

# The Linear Chromatic Number of a Sperner Family

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## Abstract

Let  $S$  be a finite set and  $\mathcal{S}$  a complete Sperner family on  $S$ , i.e. a Sperner family such that every  $x \in S$  is contained in some member of  $\mathcal{S}$ . The linear chromatic number of  $\mathcal{S}$ , defined by Civan, is the smallest integer  $n$  with the property that there exists a function  $f : S \rightarrow \{1, \dots, n\}$  such that if  $f(x) = f(y)$ , then every set in  $\mathcal{S}$  which contains  $x$  also contains  $y$  or every set in  $\mathcal{S}$  which contains  $y$  also contains  $x$ . We give an explicit formula for the number of complete Sperner families on  $S$  of linear chromatic number 2. We also prove tight bounds on the number of elements in a Sperner family of given chromatic number, and prove that complete Sperner families of maximum linear chromatic number are far more numerous those of lesser linear chromatic number.

## 1 Introduction

Let  $S$  be a set. A *Sperner family* on  $S$  is an antichain in the poset  $2^S$ ; that is, a family  $\mathcal{S}$  of subsets of  $S$  which are pairwise incomparable by inclusion. For  $x \in S$ , write  $\mathcal{S}_x = \{A \in \mathcal{S} : x \in A\}$ ;  $\mathcal{S}$  is called *complete* if  $\bigcup_{A \in \mathcal{S}} A = S$ , or equivalently  $\mathcal{S}_x \neq \emptyset$  for all  $x \in S$ . Sperner families have long been of interest in extremal set theory, beginning in the 1930s with the seminal work of Sperner [13] on the maximum size of such a family.

As an adjunct to their study [3] of multicomplexes, Civan and Yalçın defined the notion of a *linear coloring* of a simplicial complex. This concept was later generalized by Civan [2] in the context of Sperner families: given a set  $S$  and integer  $k > 0$ , a *linear  $k$ -coloring* of a Sperner family  $\mathcal{S}$  on  $S$  is a function  $f : S \rightarrow \{1, \dots, k\}$  such

that if  $f(x) = f(y)$ , then  $\mathcal{S}_x \subseteq \mathcal{S}_y$  or  $\mathcal{S}_y \subseteq \mathcal{S}_x$ . This is equivalent to requiring that for each  $i$ ,  $1 \leq i \leq k$ , there exist a linear ordering (by refinement) of the set of families  $\{\mathcal{S}_v : f(v) = i\}$ . The *linear chromatic number* of  $\mathcal{S}$ , denoted  $\lambda(\mathcal{S})$ , is the smallest  $k > 0$  for which a linear  $k$ -coloring exists. For  $m > 0$  and  $n > 0$ , we denote by  $ls(m; n)$  the number of complete Sperner families of linear chromatic number  $n$  on a set of size  $m$ ; it is easy to see that  $ls(m; 1) = 1$  for all  $m > 0$  and that  $ls(m; n) = 0$  when  $n > m$ . We also define  $ls(0; 0) = 1$  and  $ls(m; 0) = 0$  for  $m > 0$ .

It is now well known that the number of Sperner families on a set of size  $m$  is equal to the number  $D(m)$  of monotonic Boolean functions  $f : \{0, 1\}^m \rightarrow \{0, 1\}$ . The computation of  $D(m)$  was a problem first posed by Dedekind [4] in 1897, one which has generated a considerable amount of investigation. As discussed in the comprehensive survey [12], various lower and upper bounds for  $D(m)$  have been established, and its asymptotic behavior is now quite well understood ([9], [10]). Nevertheless, exact values for  $D(m)$  are still known only for  $m \leq 8$ . Even though precise formulas and recursive formulas for  $D(m)$  have been given (see for example [1], [8]), they do not give much information about its value. Thus, the determination of the value of  $D(m)$  when  $m \geq 9$  is still very much an open problem. Because every nonempty Sperner family on a set of size  $m$  corresponds to a complete Sperner family on some (uniquely determined) subset, we have the formula:

$$D(m) = 1 + \sum_{k=0}^m \sum_{n=0}^k ls(k; n).$$

(The extra 1 corresponds to the empty Sperner family and the term  $ls(0; 0)$  to the Sperner family whose only member is the empty set.) In view of this relationship, one may view the various numbers  $ls(k; n)$  as providing a refinement of the information contained in  $D(m)$ . In particular, computation (or even estimation) of the former could be helpful as a step towards the larger problem of understanding the latter.

With this goal in mind, the purpose of the present article is to study the numbers  $ls(m; n)$ , addressing several of the questions raised by Civan in [2]. Our first result, proved in Section 3, is an exact formula for  $ls(m; 2)$ . In Section 4 we study the maximum and minimum size of a complete Sperner family of specified chromatic number, generalizing the well-known theorem of Sperner [13]. Finally, in Section 5, we prove that complete Sperner families of maximum linear chromatic number far outnumber those of lesser chromatic number. This last result suggests that  $ls(m; m)$  is the parameter most likely to yield useful information about  $D(m)$ . The common idea undergirding all the results in this article is that one may study a Sperner family

of linear chromatic number  $m$  on a set of size  $n$  by associating to it to an antichain in the poset  $L_{n_1} \times \cdots \times L_{n_m}$ , where  $L_k$  is the poset  $\{0, 1, \dots, k\}$  with the usual order and  $n = n_1 + \dots + n_m$  is some partition of  $n$  into positive integer parts. The technical framework for establishing this correspondence is laid down in Section 2 and is used in each of the subsequent sections.

Throughout this article, we write  $Sp_n(S)$  for the set of Sperner families of linear chromatic number  $n$  on a set  $S$  and  $Sp(S) = \cup_n Sp_n(S)$ . We also use the notation  $[a, b]$  to mean  $\{n \in \mathbb{Z} : a \leq n \leq b\}$ .

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## 2 Representations of a Sperner Family

In this section we describe two convenient ways of representing a Sperner family of specified linear chromatic number. The first involves antichains in chain products and applies to families of arbitrary chromatic number. The second applies only to families of linear chromatic number 2 but is particularly helpful in the solution of the enumeration problem addressed in Section 3.

### 2.1 Chain Products

We begin by defining chain products and establishing some of their basic properties; we then proceed to describe their significance in the context of linear colorings.

