composition at the Paris Conservatory. By the time he retired from that post in 1978, he was an internationally recognized composer. [10] He was also a committed ornithologist and drew considerable inspiration from birdsong.

In his writing, Messiaen makes clear that his aim was not to abandon the principles of Western music but rather to enrich them. Indeed, he drew upon many sources – musical and non-musical, Western and non-Western – for inspiration. In the preface to *The Technique of my Musical Language*, he credits not only his teachers and family, but also Shakespeare, birds, Russian music, plainchant, “Hindu rhythmic”, and the mountains of Dauphiné. Mathematics makes its entrance in the very first chapter:

It is a glistening music we seek, giving to the aural sense voluptuously refined pleasures. At the same time, this music should be able to express some noble sentiments (and especially the most noble of all, the religious sentiments exalted by the theology and the truths of our Catholic faith). This charm, at once voluptuous and contemplative, resides particularly in certain mathematical impossibilities of the modal and rhythmic domains. Modes which cannot be transposed beyond a certain number of transpositions, because one always falls again into the same notes; rhythms which cannot be used in retrograde, because in such a case one finds the same order of values again – these are two striking impossibilities. [11, p.13]

By “mathematical impossibilities”, Messiaen is referring to some form of symmetry. This could be symmetry of rhythm, which he considers the horizontal dimension of music, or of pitch, which he considers its vertical dimension. A devout Catholic whose faith was the primary inspiration for his work, Messiaen believed that this cruciform symmetry within music enhanced its effect not only on the emotions but ultimately on

the very spirit of the listener, leading the latter to an imperfect but nevertheless true image of God. Later in his work, Messiaen describes the effect of these “impossibilities” on the listener:

...in spite of himself, he will submit to the strange charm of impossibilities: a certain effect of tonal ubiquity in the non-transpositions, a certain unity of movement (where beginning and end are confused because identical) in the non-retrogradation, all things which will lead him progressively to that sort of *theological rainbow* which the musical language, of which we seek edification and theory, seeks to be. [11, pp.21,63]

For Messiaen, therefore, mathematics was not merely a constituent of art, but an engine that helped it achieve its end. Readers interested in listening to works of Messiaen composed according to these principles are directed to the motet *O Sacrum Convivium* or the last two movements of the organ cycle *La Nativité du Seigneur*. Those interested in the role mathematics plays in other aspects of Messiaen’s music (i.e., beyond tonal considerations) are referred to Messiaen’s original work [11] or the survey [1].

Background from music theory

Tuning and pitch classes The Pythagoreans believed, based on philosophical considerations, that music was the foundation of mathematics. [12, p.114] This led them to examine the sounds produced by strings, the ratio of whose lengths is some fraction involving only powers of small primes. If that ratio is 2:1 (and hence the proportion of the frequencies is 1:2), the interval so produced is called the *octave*. With a ratio of 3:2, one obtains the *perfect fifth*, and with a ratio of 4:3 one gets the *perfect fourth*. Extended consideration of these and other ratios led to the development of the system of *natural tuning*. The limitations of this system can easily be seen: twelve perfect fifths should be equivalent to seven octaves; however, $2^7 = 128$ and $\left(\frac{3}{2}\right)^{12} \approx 129.746$. This discrepancy does not present too serious a problem for music played on a single instrument with a limited range. However, for an instrument with a wide range – or even music involving multiple instruments playing in different registers – tuning becomes a challenge.

One solution to the problem posed by natural tuning is to use *equal temperament*, the system used nowadays to tune most keyboard instruments. The idea behind equal temperament is to make the proportion of frequencies associated to a semitone (the smallest interval found in most Western music) equal to $\sqrt[12]{2}$. As there are twelve semitones in an octave, this results in an octave identical to that produced by natural tuning. However, because $\alpha = \sqrt[12]{2}$ is irrational, intervals that are not integer multiples of an octave will differ from those produced by natural tuning. For instance, there are five semitones in a perfect fourth and seven semitones in a perfect fifth, so equal temperament assigns the fourth a proportion of $\alpha^5 \approx 1.335$, slightly wider than the natural fourth. On the other hand, the fifth receives a proportion of $\alpha^7 \approx 1.498$, slightly narrower than the natural fifth. These calculations illustrate why it makes sense to subdivide the octave into twelve subintervals: powers of $2^{1/12}$ approximate fairly well the ratios that arise in natural tuning. As we will see later, it also helps that 12 has a plenitude of integer divisors.

