

LEFSCHETZ DECOMPOSITIONS FOR QUOTIENT VARIETIES

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ABSTRACT. In an earlier paper, the authors constructed an explicit Chow-Kunneth decomposition for quotient varieties of Abelian varieties by actions of finite groups. In the present paper, the authors extend the techniques there to obtain an explicit Lefschetz decomposition for such quotient varieties for the Chow-Kunneth projectors constructed there.

1. Introduction

It has been conjectured that every smooth projective variety X over a field k has a Chow-Künneth decomposition. Currently, Chow-Künneth decompositions are known to exist for curves and projective spaces [14], surfaces [16], abelian varieties ([5], [19]), varieties with “finite-dimensional” motives [10], and several other special classes. In an earlier paper [1], the authors proved that the quotient A/G of an abelian variety A by the action of a finite group G has a Chow-Künneth decomposition, the projectors of which can be described explicitly by pushing forward the Chow-Künneth projectors of A (as constructed by Deninger and Murre [5]) via the quotient map $A \times A \rightarrow A/G \times A/G$. Although A/G is not in general smooth, the finiteness of G ensures that the machinery of intersection theory and Chow motives can be extended to varieties of this sort, which we term *pseudo-smooth*. Moreover, there are quotient varieties of the above form which are smooth, but not abelian varieties; Igusa [9] gives such a construction, possibly due earlier to Enriques.

In [13], Künnemann proves the existence of a Lefschetz decomposition for Chow motives of abelian schemes; it seems natural to ask whether such a decomposition can be given for the quotient of an abelian variety, and, if so, if this can be given explicitly. Kahn, Murre, and Pedrini [10] have shown the existence of such a decomposition for these quotient varieties under the assumption of certain standard conjectures (which are shown to hold in characteristic 0 by [3]). The aim of this article is to construct this decomposition explicitly, in arbitrary characteristic, without assuming any conjectures.

We recall the main result of [1], which we will need in our proof. As in [1], we work throughout in the category of rational Chow motives for pseudo-smooth projective varieties.

Theorem 1.1. (See Theorem 1.1 in [1].) *Let A be an abelian variety of dimension d over a field k and G a finite group acting on A . Let $f : A \rightarrow A/G$ be the quotient map. Suppose $[\Delta_A] = \sum_{i=0}^{2d} \pi_i$ is the Beauville-Deninger-Murre Chow-Künneth decomposition for A and let $\eta_i = \frac{1}{|G|^3} (f \times f)_* \sum_{g,h \in G} (g, h)^* \pi_i$.*

Then

$$[\Delta_{A/G}] = \sum_{i=0}^{2d} \eta_i$$

is a Chow-Künneth decomposition for A/G . This decomposition satisfies Poincaré duality: that is, for any i , $\eta_{2d-i} = {}^t \eta_i$.

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In addition, η_i acts as zero on $CH_{\mathbb{Q}}^j(A/G)$ for $i < j$ and also for $i > j + d$ in general. In case $d \leq 4$, we may also conclude that η_i acts trivially on $CH_{\mathbb{Q}}^j(A/G)$ for $i < j$ and also for $i > 2j$.

Now let k be a field and A an abelian variety of dimension d over k . Let G be a finite group acting on A , and let $f : A \rightarrow A/G$ denote the quotient map. We write $\Delta : A \rightarrow A \times A$ for the diagonal map on A .

Following Beauville [4], we set $CH_s^i(A, \mathbb{Q}) = \{x \in CH^i(A, \mathbb{Q}) : \mathbf{n}^*x = n^{2i-s}x \text{ for all } n \neq 0, \pm 1\}$ and recall that

$$CH^i(A, \mathbb{Q}) = \bigoplus_{s=i-d}^i CH_s^i(A, \mathbb{Q})$$

Let \mathcal{L} be an ample line bundle on A , and set $b = c_1(\mathcal{L})$. As in [13], Assumption 2.1, we may assume without loss of generality that