In general, a poset  $P$  is called *ranked* (or *graded*) if there exists a *ranking function*  $r : P \rightarrow \mathbb{Z}$  such that  $r(y) = r(x) + 1$  whenever  $y$  covers  $x$ . The  $i$ th level of  $P$  is then defined by  $N_i = \{x \in P : r(x) = i\}$ . If the ranking function has range  $[0, m]$ ,  $P$  is called *rank-symmetric* if  $|N_i| = |N_{m-i}|$  for all  $i$  and *rank-unimodal* if there exists  $\mu \in [0, m]$  such that  $|N_i| \leq |N_j|$  for  $0 \leq i \leq j \leq \mu$  and  $|N_i| \geq |N_j|$  for  $\mu \leq i \leq j \leq m$ . Finally,  $P$  is said to have the *Sperner property* if the maximum size of an antichain in  $P$  is equal to the size of the largest level of  $P$ .

For any positive integer  $n$ , we denote by  $L_n$  the usual linear order with underlying set  $\{0, 1, \dots, n\}$ . A poset of the form  $C_{n_1, \dots, n_s} = L_{n_1} \times \cdots \times L_{n_s}$ , where  $n_1, \dots, n_s$  are positive integers, is called a *chain product*. Note that  $C_{n_1, \dots, n_s}$  has the natural

structure of a ranked poset via the function  $r(a_1, \dots, a_s) = a_1 + \dots + a_s$ . We record the following useful fact.

**Proposition 2.1.** [6, p. 230] *Chain products are rank-symmetric, rank-unimodal, and have the Sperner property.*

We now describe a natural connection between linear colorings and antichains in chain products. An *ordered partition* of a set  $S$  is an  $n$ -tuple  $\mathcal{P} = (S_1, \dots, S_n)$  such that  $\{S_1, \dots, S_n\}$  is a partition of  $S$ . Given an ordered partition  $\mathcal{P}$  as above, a *ranking* of  $\mathcal{P}$  is an  $n$ -tuple  $R = (r_1, \dots, r_n)$  of bijective functions  $r_i : S_i \rightarrow \{1, \dots, |S_i|\}$ ,  $i = 1, \dots, n$ . Now let  $M_n(S)$  denote the set of ordered triples  $(\mathcal{P}, R, \mathcal{C})$ , in which  $\mathcal{P} = (S_1, \dots, S_n)$  is an ordered partition of  $S$ ,  $R = (r_i)_{i=1}^n$  is a ranking of  $\mathcal{P}$ , and  $\mathcal{C}$  is antichain in  $C_{|S_1|, \dots, |S_n|}$ . Then there is a function

$$\Phi : M_n(S) \rightarrow Sp_n(S)$$

defined as follows: given a triple  $(\mathcal{P}, R, \mathcal{C}) \in M_n(S)$ , each element  $c \in \mathcal{C}$  corresponds to an  $n$ -tuple  $(c_1, \dots, c_n)$ , where for each  $i$ ,  $0 \leq c_i \leq |S_i|$ . Now define

$c_S = \bigcup_{i=1}^n \{s_i \in S_i : r_i(s_i) \leq c_i\}$ . It is easy to check that  $\Phi(\mathcal{P}, R, \mathcal{C}) = \{c_S : c \in \mathcal{C}\}$  is a Sperner family in  $S$  and that the coloring  $\lambda : S \rightarrow \{1, \dots, n\}$  defined by sending  $s \in S$  to the (unique)  $i$  such that  $s \in S_i$  is an  $n$ -linear coloring of this Sperner family.

**Proposition 2.2.**  $\Phi$  is surjective; moreover, if  $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$ , then  $|\mathcal{C}| = |\mathcal{S}|$ .

**Proof.**

Let  $\mathcal{S}$  be an  $n$ -linearly colorable Sperner family on  $S$ . Then there is a partition of  $S$  into color classes  $S_1, \dots, S_n$ , and for  $i = 1, \dots, n$ , there exists a bijection  $r_i : S_i \rightarrow [1, \dots, |S_i|]$  such that  $\mathcal{S}_x \subseteq \mathcal{S}_y$  if  $r_i(y) \leq r_i(x)$ . Now consider the element of  $M_n(S)$  described by the partition  $\mathcal{P} = (S_1, \dots, S_n)$ , the ranking  $R = (r_i)_{i=1}^n$ , and the antichain  $\mathcal{C} = \{C(A) : A \in \mathcal{S}\}$  in  $C_{|S_1|, \dots, |S_n|}$ , where  $C(A) = (c_1, \dots, c_n)$  is defined by  $c_i = \max\{0\} \cup r_i(A \cap S_i)$ . Clearly  $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$ , and the equality  $|\mathcal{C}| = |\mathcal{S}|$  is obvious from the construction.  $\square$

## 2.2 Directed bipartite graph representation

In this section, we introduce a convenient way of representing a Sperner family of linear chromatic number 2 by a particular labeled, directed bipartite graph. Since the results of Section 2.1 imply that each such Sperner family is represented by a

triple in  $M_2(S)$ , we show how to associate a directed graph to each such triple and vice versa.

Let  $\mathcal{B}$  denote the set of pairs  $(D, h)$ : here  $D$  is a directed bipartite graph of maximum degree 1 with bipartite sets  $X = [0, c]$  and  $Y = [0, d]$ , in which  $c + d = |S|$ ; moreover, all edges originate in  $X$  and terminate in  $Y$ , and we require that for any pair  $x_1 \rightarrow y_1, x_2 \rightarrow y_2$  of distinct directed edges,  $x_1 - x_2$  and  $y_1 - y_2$  have opposite signs. The map  $h : (X - \{0\}) \cup (Y - \{0\}) \rightarrow S$  is required to be bijective; in other words,  $h$  is a labeling by elements of  $S$  of all vertices in  $D$  except  $0 \in X$  and  $0 \in Y$ .

We now establish a bijection

$$\alpha : M_2(S) \rightarrow \mathcal{B}$$

Given a triple  $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$ , set  $\alpha(t) = (D, h)$ , where  $D$  is the directed bipartite graph with partite sets  $X = [0, |S_1|]$  and  $Y = [0, |S_2|]$  and a directed edge from  $x \in X$  to  $y \in Y$  if and only if  $(x, y) \in \mathcal{C}$ ; we define the labeling  $h : (X - \{0\}) \cup (Y - \{0\}) \rightarrow S$  by  $h(z) = r_1^{-1}(z)$  if  $z \in X - \{0\}$  or  $h(z) = r_2^{-1}(z)$  if  $z \in Y - \{0\}$ .