Among the advantages offered by equal temperament is that once one fixes a particular note at a base frequency f , the frequency of every other note in the twelve-tone

system is uniquely determined; these are

$$S_f = \{\alpha^n f : n \in \mathbb{Z}\}.$$

For instance, it is current convention for (most) orchestras to have “A above middle C” tuned to 440 Hz. This means that the respective frequencies of the notes in the octave immediately above that note are (in ascending order):

$$440, 440\alpha, 440\alpha^2, \dots, 440\alpha^{11}, 440\alpha^{12} = 880.$$

These notes are named

$$A, A\sharp = B\flat, B, C, C\sharp = D\flat, D, D\sharp = E\flat, E, F, F\sharp = G\flat, G, G\sharp = A\flat, A.$$

The pattern repeats cyclically in adjacent octaves on either side, so that the sequence of *all* notes looks like this:

$$\dots F\sharp, G, A\flat, A, B\flat, B, C, D\flat, D, E\flat, E, F, F\sharp, G, A\flat, A, B\flat, B, C, \dots$$

The reader familiar with music theory may be wondering why we chose to write $D\flat$ on the line immediately above, rather than the enharmonically equivalent $C\sharp$. The reason is that, for ease of comparison, we have adopted the same notation Messiaen uses in his book.

Thus, A – without further qualification – refers not simply to the particular note whose frequency is 440 Hz but to the set of notes corresponding to the frequencies $\{2^n \cdot 440 : n \in \mathbb{Z}\}$. In mathematical terms, a frequency is merely a positive real number. We therefore define an equivalence relation on $(0, \infty)$ by

$$g \sim h \text{ if and only if } g = 2^n h \text{ for some } n \in \mathbb{Z}$$

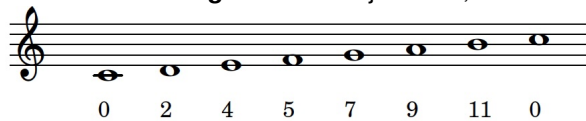
and call an equivalence class for this relation a *pitch class*.

Now let f be a chosen base frequency, and let $P_0, P_1, \dots, P_{11}$ denote the pitch classes represented by the respective frequencies $f, 2^{1/12}f, \dots, 2^{11/12}f$. In view of the obvious bijection between $\{P_i : 0 \leq i \leq 11\}$ and the group $\mathbb{Z}_{12}$ of integers modulo 12, we will henceforth consider pitch classes simply as elements of $\mathbb{Z}_{12}$. It cannot be emphasized enough that, however convenient it might be to use the “note names” $A, B\flat$, etc. to label pitch classes, the concept of a pitch class is itself independent of note names, or even of the choice of base frequency. For example, if we choose C as the pitch class 0, then $D\flat$ corresponds to 1, D to 2, and so on. However, if we choose $F\sharp$ as the pitch class 0, then G corresponds to 1, $A\flat$ to 2, A to 3, and so on.

For the balance of the article, we assume that all pitch classes are defined relative to some fixed base frequency.

Systems of tonality Readers with musical training have likely encountered *scales*. Roughly speaking, scales are tonal systems upon which most pieces of Western art music written between 1600 and 1910 were constructed. We will discuss the C major scale below, but only as an example to illustrate pitch classes and the relationships among them. Our ultimate interest lies in other systems of tonality.

The C major scale consists (within the span of one octave) of the note sequence C, D, E, F, G, A, B, C. If we take C to be the pitch class 0, then this scale can be represented by the sequence 0, 2, 4, 5, 7, 9, 11, 0 of pitch classes, as shown in Figure 1.

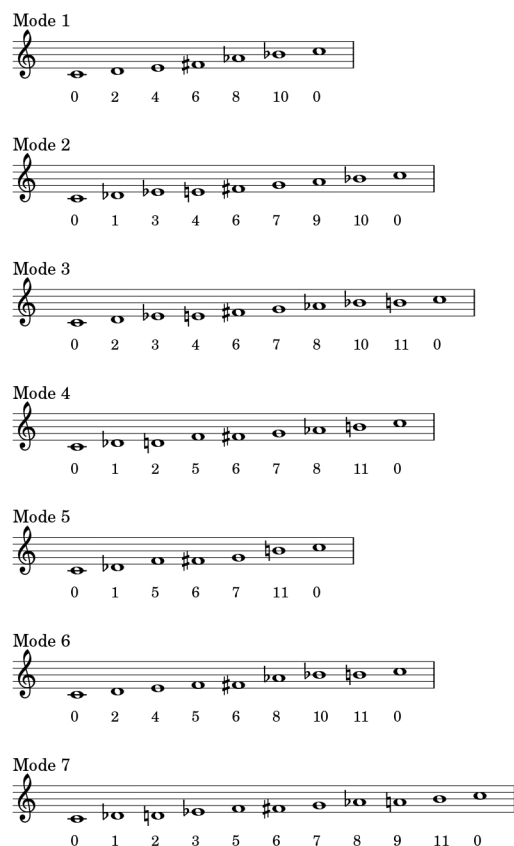
Figure 1 C major scale, labeled with pitch classes