$$(1.0.1) \quad b \in CH_0^1(A, \mathbb{Q})$$

Indeed, if $\sigma : A \rightarrow A$ is the involution $a \mapsto -a$, then $\mathcal{L}' = \mathcal{L} \otimes \sigma^*\mathcal{L}$ is another ample line bundle on A , and $c_1(\mathcal{L}') = b + \sigma^*b$. Writing $b = b_0 + b_1$ where $b_0 \in CH_0^1(A, \mathbb{Q})$ and $b_1 \in CH_1^1(A, \mathbb{Q})$, we have, for any $n \geq 0, \pm 1$, $\mathbf{n}^*(b_1 + \sigma^*b_1) = \mathbf{n}^*(b_1) + (-\mathbf{n})^*b_1 = nb_1 - nb_1 = 0$, so that $b_1 + \sigma^*b_1 = 0$. Moreover, $\mathbf{n}^*(\sigma^*b_0) = (-\mathbf{n})^*b_0 = (-n)^2b_0 = n^2b_0$, so that $\sigma^*b_0 = b_0$ and so $b + \sigma^*b = 2b_0$.

Thus

$$\mathbf{n}^*(b + \sigma^*b) = \mathbf{n}^*(2b_0) = 2n^2b_0 = n^2(b + \sigma^*b)$$

and hence $b + \sigma^*b \in CH_0^1(A, \mathbb{Q})$.

Define

$$(1.0.2) \quad \begin{aligned} \mathcal{L}_G &= \bigotimes_{g \in G} g^*(\mathcal{L}), \\ b_G &= c_1(\mathcal{L}_G) = \sum_{g \in G} c_1(g^*\mathcal{L}) = \sum_{g \in G} g^*c_1(\mathcal{L}) = \sum_{g \in G} g^*b, \\ L &= \Delta_*(b), \\ L_G &= \Delta_*(b_G) = \sum_{g \in G} (g \times g)^*L, \\ L_{GG} &= \sum_{g, h \in G} (g \times h)^*L \\ \tilde{L} &= \frac{1}{|G|^3} (f \times f)_*(L_{GG}) \end{aligned}$$

Now let F_A ($F_{\hat{A}}$) denote the Fourier transform on A (respectively, \hat{A}). We then define

$$(1.0.3) \quad \begin{aligned} c_G &= b_G^{d-1}/(d-1)!, \\ \Lambda_G &= (-1)^d [{}^t\Gamma_\sigma] \circ F_{\hat{A}} \circ \Delta_*(F_A \circ (c_G)) \circ F_A, \\ \Lambda_{GG} &= \sum_{g, h \in G} (g \times h)^*(\Lambda_G) = |G| \sum_{g \in G} (g \times 1)_*(\Lambda_G) = |G| \sum_{h \in G} (1 \times h)_*(\Lambda_G) \\ \tilde{\Lambda} &= \frac{1}{|G|^3} (f \times f)_*(\Lambda_{GG}) \end{aligned}$$

In the next section we prove the following theorem, although our primary interest is in the corollary that follows:

Theorem 1.2. *The data $(CH^*(A/G; \mathbb{Q}), \tilde{L}, \tilde{\Lambda}, (\eta_i)_{i=0}^{2d})$ define a Lefschetz algebra in the sense of [13, section 4]. In particular, this implies that there exist projectors $\{q_{i,k}|i,k\}$ refining the projectors $\{\eta_i|i\}$ on A/G such the following relations hold on the sub-algebra of A/G generated by these correspondences:*

- (a) $\tilde{L} \circ \eta_i = \eta_{i+2} \circ \tilde{L}$,
- (b) $\tilde{\Lambda} \circ \eta_i = \eta_{i-2} \circ \tilde{\Lambda}$ and
- (c) $[\tilde{\Lambda}, \tilde{L}] = \tilde{\Lambda} \circ \tilde{L} - \tilde{L} \circ \tilde{\Lambda} = \sum_{i=0}^{2d} (d-i)\eta_i$

These imply:

- (i) $\sum_k q_{i,k} = \eta_i$ for each i ,
- (ii) $q_{i,k} \circ \eta_j = \eta_j \circ q_{i,k} = q_{i,k}$ if $i = j$ or 0 otherwise,
- (iii) $q_{i,k} = 0$ for (i,k) not in $I = \{(i,k) \in \mathbb{Z} \times \mathbb{Z} \mid \max(0, i-d) \leq k \leq [i/2]\}$,
- (iv) $q_{i,k} \circ q_{j,l} = q_{i,k}$ if $i = j$ and $k = l$ and = 0 otherwise,
- (v) $q_{i,k} \circ \tilde{L} = \tilde{L} \circ q_{i-2,k-1}$,
- (vi) $\tilde{\Lambda} \circ q_{i,k} = q_{i-2,k-1} \circ \tilde{\Lambda}$,
- (vii) $\tilde{L} \circ \tilde{\Lambda} \circ q_{i,k} = k(g-i+k+1)q_{i,k}$ and
- (viii) $\tilde{\Lambda} \circ \tilde{L} \circ q_{i,k} = (k+1)(g-i+k)q_{i,k}$