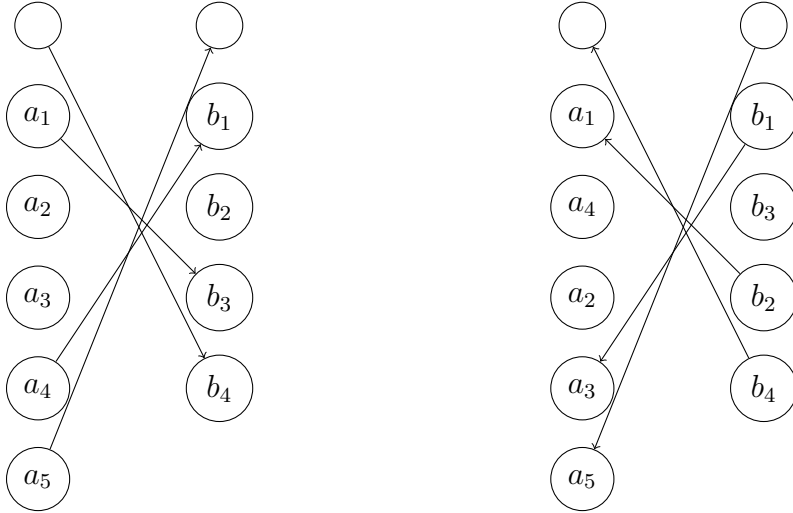
To construct an inverse  $\beta : \mathcal{B} \rightarrow M_2(S)$  for  $\alpha$ , associate to a pair  $(D, h) \in \mathcal{B}$  the triple  $(\mathcal{P}, R, \mathcal{C}) \in M_2(S)$  where  $\mathcal{P} = (h(S_1), h(S_2))$ ,  $S_1$  being the partite set of  $D$  containing (all) the initial vertices of the edges and  $S_2$  the other partite set. The ranking  $R = (r_1, r_2)$  is defined by  $r_1(x) = h^{-1}(x)$  when  $x \in h(S_1)$  and  $r_2(y) = h^{-1}(y)$  when  $y \in h(S_2)$ , and the antichain  $\mathcal{C}$  is defined by  $(x, y) \in \mathcal{C} \Leftrightarrow h^{-1}(x) \rightarrow h^{-1}(y) \in E(D)$ .

It is clear that  $\alpha$  and  $\beta$  are inverse to each other. This visual representation of elements of  $M_2(S)$  is particularly convenient in that it is easy to read off the members of the associated Sperner family  $\Phi(\beta(D, h))$ : to a directed edge  $x \rightarrow y$  of  $D$  we associate the member of  $\Phi(\beta(D, h))$  given by  $\{h(z) : z \in X, 1 \leq z \leq x\} \cup \{h(z) : z \in Y, 1 \leq z \leq y\}$ .

In Figure 1 we exhibit two representations for the Sperner family

$\mathcal{S} = \{\{b_1, b_2, b_3, b_4\}, \{a_1, b_1, b_2, b_3\}, \{a_1, a_2, a_3, a_4, b_1\}, \{a_1, a_2, a_3, a_4, a_5\}\}$  on the set  $S = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}$ . The directed graph on the left corresponds to the triple  $(\mathcal{P}, (r_1, r_2), \mathcal{C})$  where  $\mathcal{P} = (\{a_1, a_2, a_3, a_4, a_5\}, \{b_1, b_2, b_3, b_4\})$ ,  $r_1(a_i) = i$  for  $1 \leq i \leq 5$  and  $r_2(b_j) = j$  for  $1 \leq j \leq 4$ , and  $\mathcal{C} = \{(0, 4), (1, 3), (4, 1), (5, 0)\}$ . The directed graph of the right corresponds to the triple  $(\mathcal{P}', (r'_1, r'_2), \mathcal{C}')$  where  $\mathcal{P}' = (\{b_1, b_2, b_3, b_4\}, \{a_1, a_2, a_3, a_4, a_5\})$  and  $\mathcal{C}' = \{(0, 5), (1, 4), (3, 1), (4, 0)\}$ ; the ranking functions are defined by  $r'_1(b_1) = 1, r'_1(b_2) = 3, r'_1(b_3) = 2, r'_1(b_4) = 4$  and  $r'_2(a_1) = 1, r'_2(a_2) = 3, r'_2(a_3) = 4, r'_2(a_4) = 2, r'_2(a_5) = 5$ .

Figure 1: Representations of the 2-colorable Sperner family  $\mathcal{S} = \{\{b_1, b_2, b_3, b_4\}, \{a_1, b_1, b_2, b_3\}, \{a_1, a_2, a_3, a_4, b_1\}, \{a_1, a_2, a_3, a_4, a_5\}\}$  on  $S = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}$ .



### 3 On $ls(m; 2)$

The goal of this section is to give a formula for  $ls(m; 2)$ , the number of complete Sperner families of linear chromatic number 2 on a set of size  $m$ . Our first step is to reduce this calculation to a simpler problem, namely that of enumerating the so-called primitive complete Sperner families. To this end, let  $S$  be a set and  $\mathcal{S}$  a Sperner family on  $S$ . We define the set of *base points* of  $\mathcal{S}$  to be the subset of elements of  $S$  contained in every set in the family  $\mathcal{S}$ ; that is,  $B(\mathcal{S}) = \{x \in S : \mathcal{S}_x = \mathcal{S}\}$ . If  $B(\mathcal{S}) = \emptyset$ ,  $\mathcal{S}$  is called *primitive*.

It is easy to see that completeness and primitivity are dual to each other, in the sense that a Sperner family  $\mathcal{S}$  on a set  $S$  is complete if and only if the Sperner family  $\mathcal{S}^C = \{S \setminus A : A \in \mathcal{S}\}$  is primitive. Of course, we also have  $(\mathcal{S}^C)^C = \mathcal{S}$ .

**Lemma 3.1.** *Let  $S$  be a set and  $T \subseteq S$  a subset. Then there is a bijective correspondence:*

$$h : \{\mathcal{S} \in Sp(S) : B(\mathcal{S}) = T\} \rightarrow \{\mathcal{T} \in Sp(S \setminus T) : \mathcal{T} \text{ is primitive}\}$$

Moreover,  $h$  preserves linear chromatic number.

**Proof.**

Given a Sperner family  $\mathcal{S}$  on  $S$  with  $B(\mathcal{S}) = T$ , it is easy to check that  $h(\mathcal{S}) = \{A \setminus T : A \in \mathcal{S}\}$  is a primitive Sperner family on  $S \setminus T$  and that an inverse function is given by sending a primitive Sperner family  $\mathcal{T}$  on  $S \setminus T$  to  $j(\mathcal{T}) = \{A \cup T : A \in \mathcal{T}\}$ , which is a Sperner family on  $S$  whose set of base points is  $T$ . Now suppose  $f : S \rightarrow \{1, \dots, k\}$  is a  $k$ -linear coloring of  $\mathcal{S}$ ; that is, for each  $i$ ,  $\{\mathcal{S}_x : f(x) = i\}$  is linearly ordered by refinement. Then  $\{[h(\mathcal{S})]_x : x \in S \setminus T, f(x) = i\}$  is a set of families of subsets of  $S \setminus T$ , also linearly ordered by refinement. This implies that  $f|_{S \setminus T} : S \setminus T \rightarrow \{1, \dots, k\}$  is a  $k$ -linear coloring of  $h(\mathcal{S})$ , and so  $\lambda(h(\mathcal{S})) \leq \lambda(\mathcal{S})$ . Conversely, if  $g : S \setminus T \rightarrow \{1, \dots, k\}$  is a  $k$ -linear coloring of a primitive Sperner family  $\mathcal{T}$  on  $S \setminus T$ , then by extending  $g$  in any way to a function  $f : S \rightarrow \{1, \dots, k\}$ , we obtain a  $k$ -linear coloring of  $j(\mathcal{T})$ ; this is so because  $[j(\mathcal{T})]_x = j(\mathcal{T})$  for any  $x \in T$ . Hence,  $\lambda(j(\mathcal{T})) \leq \lambda(\mathcal{T})$ . Since  $j$  and  $h$  are inverse to each other, the claim is proven.  $\square$