A key feature of this example is that each pitch class in the scale bears a unique relationship to all the others. If we begin at pitch class 0, the sequence of gaps between successive pitches in the scale is 2, 2, 1, 2, 2, 2, 1. If we begin at pitch class 4, however, the sequence of gaps become 1, 2, 2, 2, 1, 2, 2. It is not hard to verify that beginning the process at a different pitch class will yield a different sequence of gaps each time. Thus, each pitch class plays a distinct “role” in the scale with respect to the others. (Readers familiar with music theory will recognize pitch class 0 as the tonic, pitch class 7 as the dominant, and so on.) One might say, loosely speaking, that the set of pitch classes in the major scale is lacking in symmetry.

In contrast, Messiaen was most interested in tonal systems that, unlike the major scale, exhibit a high degree of symmetry. Consider, for instance, the whole-tone scale, identified as Mode 1 in Figure 2. There the sequence of pitch classes is 0, 2, 4, 6, 8, 10, 0; hence, the sequence of gaps is 2, 2, 2, 2, 2, 2. It is obvious that beginning at any other pitch class yields the same sequence of gaps. In contrast to the major scale, the set $\{0, 2, 4, 6, 8, 10\}$ of pitch classes possesses a high degree of symmetry. Moreover, if one *transposes* this sequence of pitch classes – in mathematical terms, this amounts to adding some constant (modulo 12) to each pitch class – the only other set of pitch classes thus obtained is $\{1, 3, 5, 7, 9, 11\}$. This stands in stark contrast to the example of the major scale, whose set $\{0, 2, 4, 5, 7, 9, 11\}$ of pitch classes has twelve distinct transpositions. In general, we say that a nonempty, proper subset $S \subseteq \mathbb{Z}_{12}$ has *limited transposition* if the set $\{a + S : a \in \mathbb{Z}_{12}\}$ of its translates has fewer than twelve elements.

Our next task is to give a rigorous definition of the term *mode*. It is easy to see that $\mathbb{Z}_{12}$ acts by translation on the set of nonempty, proper subsets of $\mathbb{Z}_{12}$: explicitly, the action of $a \in \mathbb{Z}_{12}$ on $S \subseteq \mathbb{Z}_{12}$ is given by $a + S = \{a + s : s \in S\}$. In musical terms, this translation action corresponds to *transposing* pitches, i.e., increasing or lowering them by a fixed number of semitones, keeping in mind that pitches separated by an octave are considered equivalent. For example, transposing the sequence A, B $\flat$, F, D up one semitone yields B $\flat$, B, F $\sharp$, E $\flat$; transposing that same sequence down two semitones yields G, A $\flat$, E $\flat$, C. A *mode* is simply an orbit for this action. We can therefore extend our definition of limited transposition: we say that a mode has limited transposition if any subset of $\mathbb{Z}_{12}$ representing it has limited transposition. This is equivalent to requiring that the mode (as an orbit) contain fewer than 12 elements, or that the stabilizer of any set within that orbit be nontrivial.

Before proceeding further, several words of caution are in order. The first is that our definition of mode has nothing to do with its typical use in music theory, i.e., in reference to tonal systems of Ancient Greek or medieval music. The second is that Messiaen’s use of the word “mode” differs slightly from ours. While we use *mode* to refer to an orbit of the action just defined, Messiaen tends to use it in reference to a representative of one of those orbits. Nevertheless, for ease of exposition we will sometimes abuse terminology, using phrases like “mode with six pitch classes” when in fact we mean “a mode, any one of whose representatives has six pitch classes”. We will revisit these definitions in greater generality in the last two sections.

Figure 2 Modes of limited transposition identified by Messiaen

Messiaen's classification

In Chapter XVI of [11], Messiaen claims to classify modes of limited transposition. Figure 2 shows the modes Messiaen identified on a staff, with C representing the pitch class 0. Messiaen further asserts: “Their series is closed. It is mathematically impossible to find others of them, at least in our tempered system of twelve semitones.” [11, p. 58] What precisely he meant by that is less clear. Nonetheless, three things are evident from Messiaen’s “classification.” The first is that he does not consider the (orbit of the) full set $S = \mathbb{Z}_{12}$ a mode of limited transposition, even though its stabilizer with respect to the action defined above is nontrivial. The second is that because Messiaen aims to use these modes as tonal systems for composition (loosely understood), he seems uninterested in those containing fewer than six pitch classes. The third and most troubling issue is that Messiaen’s classification seems incomplete. There is no problem with his “first mode” (Mode 1 in Figure 2). Clearly $\{0, 2, 4, 6, 8, 10\}$ is the