Corollary 1.3. *Let A be an abelian variety of dimension d over a field k and G a finite group acting on A . Let A/G be the quotient variety and let $h^i(A/G) = \eta_i$ denote the Chow-Künneth components constructed as in the theorem above. Define $L^k P^i(A/G) = (A/G, q_{i+2k,k})$. Then the following hold:*

- (i) For $0 \leq i \leq 2d$, the Chow motive $h^i(A/G) = (A/G, \eta_i)$ has a Lefschetz decomposition

$$h^i(A/G) = \bigoplus_{k=\max\{0, g-i\}}^{[i/2]} L^k P^i(A/G)$$

- (ii) (Hard Lefschetz) The iterated composition of correspondences $\tilde{L}^i : h^{d-i}(A/G) \rightarrow h^{d+i}(A/G)$ is an isomorphism of Chow motives for all $0 \leq i \leq d$.

- (iii) If k is a finite field and $0 \leq j \leq d$, the map $g : CH^j(A/G, \mathbb{Q}) \rightarrow CH^{d-j}(A/G, \mathbb{Q})$ defined by $g(\alpha) = \tilde{L}^{d-2j} \circ \alpha$ is an isomorphism.

We remark that assertions (i) and (ii) above follow immediately from Theorem 1.2 by Sections 4 and 5 of [13]; thus, we focus our attention on proving Theorem 1.2 and statement (iii) above.

The above theorem clearly applies to the following classes of examples considered in [1].

Examples 1.4. (1) **Symmetric products of abelian varieties.** Let X denote an abelian variety and X^n/Σ_n the n -fold symmetric power of X . Observe that the action of Σ_n is not in general free so that the quotient X^n/Σ_n may not be smooth.

(2) **Example of Igusa.** (See [9]) Let X be an elliptic curve over k , with $\text{char}(k) \neq 2$. Let t denote a point of order 2 on X . Define an action of $\mathbb{Z}/2\mathbb{Z}$ on $X \times X$ by $(x, y) \mapsto (x+t, -y)$, and let Y denote

the quotient variety for this action. The resulting surface is a so-called *bi-elliptic surface*; see [4] VI, 19-20. (This example may be generalized by taking X to be an abelian variety.) Now one sees easily that the action is free so that Y is smooth. Nevertheless, in positive characteristic, Y need not be an abelian variety as shown in [9].

(3) Kummer varieties

The techniques involved in our proof are extensions of those of [1], the main advantage being that it yields explicit closed formulae for all the operators involved. In contrast, the construction of the refined projectors $\{q_{i,k}|i,k\}$ in [10] is an inductive one; when applied to finite quotients of abelian varieties, it is of exponential complexity in the dimension of the abelian variety.

2. Proofs

2.1. Preliminaries. One of the key steps in proving the main theorem of [1] was to show that given any action $\alpha : G \times A \rightarrow A$ of G on A , there exists an action $\beta : G \times A \rightarrow A$ of G on A with the following properties: first, the quotient of A by the first action of G is isomorphic to the quotient of A by the second action; second, for every g in G , $\beta(g, 0)$ is a torsion point of A . This reduction will also be useful to us in the present article, so we assume henceforth that *for every $g \in G$, $g(0)$ is a torsion point of A .*

Now let m_g be the order of $a_g = g(0)$. Next, let $m = \prod_{g \in G} m_g$, and

$$E = \{n \in \mathbf{Z} : n \equiv 1 \pmod{m}, n \neq \pm 1\}$$

Note that if $n \in E$, m_g divides $n - 1$ (for any g), so $na_g = a_g$.