Denoting by  $ls_p(m; n)$  the number of complete *primitive* Sperner families of linear chromatic number  $n$  on a set of cardinality  $m$ , the following easy consequence of Lemma 3.1 permits reduction of our enumeration problem to the primitive case.

**Corollary 3.2.**  $ls(m; n) = \sum_{k=n}^m \binom{m}{k} ls_p(k; n)$ .

We now describe the computation of  $ls_p(m; 2)$ . The main idea is to use the surjectivity of the map  $\Phi$  of Section 2.1 to argue that each complete primitive Sperner family of linear chromatic number 2 is equal to  $\Phi(t)$  for some  $t \in M_2(S)$ , and then to calculate  $|\Phi^{-1}(\Phi(t))|$  by exploiting bijectivity of the map  $\alpha$  of Section 2.2.

To begin, we establish that the color classes in any 2-linear coloring of a complete primitive Sperner family  $\mathcal{S}$  of linear chromatic number 2 are uniquely determined by  $\mathcal{S}$  itself.

**Lemma 3.3.** *With notation as above, suppose  $\Phi(\mathcal{P}, R, \mathcal{C}) = \mathcal{S}$ .*

- *If  $\mathcal{S}$  is complete, then  $(|S_1|, b) \in \mathcal{C}$  for some  $b$ ,  $0 \leq b \leq |S_2|$  and  $(a, |S_2|) \in \mathcal{C}$  for some  $a$ ,  $0 \leq a \leq |S_1|$ .*
- *If  $\mathcal{S}$  is primitive, then  $(0, b) \in \mathcal{C}$  for some  $b$ ,  $0 \leq b \leq |S_2|$  and  $(a, 0) \in \mathcal{C}$  for some  $a$ ,  $0 \leq a \leq |S_1|$ .*
- *If  $\mathcal{S}$  is both complete and primitive, then  $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$ .*

**Proof.**

If  $(|S_1|, b) \notin \mathcal{C}$  for all  $b$ ,  $0 \leq b \leq |S_2|$ , then (by construction of  $\Phi$ ), no subset in  $\mathcal{S}$  contains  $r_1^{-1}(|S_1|)$  and  $\mathcal{S}$  is not complete; a similar argument establishes that  $(a, |S_2|) \in \mathcal{C}$

for some  $a$  if  $\mathcal{S}$  is complete. If  $(0, b) \notin \mathcal{C}$  for all  $b$ ,  $0 \leq b \leq |S_2|$ , then  $r_1^{-1}(1)$  is a base point of  $\mathcal{S}$  and so  $\mathcal{S}$  is not primitive; a similar argument shows that  $(a, 0) \in \mathcal{C}$  for some  $a$  if  $\mathcal{S}$  is primitive. If  $\mathcal{S}$  is both complete and primitive, then the first two statements together with  $\mathcal{C}$  being an antichain force  $(0, |S_2|), (|S_1|, 0) \in \mathcal{C}$ .  $\square$

**Definition 3.4.** *Let  $S$  be a set and  $\mathcal{S}$  a Sperner family on  $S$ . For  $x, y \in S$ , we say that  $\mathcal{S}$  separates  $x$  from  $y$  if  $\mathcal{S}_x \not\subseteq \mathcal{S}_y$ . We say that  $x$  and  $y$  are separated by  $\mathcal{S}$  if  $\mathcal{S}_x \not\subseteq \mathcal{S}_y$  and  $\mathcal{S}_y \not\subseteq \mathcal{S}_x$ .*

Observe that in any linear coloring of  $\mathcal{S}$ , elements separated by  $\mathcal{S}$  must lie in different color classes.

**Lemma 3.5.** *Let  $S$  be a set and  $\mathcal{S}$  a complete primitive Sperner family on  $S$  of linear chromatic number 2. Then there exists a partition of  $S$  into subsets  $S_1$  and  $S_2$  such that:*

1.  $S_1, S_2 \in \mathcal{S}$ .
2.  $\mathcal{S}$  separates two elements  $x$  and  $y$  if and only if one of these elements lies in  $S_1$  and the other in  $S_2$ .
3. Every linear coloring of  $\mathcal{S}$  with two colors has color classes  $S_1$  and  $S_2$ , and so  $S_1$  and  $S_2$  are uniquely determined (up to interchanging).

**Proof.**

Fix a linear coloring  $f : S \rightarrow \{1, 2\}$ ; let  $S_1 = f^{-1}(1) = \{x_1, \dots, x_p\}$ ,  $S_2 = f^{-1}(2) = \{y_1, \dots, y_q\}$ , where  $\mathcal{S}_{x_i} \supseteq \mathcal{S}_{x_j}$  and  $\mathcal{S}_{y_i} \supseteq \mathcal{S}_{y_j}$  for all  $i < j$ . By definition, no element of  $S_1$  is separated by  $\mathcal{S}$  from any other, and similarly for  $S_2$ . Since  $\mathcal{S}$  is primitive,  $y_1$  is not a base point of  $\mathcal{S}$ ; so, there exists some  $A \in \mathcal{S}$  such that  $y_1 \notin A$ . This implies that  $y_j \notin A$  for all  $1 \leq j \leq q$ ; thus, we must have  $A \subseteq S_1$ . Since  $\mathcal{S}$  is complete, there exists some  $B \in \mathcal{S}$  such that  $x_p \in B$ . Hence  $B$  contains all  $x_i$ ,  $1 \leq i \leq p$ , and so  $B \supseteq S_1$ . Since  $A \subseteq S_1 \subseteq B$ , and  $A$  and  $B$  are both members of the Sperner family  $\mathcal{S}$ , it follows that  $A = B = S_1$ , and hence  $S_1 \in \mathcal{S}$ . A similar argument establishes that  $S_2 \in \mathcal{S}$ . It is now clear that  $\mathcal{S}$  separates  $x$  from  $y$  if and only if one lies in  $S_1$  and the other in  $S_2$ . It follows that the color classes associated to any linear 2-coloring must be  $S_1$  and  $S_2$ .  $\square$