only subset of $\mathbb{Z}_{12}$ (up to translation) with a stabilizer of size 6. Mode 2 is in fact *not* the only mode with stabilizer of size 4. Still, the only other one corresponds to the set $\{0, 3, 6, 9\}$ of pitch classes, which in mathematical terms is the unique subgroup of $\mathbb{Z}_{12}$ of order 4 and in music corresponds to the diminished seventh chord. Messiaen seems to acknowledge this implicitly ([11, p.59]), so one could justify its exclusion from the list on the ground that it contains too few pitch classes. One can similarly argue for the exclusion of the augmented triad from the list of modes with stabilizer 3. [11, p.59] However, Donald Street [13, p.819] notes that Messiaen omits without mention the mode corresponding to the set $\{0, 3, 4, 7, 8, 11\}$, whose stabilizer clearly has size 3. The exclusion of this mode is puzzling, especially in view of Messiaen's inclusion of Mode I, which also has six pitch classes. On what basis does he retain some modes with six pitch classes but discard others? A comment he makes upon furnishing an example (from his own work) of the use of the fifth mode provides a hint: "This mode 5, being a truncated mode 4, has the right of quotation here only because it engenders the melodic formula (already seen in Chapter X):" [11, p.62]. The observation that mode 5 is a "truncation" of mode 4 is by itself hardly compelling as an explanation for the exclusion of the missing mode. In fact, many of the modes Messiaen lists are truncations of others in that same sense. It seems, rather, that he is applying both aesthetic and mathematical criteria to identify modes acceptable to him for use.

In view of Messiaen's choices – including his apparent ambivalence regarding modes with exactly six pitch classes – it seems natural to study the following generalization of the mathematical question he considered. Given an integer n and an equal subdivision of the octave into n pitch classes, how many modes of limited transposition are there, and can those be classified? This question has been of interest to music theorists for some time. Messiaen himself mentions such a classification for quarter-tones (the case $n = 24$) but laments that he "cannot busy himself here with it." [11, p.58-59] (Perhaps the margin of his page was too small to accommodate all his thoughts on the topic.) Other studies (see [14]) provide algorithms to compute modes of limited transposition in the case of n subdivisions. Jedrzejewski [15, p.77, Theorem 94] applies Pólya's Theorem to provide a formula for the number of such modes. (Jedrzejewski's definition of mode differs very slightly from ours in that he considers the subset of all pitch classes to have limited transposition, so his calculation yields one more than ours.) Related questions have been studied in [16]. Our contribution is to apply techniques from elementary number theory to derive an expression for the number of modes of limited transposition in terms of common arithmetic functions.

Mathematical preliminaries

In preparation for the main result, we recall some properties of the Möbius function $\mu : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{if } n \text{ is not squarefree.} \end{cases}$$

Instead of developing the theory from scratch, we will direct the reader to [17] for details and simply state the main result, the so-called Möbius Inversion Theorem.

Theorem 1. (See [17, Theorem 2, p. 20]) Let $f : \mathbb{Z}^+ \rightarrow \mathbb{C}$ be a function. Define $F : \mathbb{Z}^+ \rightarrow \mathbb{C}$ by $F(n) = \sum_{d|n} f(d)$, where the sum is over all positive integer divisors d .

Then

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d).$$

Recall also Euler's totient function $\phi : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, defined by

$$\phi(n) = |\{a : 1 \leq a \leq n, \gcd(a, n) = 1\}|.$$

We now apply Theorem 1 to derive a formula for $\phi(n)$. Given n , define, for every positive divisor d of n , $T_d = \{m \in \mathbb{Z} : 1 \leq m \leq n, \gcd(m, n) = d\}$. Then $T_d = \{cd : c \in \mathbb{Z}, 1 \leq c \leq \frac{n}{d}, \gcd(c, \frac{n}{d}) = 1\}$, so by definition $|T_d| = \phi\left(\frac{n}{d}\right)$. In view of the disjoint union $\{1, \dots, n\} = \bigsqcup_{d|n} T_d$ and the fact that d divides n if and only if n/d divides n , we have $n = \sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} \phi(d)$. Applying Theorem 1 (with $f(n) = \phi(n)$) then yields

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}. \quad (1)$$

In the next section, we will use Theorem 1 repeatedly in the proof of our main result.

Enumerating modes of limited transposition

Returning to the discussion of Messiaen's work, it is evident that the modes he was interested in are orbits under translation of nonempty, proper subsets of $\mathbb{Z}_{12}$ containing at least six (perhaps at least seven) elements. We wish to generalize Messiaen's idea to tonal systems in which the octave is split into n subintervals of equal size, where $n \geq 2$.