For each $g \in G$, the map $g : A \rightarrow A$ defined by $a \mapsto g \cdot a$ factors uniquely as $g = \tau_{a_g} \circ g_0$, where $g_0 : A \rightarrow A$ is a homomorphism, and $\tau_{a_g} : A \rightarrow A$ is the translation $a \mapsto a + a_g$. Thus, if $n \in E$ and $a \in A$, $(\mathbf{n} \circ g)(a) = n(g(a)) = n(g_0(a) + a_g) = g_0(na) = (g \circ \mathbf{n})a$; that is,

$$\mathbf{n} \circ g = g \circ \mathbf{n}.$$

Recall our choice of ample line bundle \mathcal{L} from Section 1. It follows from [7], Exercise II.7.5 that $\mathcal{L}_G = \otimes_{g \in G} g^*(\mathcal{L})$ is also an ample line bundle. Moreover, because $b = c_1(\mathcal{L}) \in CH_0^1(A, \mathbb{Q})$, $\mathbf{n}^*b = n^2b$ for all $n \neq 0, \pm 1$. Now if $n \in E$ and $g \in G$, $\mathbf{n}^*(g^*b) = g^*(\mathbf{n}^*b) = n^2g^*b$. By Proposition 2.5, this suffices to show that $g^*b \in CH_0^1(A, \mathbb{Q})$ and hence that $d_G = \sum_{g \in G} g^*d \in CH_0^1(A, \mathbb{Q})$ is an ample line bundle satisfying the hypothesis in 1.0.1.

2.2. Technical Lemmas. We now make some general computations which will be helpful in proving the main theorem. We will use the following lemma repeatedly in the sequel – often without explicit mention.

Lemma 2.1. *Let $g, h \in G$ and $\alpha, \beta \in CH^*(A \times_k A, \mathbb{Q})$. Then for any $k \in G$,*

$$(i) (g \times h)^*\alpha \circ \beta = (k \times h)^*(\alpha \circ (k^{-1} \times g^{-1})^*\beta)$$

$$(ii) \alpha \circ (g \times h)^*\beta = (g \times k)^*((h^{-1} \times k^{-1})^*\alpha \circ \beta)$$

$$(iii) (g \times g)^*(\alpha \circ \beta) = (g \times g)^*\alpha \circ (g \times g)^*\beta.$$

Proof.

The third statement follows from the first by taking $h = k = g$ and replacing β by $(g \times g)^*\beta$. We prove the first statement only, as the second is similar.

$$\begin{aligned}
(g \times h)^*\alpha \circ \beta &= p_{13*}(p_{12}^*\beta \bullet p_{23}^*(g \times h)^*\alpha) = p_{13*}(p_{12}^*\beta \bullet (k \times g \times h)^*p_{23}^*\alpha) \\
&= p_{13*}(k \times g \times h)^*((k^{-1} \times g^{-1} \times h^{-1})^*p_{12}^*\beta \bullet p_{23}^*\alpha) = p_{13*}(k^{-1} \times g^{-1} \times h^{-1})_*(p_{12}^*(k^{-1} \times g^{-1})^*\beta \bullet p_{23}^*\alpha) \\
&= (k^{-1} \times h^{-1})_*p_{13*}(p_{12}^*(k^{-1} \times g^{-1})^*\beta \bullet p_{23}^*\alpha) = (k \times h)^*p_{13*}(p_{12}^*(k^{-1} \times g^{-1})^*\beta \bullet p_{23}^*\alpha) \\
&= (k \times h)^*(\alpha \circ (k^{-1} \times g^{-1})^*\beta)
\end{aligned}$$

Following the notation of (1.0.2), we define, for $\alpha \in CH^*(A \times A, \mathbb{Q})$, $\alpha_G = \sum_{g \in G} (g \times g)^*\alpha$ and $\alpha_{GG} = \sum_{g, h \in G} (g \times h)^*\alpha$. Observe that $\alpha_{GG} = \sum_{g, h \in G} (g \times h)^*\alpha = \sum_{g \in G} (g \times 1)^*(\sum_{h \in G} (h \times h)^*\alpha) = \sum_{g \in G} (g \times 1)^*\alpha_G$

and similarly $\alpha_{GG} = \sum_{h \in G} (1 \times h)^*\alpha_G$.