If  $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$  is a triple with  $\mathcal{P} = (S_1, S_2)$ ,  $R = (r_1, r_2)$ , we define the *opposite triple*  $t^{op} = (\mathcal{P}^{op}, R^{op}, \mathcal{C}^{op})$  by  $\mathcal{P}^{op} = (S_2, S_1)$ ,  $R^{op} = (r_2, r_1)$  and  $\mathcal{C}^{op} = \{(y, x) \in C_{|S_2|, |S_1|} : (x, y) \in \mathcal{C}\}$ . In terms of the directed graph representation, if



$\alpha(t) = (D, h)$ , then  $\alpha(t^{op}) = (D^R, h)$ , where  $D^R$  is the reverse of  $D$ . It is clear from the definition that  $\Phi(t^{op}) = \Phi(t)$ . In light of this interpretation, the next result essentially asserts that the directed graph associated to a triple  $t$  is determined (up to reversal of arrows) by the Sperner family  $\Phi(t)$ .

**Lemma 3.6.** *Suppose  $\mathcal{S}$  is a complete primitive Sperner family of linear chromatic number 2 on a set  $S$ . If  $\mathcal{S} = \Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}', R', \mathcal{C}')$ , then either  $\mathcal{P} = \mathcal{P}'$  and  $\mathcal{C} = \mathcal{C}'$  or  $\mathcal{P} = \mathcal{P}^{op}$  and  $\mathcal{C}' = \mathcal{C}^{op}$ .*

**Proof.**

The fact that  $\mathcal{P}'$  equals  $\mathcal{P}$  or  $\mathcal{P}^{op}$  follows from Lemma 3.5. Since  $\Phi(\mathcal{P}', R', \mathcal{C}') = \Phi(\mathcal{P}'^{op}, R'^{op}, \mathcal{C}'^{op})$ , we may assume that  $\mathcal{P}' = \mathcal{P}$ . First observe that  $|\mathcal{C}| = |\mathcal{S}| = |\mathcal{C}'|$  by Proposition 2.2. We proceed to prove by induction on  $k = |\mathcal{C}| = |\mathcal{C}'|$  that if  $\Phi(\mathcal{P}, R, \mathcal{C}) = \Phi(\mathcal{P}, R', \mathcal{C}') = \mathcal{S}$ , then  $\mathcal{C} = \mathcal{C}'$ . By Lemma 3.3,  $(0, |S_2|)$  and  $(|S_1|, 0)$  are in both  $\mathcal{C}$  and  $\mathcal{C}'$ ; so  $k \geq 2$ , and if  $k = 2$ , then  $\mathcal{C} = \mathcal{C}'$ . Now suppose  $k > 2$ . Let  $c = (x, y) \in \mathcal{C}$  be the element for which  $x$  is positive and minimal (among all such choices); likewise, let  $c' = (x', y') \in \mathcal{C}'$  be the element for which  $x'$  is positive and minimal. Then the elements of  $Z_1 = \{z \in S_1 : 1 \leq r_1(z) \leq x\}$  are exactly those which appear in all but one element of  $\mathcal{S}$ ; yet the same is true of  $Z'_1 = \{z \in S_1 : 1 \leq r'_1(z) \leq x'\}$ , and so it must be the case that  $x = x'$ . Similarly, the elements of  $W = \{w \in S_2 : y \leq r_2(w) \leq |S_2|\}$  and  $W' = \{w \in S_2 : y' \leq r_2(w) \leq |S_2|\}$  are exactly those which appear in a unique element of  $\mathcal{S}$ , so  $y = y'$ . Then  $\mathcal{D} = \mathcal{C} \setminus \{(x, y)\}$  and  $\mathcal{D}' = \mathcal{C}' \setminus \{(x, y)\}$  are antichains of size  $k-1$  with  $\Phi(\mathcal{P}, R, \mathcal{D}) = \Phi(\mathcal{P}, R', \mathcal{D}') = \mathcal{S} \setminus \{A\}$ ; so, by induction,  $\mathcal{D} = \mathcal{D}'$ . Thus,  $\mathcal{C} = \mathcal{C}'$ , as desired.  $\square$

We now come to our main result. For convenience, write  $S \sqsubseteq [a, b]$  to mean a subset  $S \subseteq [a, b]$  containing both  $a$  and  $b$ . If  $c_0 < \dots < c_k$ , we also define the *elasticity* of the subset  $C = \{c_0, \dots, c_k\} \subseteq [0, n]$  by  $e(C) = \prod_{i=1}^k (c_i - c_{i-1})!$ .

**Theorem 3.7.** *Suppose  $m \geq 2$ . Then*

$$\begin{aligned} ls_p(m; 2) &= \frac{m!}{2} \sum_{1 \leq k \leq m-1} \sum_{I \sqsubseteq [0, k]} \sum_{\substack{J \sqsubseteq [0, m-k] \\ |J|=|I|}} \frac{1}{e(I)e(J)} \\ &= m! \left( \sum_{1 \leq k \leq \lfloor m/2 \rfloor} \sum_{I \sqsubseteq [0, k]} \sum_{\substack{J \sqsubseteq [0, m-k] \\ |J|=|I|, J \neq I}} \frac{1}{e(I)e(J)} + \frac{1}{2} \sum_{I \sqsubseteq [m/2]} \frac{1}{e(I)^2} \right) \end{aligned}$$

**Proof.**

We prove the first formula, as the second then follows immediately. Given any complete primitive Sperner family  $\mathcal{S}$  on  $S$  of linear chromatic number 2, Lemma 3.6 implies that if we fix a triple  $t = (\mathcal{P}, R, \mathcal{C}) \in M_2(S)$  with  $\Phi(t) = \mathcal{S}$ , then

$$|\Phi^{-1}(\mathcal{S})| = 2|\{R' : \Phi(\mathcal{P}, R', \mathcal{C}) = \mathcal{S}\}|$$

Now if we let  $\alpha(\mathcal{P}, R, \mathcal{C}) = (D, h)$  then for  $(\mathcal{P}, R', \mathcal{C}) \in \Phi^{-1}(\mathcal{S})$ , we must have  $\alpha(\mathcal{P}, R', \mathcal{C}) = (D, h')$  for some labeling  $h' : V(D) \rightarrow S$ . Thus,

$$|\Phi^{-1}(\mathcal{S})| = 2|\{h' : V(D) \rightarrow S, h' \text{ is bijective, } \Phi(\beta(D, h')) = \mathcal{S}\}|$$