For any integer $n \geq 2$, the group $\mathbb{Z}_n$ acts by translation on the set of nonempty, proper subsets of itself. Note that for each k , $1 \leq k \leq n-1$, there is an induced subaction of $\mathbb{Z}_n$ on $P_{k,n} = \{S \subseteq \mathbb{Z}_n : |S| = k\}$. The stabilizer (for this action) of a subset $S \subseteq \mathbb{Z}_n$ is denoted $\text{Stab}(S)$, and we call S *primitive* if $\text{Stab}(S)$ is the trivial subgroup. We then define an n -*mode* (or simply *mode*, if there is no ambiguity) to be an orbit for this action. The *size* of a mode is the cardinality of any subset representing the orbit, and the *transposition length* of that mode is the cardinality of the orbit itself.

A mode is said to be of *limited transposition* if its transposition length is strictly less than n ; this is equivalent to requiring it to be the orbit of some subset S such that $\text{Stab}(S)$ is nontrivial. A mode of limited transposition is a *strong Messiaen mode* if its size is strictly greater than $n/2$ or a *weak Messiaen mode* if its size is exactly $n/2$.

Our goal is to enumerate modes of limited transposition, as well as strong and weak Messiaen modes. Our method follows the algorithm of Baratè and Ludovico [14] to arrive at a formula in closed form. A key observation is that, given a subgroup H of an abelian group G , taking the quotient by H establishes a bijection between $\{S \subseteq G : \text{Stab}(S) = H\}$ and $\{T \subseteq G/H : T \text{ is primitive}\}$. This correspondence allows us to reduce the problem of enumerating the modes of limited transposition to that of enumerating primitive subsets.

Theorem 2. *Let $n \geq 2$ be an integer, and let ℓ_n denote the number of n -modes of limited transposition. Let s_n and w_n denote, respectively, the number of strong and weak Messiaen n -modes. Then*

1. For all n , $\ell_n = w_n + 2s_n$. If n is odd, then $w_n = 0$.
2. For all n , $\ell_n = \frac{1}{n} \sum_{d|n} (2^d - 2) \left(\phi\left(\frac{n}{d}\right) - \mu\left(\frac{n}{d}\right) \right)$.
3. If n is even, then $w_n = \frac{1}{n} \sum_{d|\frac{n}{2}} \binom{2d}{d} \left(\phi\left(\frac{n}{2d}\right) - \mu\left(\frac{n}{2d}\right) \right)$.

Proof. Observe first that if H is the stabilizer of $S \subseteq \mathbb{Z}_n$, then S is a union of H -cosets; thus, $\mathbb{Z}_n \setminus S$ is also a union of H -cosets. It follows that complementation establishes a bijection between modes of limited transposition of size strictly less than $n/2$ and strong Messiaen modes. Thus, $\ell_n = 2s_n + w_n$. If n is odd, then there are no modes of size $n/2$, so $w_n = 0$.

We now consider the translation action of $\mathbb{Z}_n$ on $\mathcal{Q}_n = \cup_{k=1}^n P_{k,n}$. For each positive divisor d of n , define $q_{d,n}$ to be the number of orbits of transposition length n/d . If $S \in \mathcal{Q}_n$ is a representative of any such orbit, then $H = \text{Stab}(S)$ has order d and S is a union of H -cosets. Furthermore, the image of S under the quotient map $\pi : \mathbb{Z}_n \rightarrow \mathbb{Z}_n/H$ is a primitive subset of $\mathbb{Z}_n/H \cong \mathbb{Z}_{n/d}$. Conversely, if $T \subseteq \mathbb{Z}_n/H$ is primitive, then $\pi^{-1}(T)$ has stabilizer H . It follows that

$$q_{d,n} = q_{1,n/d}. \quad (2)$$

Because $\mathcal{Q}_n$ is a disjoint union of orbits, we have

$$|\mathcal{Q}_n| = 2^n - 2 = \sum_{d|n} \frac{n}{d} q_{d,n} = \sum_{d|n} \frac{n}{d} q_{1,\frac{n}{d}} = \sum_{d|n} d q_{1,d}.$$

By Theorem 1, $n q_{1,n} = \sum_{d|n} \mu\left(\frac{n}{d}\right) (2^d - 2)$, i.e.,

$$q_{1,n} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) (2^d - 2). \quad (3)$$

Now we can compute the number of modes of limited transposition:

$$\ell_n = \sum_{d|n, d>1} q_{d,n} = \sum_{d|n, d>1} q_{1,n/d} = \sum_{d|n} q_{1,d} - q_{1,n}.$$

From (3) this equals

$$\sum_{d|n} \sum_{c|d} \frac{1}{d} \mu\left(\frac{d}{c}\right) (2^c - 2) - \frac{1}{n} \sum_{c|n} \mu\left(\frac{n}{c}\right) (2^c - 2).$$