Lemma 2.2. *If $\alpha, \beta \in CH^*(A \times A, \mathbb{Q})$, then $\alpha_{GG} \circ \beta_{GG} = (\alpha_G \circ \beta_G)_{GG}$.*

Proof. From the equations above,

$$\begin{aligned}
\alpha_{GG} \circ \beta_{GG} &= \sum_{h \in G} (1 \times h)^*\alpha_G \circ \sum_{g \in G} (g \times 1)^*\beta_G \\
&= \sum_{g, h \in G} p_{13*}(p_{12}^*(g \times 1)^*\beta_G \bullet p_{23}^*(1 \times h)^*\alpha_G) \\
&= \sum_{g, h \in G} p_{13*}(g \times 1 \times h)^*(p_{12}^*\beta_G \bullet p_{23}^*\alpha_G) \\
&= \sum_{g, h \in G} (g \times h)^*p_{13*}(p_{12}^*\beta_G \bullet p_{23}^*\alpha_G) = \sum_{g, h \in G} (g \times h)^*(\alpha_G \circ \beta_G) = (\alpha_G \circ \beta_G)_{GG}
\end{aligned}$$

2.3. We now return to the proof of Theorem 1.3. By Lemma 2.4 of [1], the elements $\rho_i = \sum_{g, h \in G} (g, h)^*\pi_i = (\pi_i)_{GG} \in CH^d(A \times A, \mathbb{Q})^{G \times G}$, $0 \leq i \leq 2d$ satisfy $\rho_i \circ \rho_i = |G|^2\rho_i$ and $\rho_i \circ \rho_j = 0$ if $i \neq j$.

In [13], Künnemann defines correspondences $L, \Lambda \in CH^*(A \times A, \mathbb{Q})$ with respect to a choice of symmetric ample line bundle and shows that $(CH^*(A \times A, \mathbb{Q}), L, \Lambda, (\pi_i)_{i=0}^{2d})$ is a Lefschetz algebra. Since \mathcal{L}_G is such a line bundle, we see immediately that

$$(2.3.1) \quad (CH^*(A \times A, \mathbb{Q}), L_G, \Lambda_G, (\pi_i)_{i=0}^{2d}) \text{ is a Lefschetz algebra in the sense of [13].}$$

In particular, the following hold (for any j):

$$(2.3.2) \quad L_G \circ \pi_j = \pi_{j+2} \circ L_G$$

$$(2.3.3) \quad \Lambda_G \circ \pi_j = \pi_{j-2} \circ \Lambda_G$$

$$(2.3.4) \quad [L_G, \Lambda_G] = \sum_{i=0}^{2d} (d-i)\pi_i$$

In this light, the following Proposition 2.3 may be viewed as an equivariant (almost) analogue of the statement (2.3.1).

Proposition 2.3. *The following properties hold:*

- (i) $CH^*(A \times A, \mathbb{Q})^{G \times G}$ is a graded \mathbb{Q} -algebra with unit element $\frac{1}{|G|^2}[\Delta_A]_{GG}$.
- (ii) $\sum_{i=0}^{2d} \rho_i = [\Delta_A]_{GG}$
- (iii) $\rho_i \circ \rho_j = \begin{cases} |G|^2 \rho_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- (iv) $L_{GG} \circ \rho_i = \rho_{i+2} \circ L_{GG}$
- (v) $\Lambda_{GG} \circ \rho_i = \rho_{i-2} \circ \Lambda_{GG}$
- (vi) $[\Lambda_{GG}, L_{GG}] = |G|^2 \sum_{i=0}^{2d} (d-i) \rho_i$.

Proof.

Lemma 2.1 shows that $CH^*(A \times A, \mathbb{Q})^{G \times G}$ is a ring under cycle addition and composition of correspondences, with unit element $\frac{1}{|G|^2}[\Delta_A]_{GG}$. The second and third statements follow from the construction of the ρ_i . It remains to prove that last three statements.

Recall that $\rho_i = (\pi_i)_{GG}$. By Lemmas 2.1 and 2.2, we have

$$L_{GG} \circ \rho_i = L_{GG} \circ (\pi_i)_{GG} = (L_G \circ (\pi_i)_G)_{GG} = (L_G \circ \sum_{g \in G} (g \times g)^* \pi_i)_{GG} = \sum_{g \in G} ((g \times g)^* (L_G \circ \pi_i))_{GG}$$

On the other hand,

$$\rho_{i+2} \circ L_{GG} = (\pi_{i+2})_{GG} \circ L_{GG} = ((\pi_{i+2})_G \circ L_G)_{GG} = \sum_{g \in G} [(g \times g)^* \pi_{i+2} \circ L_G]_{GG} = \sum_{g \in G} [(g \times g)^* (\pi_{i+2} \circ L_G)]_{GG}$$

The two quantities are equal by (2.3.2), thereby proving the fourth statement. The proof of the fifth statement is virtually identical. .