To compute the term on the right, let  $I = \{0 = a_0, a_1, \dots, a_k = |S_1|\} \subseteq [0, |S_1|]$  be the set of initial vertices of edges in  $D$  and  $J = \{0 = b_0, b_1, \dots, b_k = |S_2|\} \subseteq [0, |S_2|]$  the set of terminal vertices, where the notation is chosen so that the sequences  $\{a_i\}$  and  $\{b_i\}$  are increasing. By construction, if  $1 \leq i \leq k$  and  $a_{i-1} < x \leq a_i$ , the element of  $S$  labeled  $h(x)$  appears in exactly  $i$  members of the Sperner family  $\mathcal{S}$ ; similarly, if  $b_{i-1} < y \leq b_i$ , then  $h(y)$  appears in exactly  $i$  members of  $\mathcal{S}$ . If one obtains the same Sperner family upon replacing  $h$  by  $h'$ , it must be the case that for all  $i$ ,  $1 \leq i \leq k$ ,  $h'$  must be obtained from  $h$  by permutation of the labels on the various sets of vertices  $A_i = \{x \in X : a_{i-1} < x \leq a_i\}$  and  $B_i = \{y \in Y : b_{i-1} < y \leq b_i\}$ ,  $1 \leq i \leq k$ . Conversely, it is clear that any labeling obtained from  $h$  by permutation of these sets of labels will yield the same Sperner family. The number of such labelings is precisely  $e(I)e(J)$ , so  $|\Phi^{-1}(\mathcal{S})| = 2e(I)e(J)$ .

Now if we wish to sum over *all* complete primitive Sperner families, we need to consider all such possible bipartite directed graphs and all possible labelings of such graphs. The bipartition is determined uniquely by an integer  $1 \leq k \leq m - 1$ ; having fixed this, the edges are determined by specifying subsets  $I \subseteq [0, k]$  and  $J \subseteq [0, m - k]$  of equal size, and a labeling is simply a permutation of  $\{1, \dots, m\}$ .

Finally, for convenience we write  $Sp'_2(S)$  for the set of complete primitive Sperner families of linear chromatic number 2,  $M'_2(S) = \Phi^{-1}(Sp'_2(S))$ , and  $\mathcal{B}' = \alpha(M'_2(S))$ . Then

$$\begin{aligned} ls_p(m; 2) &= \sum_{\mathcal{S} \in Sp'_2(S)} 1 = \sum_{t \in M'_2(S)} \frac{1}{|\Phi^{-1}(\Phi(t))|} = \sum_{(D, h) \in \mathcal{B}'} \frac{1}{|\Phi^{-1}(\Phi(\beta(D, h)))|} \\ &= \sum_{k=1}^{m-1} \sum_{I \subseteq [0, k]} \sum_{J \subseteq [0, m-k]: |J|=|I|} \frac{m!}{2e(I)e(J)} = \frac{m!}{2} \sum_{I \subseteq [0, k]} \sum_{J \subseteq [0, m-k]: |J|=|I|} \frac{1}{e(I)e(J)} \end{aligned}$$

Table 1:

$m$	$ls_p(m; 2)$	$ls(m; 2)$
2	1	1
3	3	6
4	19	37
5	135	270
6	1351	2521
7	11823	24906

□

When making computations, the second formula of Theorem 3.7 is usually more convenient. Table 1 displays the values of  $ls_p(m; 2)$  and  $ls(m; 2)$  for small values of  $m$ , confirming the calculations in [2, Section 2].

## 4 On the size of Sperner families of specified chromatic number

A well-known result of Sperner asserts that the size of a maximum antichain in the Boolean lattice  $B_m$  (or equivalently the maximum size of a Sperner family on  $S = [m]$ ) is  $\binom{m}{\lfloor m/2 \rfloor}$ ; this is met by considering the family of subsets of  $S$  of size  $\lfloor m/2 \rfloor$ . A trivial lower bound of 1 is met by the Sperner family  $\{S\}$ . In this section we address the analogous question, raised in [2], of finding upper and lower bounds for the size of a Sperner family with specified linear chromatic number. For convenience, we introduce some notation. Given  $m \geq n$ , write  $m = qn + r$  for some integers  $q, r$ ,  $0 \leq r < n$ , and define:

$$s(m, n) = |\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i = \lfloor m/2 \rfloor \text{ and there exists } r \text{ such that } 0 \leq x_i \leq \lfloor m/n \rfloor$$

$$\text{for } 1 \leq i \leq r \text{ and } 0 \leq x_i \leq \lfloor m/n \rfloor \text{ for } r+1 \leq i \leq n\}|$$

and

$$t(n) = \min\{t : \binom{t}{\lfloor t/2 \rfloor} \geq n\}.$$

**Theorem 4.1.** *Let  $S$  be a set of cardinality  $m$  and  $\mathcal{S}$  a complete Sperner family on  $S$  of linear chromatic number  $n$ . Then  $t(n) \leq |\mathcal{S}| \leq s(m, n)$ .*

**Proof.**

For the lower bound, let  $S$  be a set with  $m$  elements and suppose  $\mathcal{S}$  is a Sperner family of cardinality  $t$ , where  $\ell = \binom{t}{\lfloor t/2 \rfloor} < n$ . After fixing a bijection between  $\mathcal{S}$  and  $[t]$ , we may identify, for each  $x \in S$ ,  $\mathcal{S}_x$  with a subset  $T_x \subseteq [t]$ . By a now well-known result [5], there exists a partition of the poset  $2^{[t]}$  into disjoint chains  $C_1, \dots, C_\ell$ ; thus, we may construct a linear  $\ell$ -coloring  $f : S \rightarrow [\ell]$  of  $\mathcal{S}$  by declaring  $f(x)$  to be the unique index  $i$  such that  $T_x$  is a member of  $C_i$ . Hence  $\lambda(\mathcal{S}) \leq \ell < n$ .