Interchanging the order of summation in the first expression, this becomes

$$\begin{aligned} & \sum_{c|n} \sum_{d'|\frac{n}{c}} \frac{\mu(d')}{d'c} (2^c - 2) - \frac{1}{n} \sum_{c|n} \mu\left(\frac{n}{c}\right) (2^c - 2) \\ &= \sum_{c|n} \frac{2^c - 2}{c} \sum_{d'|\frac{n}{c}} \frac{\mu(d')}{d'} - \frac{1}{n} \sum_{c|n} \mu\left(\frac{n}{c}\right) (2^c - 2). \end{aligned}$$

Using (1) to rewrite the second sum, we obtain

$$\begin{aligned}\ell_n &= \sum_{c|n} \frac{2^c - 2}{c} \cdot \frac{\phi(n/c)}{n/c} - \frac{1}{n} \sum_{c|n} \mu\left(\frac{n}{c}\right) (2^c - 2) \\ &= \frac{1}{n} \sum_{c|n} (2^c - 2) \phi(n/c) - \frac{1}{n} \sum_{c|n} \mu\left(\frac{n}{c}\right) (2^c - 2) \\ &= \frac{1}{n} \sum_{c|n} (2^c - 2) \left(\phi\left(\frac{n}{c}\right) - \mu\left(\frac{n}{c}\right) \right).\end{aligned}$$

This establishes the formula for ℓ_n .

We now enumerate w_n for even values of n . This time, we consider the action of $\mathbb{Z}_n$ on $\mathcal{R} = P_{\frac{n}{2}, n}$. For each positive divisor d of n , let $r_{d,n}$ be the number of orbits of length n/d ; arguing as before, we see that $r_{d,n} = r_{1,n/d}$. Since $\mathcal{R}$ is a disjoint union of orbits, we have

$$|R| = \binom{n}{n/2} = \sum_{d|n} \frac{n}{d} r_{d,n} = \sum_{d|n} \frac{n}{d} r_{1, \frac{n}{d}} = \sum_{d|n} d r_{1,d}.$$

When n is odd, there is obviously no such action, so $r_{d,n} = 0$ for all positive divisors d of n and hence $\sum_{d|n} d r_{1,d} = 0$. Defining $\beta(n)$ to be 1 when n is even or 0 when n is odd, we have, in all cases,

$$\sum_{d|n} d r_{1,d} = \beta(n) \binom{n}{n/2}.$$

Therefore, by Theorem 1,

$$r_{1,n} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \beta(d) \binom{d}{d/2}.$$

As before, the number of weak Messiaen modes is:

$$w_n = \sum_{d|n, d>1} r_{d,n} = \sum_{d|n, d>1} r_{1,n/d} = \sum_{d|n} r_{1,d} - r_{1,n}.$$

A computation identical to the preceding one, but replacing $2^c - 2$ with $\beta(c) \binom{c}{c/2}$, shows ultimately that

$$w_n = \frac{1}{n} \sum_{c|n} \beta(c) \binom{c}{c/2} \left(\phi\left(\frac{n}{c}\right) - \mu\left(\frac{n}{c}\right) \right). \quad (4)$$

As expected, $w_n = 0$ when n is odd. When n is even, we have:

$$w_n = \frac{1}{n} \sum_{c|\frac{n}{2}} \binom{2c}{c} \left(\phi\left(\frac{n}{2c}\right) - \mu\left(\frac{n}{2c}\right) \right). \quad (5)$$

■

Remark. The reasoning by which we arrived at the formula (2) leads itself readily to a recursive procedure for finding (as opposed to merely enumerating) the modes of limited transposition associated with a given stabilizer subgroup. When n is prime, there are none: all non-trivial, proper subsets of $\mathbb{Z}_n$ are in this case primitive. Now suppose n is composite. For every nontrivial divisor d of n , let H_d denote the unique subgroup of $\mathbb{Z}_n$ of order d and $\pi_{n,d} : \mathbb{Z}_n \rightarrow \mathbb{Z}_n/H_d \cong \mathbb{Z}_{n/d}$ the corresponding quotient map. Then the modes of limited transposition in $\mathbb{Z}_n$ associated with the stabilizer H_d are exactly those of the form $\pi_{n,d}^{-1}(P)$, where P is a primitive subset of $\mathbb{Z}_{n/d}$.

Return to Messiaen's modes In the case $n = 12$ considered by Messiaen, direct computation using Theorem 2 shows that $\ell_{12} = 15$ and $w_{12} = 5$. It follows that $s_{12} = 5$. Thus, we expect five modes of size six and five of size strictly greater than six. All ten are listed in Figure 3, ordered first by the size of the stabilizer subgroup and then by the size of the mode. All three modes absent from Messiaen's list have size 6. The first (IV) is a translation of the one identified by Street [13, Ex. 5(f), p. 819] and has stabilizer of size 3. The remaining two (VI and VII) have stabilizer of size 2.