It remains to prove the commutator relation. Again, using Lemmas 2.1 and 2.2 together with (2.3.4),

$$[\Lambda_{GG}, L_{GG}] = \Lambda_{GG} \circ L_{GG} - L_{GG} \circ \Lambda_{GG} = (\Lambda_G \circ L_G)_{GG} - (L_G \circ \Lambda_G)_{GG} = [\Lambda_G, L_G]_{GG} = \sum_{i=0}^{2d} (d-i) (\pi_i)_{GG} = \sum_{i=0}^{2d} (d-i) \rho_i$$

2.4. Proof of Theorem 1.1. The projections $A \times A \times A \rightarrow A \times A$ and $A \times A \rightarrow A$ will be denoted p with the appropriate superscripts and subscripts to indicate which factors are the source and the target; the corresponding projections for A/G will be denoted q with the corresponding superscripts and subscripts. For convenience of notation set $r = (f \times f \times f) : A \times_k A \times_k A \rightarrow A/G \times_k A/G \times_k A/G$.

Lemma 2.4. *Let $\alpha, \beta \in CH^*(A \times A, \mathbb{Q})^{G \times G}$. Then $(f \times f)_*(\alpha) \circ (f \times f)_*(\beta) = |G|(f \times f)_*(\alpha \circ \beta)$. In particular, $((f \times f)_*(\alpha))^i = |G|^{i-1} (f \times f)_*(\alpha^i)$, where the exponent i denotes the i -fold iterated composition of correspondences.*

Proof. Now,

$$(2.4.1) \quad (f \times f)_*(\alpha) \circ (f \times f)_*(\beta) = q_{13}^{123} \circ (q_{12}^{123})^* (f \times f)_*(\alpha) \bullet q_{23}^{123} (f \times f)_*(\beta)$$

Since the degree of r is $|G|^3$, r_*r^* corresponds to multiplication by $|G|^3$, and therefore, the last expression equals:

$$(2.4.2) \quad \frac{1}{|G|^3} q_{13}^{123} {}_*(r_*r^* q_{12}^{123*} (f \times f)_*(\alpha) \bullet q_{23}^{123*} (f \times f)_*(\beta))$$

Because $q_{12}^{123} \circ r = (f \times f) \circ p_{12}^{123}$, the above simplifies to:

$$\frac{1}{|G|^3} q_{13}^{123} {}_*(r_*p_{12}^{123*} (f \times f)^*(f \times f)_*(\alpha) \bullet q_{23}^{123*} (f \times f)_*(\beta))$$

Since α is $G \times G$ -invariant, $(f \times f)^*(f \times f)_*$ is multiplication by $|G|^2$, so the expression equals:

$$\frac{1}{|G|^3} q_{13}^{123} {}_*(r_*p_{12}^{123*} |G|^2 \alpha \bullet q_{23}^{123*} (f \times f)_*(\beta))$$

Finally, applying the projection formula, the formula $q_{ij}^{123} \circ r = (f \times f) \circ p_{ij}^{123}$ and $(G \times G)$ -invariance of both α and β , one may identify the last expression with:

$$\begin{aligned} \frac{1}{|G|} q_{13}^{123} {}_*(r_*p_{12}^{123*} (\alpha) \bullet r_*q_{23}^{123*} (f \times f)_*(\beta)) &= \frac{1}{|G|} (f \times f)_* p_{13}^{123} {}_*(p_{12}^{123*} (\alpha) \bullet p_{23}^{123*} (f \times f)^*(f \times f)_*(\beta)) \\ &= |G| (f \times f)_* p_{13}^{123} {}_*(p_{12}^{123*} (\alpha) \cdot p_{23}^{123*} (\beta)) \\ &= |G| (f \times f)_*(\alpha \circ \beta) \end{aligned}$$

This proves the first statement of the lemma and the second follows readily.