For the upper bound, note that Proposition 2.2 allows us to reduce our question to that of finding the size of a maximum antichain in a chain product  $C_{m_1, \dots, m_n}$ , where  $m_1, \dots, m_n$  are positive integers whose sum is  $m$ . By Proposition 2.1, the size of the largest antichain is equal to  $|\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i = \lfloor m/2 \rfloor, 0 \leq x_i \leq m_i\}|$ . Hence, the problem is reduced to finding the values of  $m_1, \dots, m_n$  satisfying  $\sum_{i=1}^n m_i = m$  which maximize this quantity. We claim that this is achieved by choosing each  $m_i$  to be either  $\lfloor m/n \rfloor$  or  $\lceil m/n \rceil$ ; clearly there is only one way of achieving this, up to permutation of the  $m_i$ . Suppose instead that  $m_1 \leq \lfloor m/n \rfloor - 1$  and  $m_2 \geq \lceil m/n \rceil + 1$ ; we will show that any such choice of  $m_1, m_2, \dots, m_n$  does not yield a maximum. If  $n = 2$ ,  $s(m_1, m_2) = m_1 + 1 < m_1 + 2 = s(m_1 + 1, m_2 - 1)$ . If  $n > 2$ , then

$$\begin{aligned}
& s(m_1 + 1, m_2 - 1, m_3, \dots, m_n) - s(m_1, m_2, m_3, \dots, m_n) \\
&= |\{(x_2, \dots, x_n) : \sum_{i=2}^n x_i = \lfloor m/2 \rfloor - m_1 - 1 : 0 \leq x_2 \leq m_2 - 1, 0 \leq x_i \leq m_i \text{ for } i \geq 3\}| \\
&\quad - |\{(x_1, x_3, \dots, x_n) : x_1 + \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_2 : 0 \leq x_i \leq m_i \text{ for all } i\}| \\
&= \sum_{j=0}^{m_2-1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_1 - 1 - j\}| \\
&\quad - \sum_{j=0}^{m_1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - m_2 - j\}| \\
&= \sum_{j=m_1+1}^{m_2-1} |\{(x_3, \dots, x_n) : \sum_{i=3}^n x_i = \lfloor m/2 \rfloor - j\}|
\end{aligned}$$

Since  $m_1 \leq m_2 - 2$ , this sum has at least one term in it and hence is positive.  $\square$

**Remarks.**

1. If  $\ell = \binom{t(n)}{\lfloor t(n)/2 \rfloor} \leq m$ , we can show that the lower bound is tight. Consider the family  $\mathcal{A} = \{A \subseteq [t(n)] : |A| = \lfloor t(n)/2 \rfloor\}$  of subsets of  $[t(n)]$ . For each  $x \in [t(n)]$ , let  $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$ . Then  $\mathcal{T} = \{\mathcal{A}_x : x \in [t(n)]\}$  is a Sperner family on  $\mathcal{A}$ . Since any two elements of  $\mathcal{A}$  are separated by  $\mathcal{T}$ , the linear chromatic number of  $\mathcal{T}$  must be exactly  $\ell$ . Finally, if  $S$  is any set with  $m$  elements, fix a bijection  $h : [\ell] \rightarrow L$  where  $L \subseteq S$  is any subset of cardinality  $\ell$ . By Lemma 3.1, the complete Sperner family  $\mathcal{S} = \{h(X) \cup (S - L) : X \in \mathcal{T}\}$  of subsets of  $S$  has linear chromatic number  $\ell$ . To show tightness of the upper bound, write  $m = qn + r$  for  $q, r \in \mathbb{Z}$  and  $0 \leq r < n$ , and partition  $S$  into  $n$  subsets  $S_1, \dots, S_n$  such that  $|S_i| = \lceil m/n \rceil$  for  $1 \leq i \leq r$  and  $|S_i| = \lfloor m/n \rfloor$  for  $r + 1 \leq i \leq n$ . Then  $\mathcal{S} = \{\cup_{i=1}^n T_i : T_i \subseteq S_i \text{ for all } i, \sum_{i=1}^n |T_i| = \lfloor m/2 \rfloor\}$  is a Sperner family on  $S$  of size  $s(m, n)$ .

2. Note that when  $n = m$ , the upper bound of Theorem 4.1 is merely  $\binom{m}{\lfloor m/2 \rfloor}$ . This is not surprising in view of the fact the Sperner family on  $S$  consisting of all subsets of size  $\lfloor m/2 \rfloor$  has linear chromatic number  $m$ .

## 5 Sperner families of maximum linear chromatic number

In this last section we address a question raised in [2] concerning unimodality of the numbers  $ls(m; n)$ : does there exist  $k \leq m$  such that  $ls(m; 1) \leq \dots \leq ls(m; k) \geq ls(m; k + 1) \geq \dots \geq ls(m; m)$ ? We show that for sufficiently large  $m$ ,  $ls(m; m)$  is much bigger than  $\sum_{n < m} ls(m; n)$ ; thus the only hope for unimodality of the numbers  $ls(m; n)$  would lie in a proof that they are increasing as a function of  $n$ .

Given a poset  $P$ , we denote by  $a(P)$  the number of antichains in  $P$ . If  $P$  and  $Q$  are posets with the same underlying set, we say that  $Q$  is a *relaxation* of  $P$  if whenever  $y$  covers  $x$  in  $Q$ ,  $y$  also covers  $x$  in  $P$ . In this situation, it is clear that  $a(Q) \geq a(P)$ . The main tool we will need is:

**Theorem 5.1.** ([7, Theorem 1.3]) *Let  $P$  be a graded poset with levels  $P_1, \dots, P_k$ , with  $|P_k| \leq M$ . Suppose  $d_+(x) \geq n$  for all  $x \in P_1 \cup \dots \cup P_{k-1}$  and  $d_-(x) \leq n$  for all  $x \in P_2 \cup \dots \cup P_k$ . Then  $a(P) \leq (k2^n - (k - 1))^{M/n}$ .*

For our purposes, the weaker inequality  $a(P) \leq k^{M/n} 2^M = 2^{M(1 + \frac{\log_2 k}{n})}$  suffices.

**Proposition 5.2.** *There are at least  $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor}}$  complete Sperner families on a set of cardinality  $m$ .*

**Proof.**

Suppose  $|S| = m$  and let  $\mathcal{B} = \{B \subseteq S : |B| = \lfloor m/2 \rfloor\}$ . Observe that each of the  $2^{\mathcal{B}} = 2^{\binom{m}{\lfloor m/2 \rfloor}}$  subsets  $\mathcal{S} \subseteq \mathcal{B}$  is a Sperner family; hence it suffices to count the primitive families among these. If  $\mathcal{S}$  is imprimitive, some  $s \in S$  must be a base point of  $\mathcal{S}$ ; that is,  $\mathcal{S} \subseteq \mathcal{B}_s = \{B \in \mathcal{B} : s \in B\}$ . Since  $|\mathcal{B}_s| = \binom{m-1}{\lfloor m/2 \rfloor - 1}$  and there are  $m$  choices for  $s$ , it follows that there are at least  $2^{\binom{m}{\lfloor m/2 \rfloor}} - m2^{\binom{m-1}{\lfloor m/2 \rfloor - 1}}$  complete Sperner families on  $S$ .  $\square$

**Remark.**

The result of Proposition 5.2 indicates that – as one might expect – most Sperner families are complete. Indeed,  $\binom{m-1}{\lfloor m/2 \rfloor - 1} = \frac{\lfloor m/2 \rfloor}{m} \binom{m}{\lfloor m/2 \rfloor} \leq \frac{1}{2} \binom{m}{\lfloor m/2 \rfloor}$ ; so for large values of  $m$ ,  $m2^{\binom{m-1}{\lfloor m/2 \rfloor}} \leq 2^{\log_2 m + \frac{1}{2} \binom{m}{\lfloor m/2 \rfloor}}$  is much smaller than  $2^{\binom{m}{\lfloor m/2 \rfloor}}$ . A much more precise result in this direction was proven by Korshunov in [10] (see also [11], [12]).