Generalization to finite abelian groups

The results in the preceding section can all be extended, in an appropriate sense, to the setting of finite abelian groups. As before, a finite abelian group A acts by translation on the set of its nonempty, proper subsets; moreover, any subset $S \subseteq A$ is a disjoint union of cosets of its stabilizer $\text{Stab}(S)$. One then defines a *mode* to be an orbit for this action; it is said to be of *limited transposition* if the stabilizer of any of its representatives is nontrivial. The *size* and *transposition length* of a mode, along with the term *primitive*, are defined as before. A *strong Messiaen mode* is a mode of limited transposition of size strictly greater than $|A|/2$, and a *weak Messiaen mode* is one of size exactly $|A|/2$. It is easy to show that the subsets of A with stabilizer H are in bijective correspondence with the primitive subsets in A/H . The only things missing are analogues of the Möbius function and totient function for abelian groups. These are supplied by a construction of Louis Weisner.

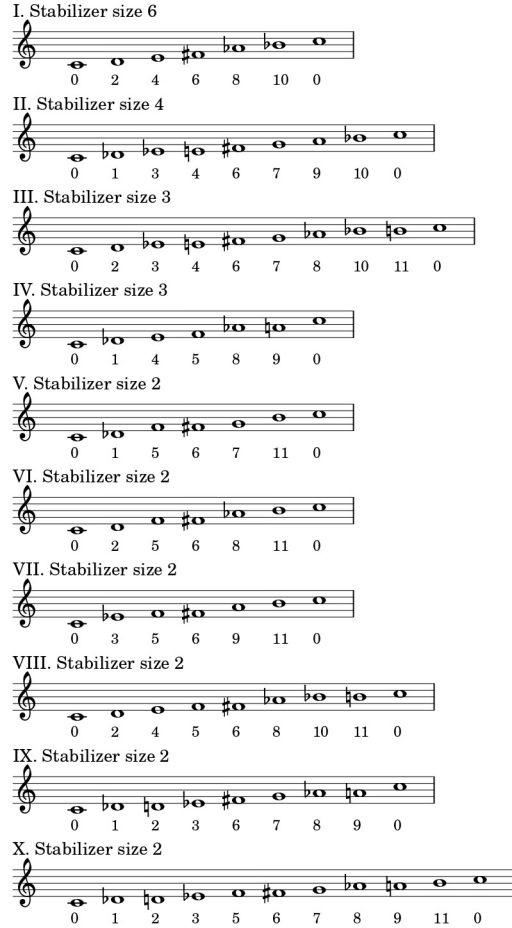
Definition 1. ([18, pp.475,483]) A *hierarchy* is pair $(S, \prec)$, where S is a nonempty set and $\prec$ is a relation on S satisfying:

1. For all $a \in S$, $a \prec a$.
2. If a and b are elements of S such that $a \prec b$ and $b \prec a$, then $a = b$.
3. If a, b , and c are elements of S such that $a \prec b$ and $b \prec c$, then $a \prec c$.
4. For all $a, b \in S$, there exists $d \in S$ such that $d \prec a$ and $d \prec b$; and such that if $c \in S$ satisfies $c \prec a$ and $c \prec b$, then $c \prec d$.
5. For all $a, b \in S$, there exists $\ell \in S$ such that $a \prec \ell$ and $b \prec \ell$; and such that if $m \in S$ satisfies $a \prec m$ and $b \prec m$, then $\ell \prec m$.
6. For every pair of elements $a, b \in S$, there are only finitely many $x \in S$ such that $a \prec x \prec b$.

A *unit element* of S is an element $u \in S$ such that $u \prec a$ for all $a \in S$.

The prototype for this definition is the set of positive integers, where $\prec$ corresponds to divisibility, i.e., $a \prec b$ if and only if $a|b$. The first three items require that $(S, \prec)$ be a partially ordered set. The fourth and fifth items assert the existence of something analogous to a greatest common divisor and least common multiple, respectively.

Figure 3 Strong and weak Messiaen modes of limited transposition for $n = 12$. Messiaen identified I, II, III, V, VIII, IX, and X as (respectively) his modes 1, 2, 3, 5, 6, 4, and 7. Modes IV, VI, and VII do not appear in his classification.



Now let A be a finite abelian group. It is easy to see that the set $\mathcal{S}_A$ of all subgroups of A is a hierarchy under the relation of set inclusion. The first three conditions are trivial; moreover, given subgroups H and K of A , the intersection $H \cap K$ satisfies the fourth condition and the sum $H + K$ satisfies the fifth. The sixth condition follows directly from finiteness of A , and the unit element for this hierarchy is the trivial subgroup.