Returning to the proof of Theorem 1.2, define $\tilde{\Lambda} = 1/|G|^3 (f \times f)_*(\Lambda_{GG})$, $\tilde{L} = 1/|G|^3 (f \times f)_*(L_{GG})$ and $\eta_i = 1/|G|^3 (f \times f)_*(\rho_i)$.

Now Lemma 2.4, together with Proposition 2.3 implies that $(CH^*(A/G), \tilde{L}, \tilde{\Lambda}, (\eta_i)_{i=0}^{2d})$ forms a Lefschetz algebra. The first two assertions of Corollary 1.3 follow immediately from the formalism of Sections 4 and 5 of [13].

Next we assume that k is a finite field and prove statement (iii) of Corollary 1.3. Since L_{GG} is $G \times G$ -invariant, Lemma 2.1 shows that the map $\phi : CH^p(A; \mathbb{Q}) \rightarrow CH^{d-p}(A, \mathbb{Q})$ defined by $\alpha \mapsto L_{GG}^{d-2p} \circ \alpha$ leaves the G -invariant part-stable: i.e. ϕ restricts to a map

$$\phi^G : CH^p(A, \mathbb{Q})^G \rightarrow CH^{d-p}(A, \mathbb{Q})^G$$

Observe that the projection $\pi : CH^*(A, \mathbb{Q}) \rightarrow CH^*(A, \mathbb{Q})^G$ sends $\alpha \mapsto \sum_{g \in G} g^*(\alpha)$. Thus one may readily verify the commutativity of the two squares:

$$\begin{array}{ccc} CH^p(A, \mathbb{Q})^G & \xrightarrow{\phi^G} & CH^{d-p}(A, \mathbb{Q})^G \\ \downarrow & & \downarrow \\ CH^p(A, \mathbb{Q}) & \xrightarrow{\phi} & CH^{d-p}(A, \mathbb{Q}) \\ \downarrow \pi & & \downarrow \pi \\ CH^p(A, \mathbb{Q})^G & \xrightarrow{\phi^G} & CH^{d-p}(A, \mathbb{Q})^G \end{array}$$

The middle row is an isomorphism by Hard Lefschetz for the Chow-groups of Abelian varieties as proved in [13]. Therefore the top row is injective while the bottom row is surjective, thereby proving statement (ii) in Theorem 1.1.

2.5. Appendix. In this appendix we discuss briefly the arguments to show that the main results of [13] go through for line bundles \mathcal{L} on an Abelian variety that satisfy the condition that $\mathbf{n}^*(c_1(\mathcal{L})) = n^2(c_1(\mathcal{L}))$. It is shown there that the only hypothesis one needs to impose on \mathcal{L} is that $c_1(\mathcal{L})$ belong to $CH_0^1(A, \mathbb{Q})$. Therefore, it suffices to prove the following result.

Proposition 2.5. *Let E denote the infinite subset of the integers chosen as earlier. Let $\alpha \in CH^1(A, \mathbb{Q})$ be such that $\mathbf{n}^*(\alpha) = n^2(\alpha)$ for all $n \in E$. Then $\alpha \circ \pi_2 = \alpha$ and $\alpha \circ \pi_j = 0$ for all $j \neq 2$ and hence $\alpha \in CH_0^1(A, \mathbb{Q})$.*

Proof. This is a standard computation:

$$\begin{aligned}
 (2.5.1) \quad \mathbf{n}^*(\alpha) &= \mathbf{n}^*(\alpha \circ [\Delta_A]) \\
 &= \sum_{i=0}^{2d} \mathbf{n}^*(\alpha \circ \pi_j) \\
 &= \sum_{i=0}^{2d} \mathbf{n}^*(p_{2*}(p_1^*(\alpha) \bullet \pi_j)) \\
 &= \sum_{i=0}^{2d} p_{2*}(p_1^*(\alpha) \bullet (1 \times n)^*(\pi_j)) \\
 &= \sum_{i=0}^{2d} p_{2*}(p_1^*(\alpha) \bullet n^j \pi_j) \\
 &= \sum_{i=0}^{2d} n^j (\alpha \circ \pi_j)
 \end{aligned}$$

Therefore, we obtain:

$$n^2(\alpha - \alpha \circ \pi_2) + \sum_{j \neq 2} n^j (\alpha \circ \pi_j) = 0$$

for all $n \in E$. Since E is an infinite subset of the integers with infinitely many primes in it, the required conclusion follows.

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