To study complete linear Sperner families  $\mathcal{S}$  of chromatic number at most  $m-1$  on a set  $S$  with  $m$  elements, first note that any such family, being  $(m-1)$ -linearly colorable, is determined by a partition of  $S$  into  $m-2$  singletons and one doubleton, an ordering of the two elements within that doubleton, and an antichain in the poset  $P = C_2, \underbrace{1, \dots, 1}_{m-2}$ . Clearly there are  $\frac{m(m-1)}{2}$  such doubletons and two orderings for each choice, so to obtain an upper bound on the number of such  $\mathcal{S}$ , it remains to bound the number of antichains in  $P$ . To this end, we use Theorem 5.1, following a construction similar to that found in [7, Prop. 5.1]. Note that  $P$  is a ranked poset with levels  $L_i$ ,  $i = 0, \dots, m$ , where  $\ell_i = |L_i| = \binom{m-2}{i} + \binom{m-2}{i-1} + \binom{m-2}{i-2} = \binom{m-1}{i} + \binom{m-2}{i-2}$ . By Proposition 2.1, level  $m/2$  is the largest level when  $m$  is even and levels  $\lfloor m/2 \rfloor$  and  $\lceil m/2 \rceil$  are the largest when  $m$  is odd.

Viewing  $P$  as a directed graph whose arcs originate at vertices in  $L_i$  and terminate at vertices in  $L_{i+1}$ , observe that for  $x \in L_i$ ,  $i-1 \leq d_-(x) \leq i$  and  $m-i \leq d_+(x) \leq m-i+1$ . Now suppose  $i > \lceil m/2 \rceil$  and  $x \in L_i$  with  $d_-(x) > \lceil m/2 \rceil$ ; write  $x = (a_1, \dots, a_{m-1})$ , where  $0 \leq a_1 \leq 2$  and  $0 \leq a_j \leq 1$  for  $2 \leq j \leq m-1$ . Then let  $J_x = \{j : a_j \neq 0\}$  and

let  $K_x \subseteq J_x$  consist of the  $i - \lceil m/2 \rceil$  largest elements of  $J_x$ . For each  $k \in K_x$ , let  $x(k)$  denote the vertex  $(b_1, \dots, b_{m-1})$  in level  $i - 1$ , where  $b_j = a_j$  for  $j \neq k$  and  $b_k = 0$ , and let  $\mathcal{A}_x$  denote the set of arcs connecting  $x(k)$  to  $x$  for all  $k \in K_x$ . Define  $P'$  to be the poset obtained from  $P$  by deleting the set of arcs  $\bigcup_{i > \lceil m/2 \rceil} \bigcup_{x \in L_i} \mathcal{A}_x$ . Observe that  $P'$  is a relaxation of  $P$  and that every vertex in  $P'$  has in-degree at most  $\lceil m/2 \rceil$ . Moreover, if  $i > \lceil m/2 \rceil$ , then every vertex in  $L'_i$ , the  $i$ th level of  $P'$ , has in-degree exactly  $\lceil m/2 \rceil$ .

We now describe the construction of a poset  $Q$  of which  $P'$  is a subposet. For each  $i > \lceil m/2 \rceil$ , add  $\ell_{\lceil m/2 \rceil} - \ell_i$  new vertices to  $L'_i$ , and add new arcs originating at vertices in  $L'_{i-1}$  and terminating at these new vertices in such a way that all vertices in the (new)  $i - 1$ st level have out-degree  $\lceil m/2 \rceil$  and all vertices in the (new)  $i$ th level have in-degree  $\lceil m/2 \rceil$ . Denote by  $Q$  the resulting poset. Now  $Q$  satisfies the hypotheses of Theorem 5.1 with  $k = m + 1$ ,  $n = \lceil m/2 \rceil$ , and

$$M = \ell_{\lceil m/2 \rceil} = \binom{m-1}{\lceil m/2 \rceil} + \binom{m-2}{\lceil m/2 \rceil - 2} \leq \frac{3}{4} \left(1 + \frac{5}{4(m-2)}\right) \binom{m}{\lceil m/2 \rceil}$$

Therefore,  $a(P) \leq a(P') \leq a(Q) \leq 2^{\frac{3}{4}(1 + \frac{5}{4(m-2)})} \binom{m}{\lceil m/2 \rceil}^{(1 + 2\frac{\log_2(m+1)}{m})}$ .

Thus we have proven the following:

**Proposition 5.3.** *Let  $S$  be a set of size  $m$ . The number of complete Sperner families on  $S$  of linear chromatic number at most  $m - 1$  is bounded above by:*

$$m(m-1) 2^{\frac{3}{4}(1 + \frac{5}{4(m-2)})} \binom{m}{\lceil m/2 \rceil}^{(1 + 2\frac{\log_2(m+1)}{m})}$$

*In particular, for any  $\varepsilon > 0$ , the number of such families is at most  $2^{(\frac{3}{4} + \varepsilon) \binom{m}{\lceil m/2 \rceil}}$  for sufficiently large  $m$ .*

Combining this with the estimate of Proposition 5.2 and the succeeding remark, we have:

**Corollary 5.4.** *On a set of cardinality  $m$ , there are at least*

$$2^{\binom{m}{\lceil m/2 \rceil}} - m 2^{\frac{1}{2} \binom{m}{\lceil m/2 \rceil}} - m(m-1) 2^{\frac{3}{4}(1 + \frac{5}{4(m-2)})} \binom{m}{\lceil m/2 \rceil}^{(1 + 2\frac{\log_2(m+1)}{m})}$$

*complete Sperner families of linear chromatic number  $m$ . In particular, for any  $\varepsilon > 0$ , the number of such families is at least*

$$2^{\binom{m}{\lceil m/2 \rceil}} - 2^{(3/4 + \varepsilon) \binom{m}{\lceil m/2 \rceil}} - 2^{(1/2 + \varepsilon) \binom{m}{\lceil m/2 \rceil}}$$

*for sufficiently large  $m$ .*

Therefore, the number of Sperner families of maximum linear chromatic number grows much faster (as a function of  $m$ ) than the number of those of lesser chromatic number.

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