The definition of the generalized Möbius function is then formulated as follows. (Weisner actually works in a much more general setting; for simplicity, we provide only the specialization to this case.) Given a positive integer m and subgroups H and K of A with $H \subseteq K$, denote by $Q_m(H; K)$ the number of sets of m distinct subgroups $L_1, \dots, L_m$ such that (i) for all i , L_i is a proper subgroup of K , and (ii) $\cap_{i=1}^m L_i = H$. We then define the Möbius function (cf. [18, p.480]) thus:

$$\mu(H; K) = \begin{cases} 1 & \text{if } H = K \\ \sum_{m=1}^{\infty} (-1)^m Q_m(H; K) & \text{otherwise.} \end{cases}$$

In view of the sixth condition of Definition 1, the above sum is actually finite. The analogue of Theorem 1 can then be formulated:

Theorem 3. [18, Theorem 13] *Let A and B be abelian groups and $f : \mathcal{S}_A \rightarrow B$ a function. Define $g : \mathcal{S}_A \rightarrow B$ by*

$$g(K) = \sum_{H \in \mathcal{S}_K} f(H).$$

Then

$$f(K) = \sum_{H \in \mathcal{S}_K} \mu(H; K)g(H).$$

(In both formulae, summation refers to the group operation on B .)

We now define the generalized totient function $\tilde{\phi}$ by stipulating that its value on an abelian group A be

$$\tilde{\phi}(A) = |A| \sum_{H \in \mathcal{S}_A} \frac{\mu(1; H)}{|H|}. \quad (6)$$

If $A = \mathbb{Z}_n$, then $\tilde{\phi}(A) = \phi(n)$. This definition, however, is intrinsically unsatisfying, despite its resemblance to formula (1). Indeed, the formula (1) was derived from the definition of the totient function, not the other way around! One might ask, then, if there exists an “organic” definition of $\tilde{\phi}$ from which (6) can be derived. This seems unlikely, considering that $\tilde{\phi}(\mathbb{Z}_2 \times \mathbb{Z}_2) = 0$.

The generalization of Theorem 2 to the setting of finite abelian groups appears as Theorem 4. Note, however, that the third item of Theorem 2 needs to be replaced by the analogue of (4). In the situation of Theorem 2, A is a cyclic group of order n , which has a unique element of order 2. This means that the even order subgroups of A are in bijection with the subgroups of its (unique) subgroup of order $n/2$, allowing us to proceed from (4) to (5). In general, an abelian group may have more than one element of order 2, so the above reasoning does not apply.

Theorem 4. *Let A be an abelian group with at least 2 elements. Let ℓ_A denote the number of modes of limited transposition in A , s_A the number of strong Messiaen modes, and w_A the number of weak Messiaen modes. Then*

1. $\ell_A = w_A + 2s_A$. *If n is odd, then $w_A = 0$.*
2. $\ell_A = \frac{1}{|A|} \sum_{H \in \mathcal{S}_A} (2^{|H|} - 2) \left(\tilde{\phi}(A/H) - \mu(1; A/H) \right).$
3. $w_A = \frac{1}{|A|} \sum_{H \in \mathcal{S}_A} \beta(H) \binom{|H|}{|H|/2} \left(\tilde{\phi}(A/H) - \mu(1; A/H) \right),$
where $\beta(H) = \begin{cases} 1 & |H| \text{ even} \\ 0 & |H| \text{ odd.} \end{cases}$

The proof of Theorem 4 follows that of Theorem 2, *mutatis mutandis*, using the generalization of Möbius inversion articulated in Theorem 3.

It is natural to wonder if a mode of limited transposition has any musical meaning in this more general context. It is well-known that every finite abelian group A is isomorphic to a direct product $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ of cyclic groups. One might imagine an

ensemble of k musical instruments, with the i th instrument playing notes within a subdivision of the octave into n_i equal parts. A subset of the form $S = B_1 \times \cdots \times B_k$, where $B_i \subseteq \mathbb{Z}_{n_i}$ for each i , could be understood as a set of chords in which the i th instrument is allowed to play notes within the subset B_i . It is harder to imagine what a general subset of the direct product would “sound” like, as the choice of “permissible” notes for one instrument would not be independent of those for the others. Needless to say, this is a somewhat artificial interpretation. Theorem 4 is more likely a mathematical curiosity than something of serious interest to music theorists.

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Summary The French composer Olivier Messiaen invented modes of limited transposition and used them in his own musical compositions. Roughly speaking, a mode of limited transposition is a tonal system that exhibits a high degree of symmetry with respect to (musical) transposition. Although Messiaen himself did not have extensive training in higher mathematics, the written commentary he left on his own musical language reflects a deep appreciation for mathematics and the role it plays in enhancing the art form. In this article, we investigate a claim made by Messiaen about enumeration, and apply basic abstract algebra and number theory to study modes of limited transposition within a tonal system in which the octave is split into n equal subdivisions.