# Homology of zero-divisors

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**Abstract**

Let $R$ be a commutative ring with unity. We define a semisimplicial abelian group based on the structure of the semigroup of ideals of $R$ and investigate various properties of the homology groups of the associated chain complex.

## 1 Introduction

Let $R$ be a commutative ring with unity. The set $Z(R)$ of zero-divisors in a ring does not possess any obvious algebraic structure; consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [2], [3] have focused on studying the so-called *zero-divisor graph* $\Gamma_R$, whose vertices are the zero-divisors of $R$, with $xy$ being an edge if and only if $xy = 0$. This object $\Gamma_R$ is somewhat unwieldy in that it has many symmetries; for example, if $u \in R^*$ is any unit, then $x \mapsto ux$ induces a (graph) automorphism of $\Gamma_R$. One way of treating this issue – following an idea of Lauve [5] – is to work with the *ideal zero-divisor graph* $\mathcal{I}_R$. In effect, one replaces zero-divisors of $R$ by proper ideals with nonzero annihilator; this is the approach adopted by the authors in [1]. Such a perspective also has its shortcomings; for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals $I, J, K$ in $R$ with $IJK = 0$, but $IJ \neq 0$, $IK \neq 0$, $JK \neq 0$.

In this paper we adopt a different philosophy, using a new type of homology to study $Z(R)$ and capture the situation described above. Roughly speaking, if we denote by $\mathbf{Z}_n(R)$ the free abelian group generated by the set of $(n+1)$-tuples $(I_0, \dots, I_n)$ of distinct ideals of $R$ such that $I_0 \cdot \ldots \cdot I_n \neq 0$, there are obvious maps $\mathbf{Z}_n(R) \longrightarrow \mathbf{Z}_{n-1}(R)$ obtained by forgetting one of the factors. This gives $\mathbf{Z}_.(R)$ the structure of a semi-simplicial abelian group; hence we may speak of its associated chain complex $\mathbf{C}_.(R)$. Our homology groups $H_*(R)$ are then defined as the homology groups of a certain quotient of $\mathbf{C}_.(R)$. The idea behind this construction was sketched by Lauve in [5], although the precise definition is due to the authors.

After giving a precise definition of these homology groups $H_*(R)$, we study the group $H_0(R)$ in depth and compute $H_1(\mathbb{Z}/p^r\mathbb{Z})$ when $p$ is a prime and $r \geq 1$ is an integer. We then give some conditions on $R$ sufficient to ensure that $H_n(R) = 0$ for $n > 0$. In the last section we consider the *Euler characteristic* $\chi(R) = \sum_{n=0}^{\infty}(-1)^n \text{rk } H_n(R)$. Using some ideas from partition theory, we prove the surprising result that $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is always either 0, 1, or 2, depending on the value of $r$ relative to the "pentagonal" numbers $m(3m-1)/2$ and the related numbers $m(3m+1)/2$. We also derive formulas for the Euler characteristic for some other special types of finite rings.

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# 2 Preliminaries

Let $R$ be a commutative ring and $\mathcal{P}$ the set of proper ideals of $R$. For each $n \geq 0$, let $S_n(R)$ be the set of ordered $(n+1)$-tuples $(I_0, \ldots, I_n)$, where $I_0, \ldots, I_n$ are distinct proper ideals of $R$ and $I_0 I_1 \ldots I_n \neq 0$; let $S_{-1}(R)$ be a set consisting of one element. If there is no danger of ambiguity, we simply write $S_n$ instead of $S_n(R)$. Observe that for each $i$, $0 \leq i \leq n$, there is a "face map" $\phi_i^n : S_n \to S_{n-1}$ defined by $\phi_i^n(I_0, \ldots, I_n) = (I_0, \ldots, \hat{I}_i, \ldots, I_n)$. Moreover, $S_0(R) = \emptyset$ if and only if $R$ is a field, so when $R$ is not a field, there is a unique "augmentation" map $\varepsilon : S_0(R) \to S_{-1}(R)$. Now for each $n \geq -1$, let $Z_n$ be the free abelian group generated by $S_n$. We denote by $[I_0, \ldots, I_n]$ the basis element corresponding to $(I_0, \ldots, I_n) \in S_n$. Likewise, the various face maps $\phi_i^n$ extend $\mathbb{Z}$-linearly to maps $\phi_i^n : Z_n \to Z_{n-1}$; moreover, if $S_0 \neq \emptyset$, there is a unique $\mathbb{Z}$-linear map $\varepsilon : Z_0 \to Z_{-1} = \mathbb{Z}$ defined by $\varepsilon(\sum n_i(I_i)) = \sum n_i$. Thus, there is a semisimplicial abelian group:

$$\mathbf{Z}_{\cdot}(R): \quad \ldots \overset{\to}{\underset{\to}{\to}} Z_1 \rightrightarrows Z_0$$

with augmentation $\varepsilon : Z_0 \to \mathbb{Z}$ if $R$ is not a field.

This in turn gives rise to an (augmented) chain complex in the standard manner by taking an alternating sum of face maps. For each $n \geq 0$, define $\delta_n = \sum_{i=0}^{n}(-1)^i \phi_i^n$; then we have a complex:

$$\mathbf{C}_{\cdot}(R): \qquad \ldots \xrightarrow{\delta_1} Z_1 \xrightarrow{\delta_0} Z_0$$

of abelian groups.

In practice, the $Z_n$ are too large to be useful invariants; in particular, we chose $Z_n$ to be the free $\mathbb{Z}$-module with basis $S_n$, which consisted of *ordered* $(n+1)$-tuples of ideals of $R$ having nonzero product. Because multiplication in $R$ is commutative, the order of the ideals in this $(n+1)$-tuple ought not to matter; it might appear more natural to work with *unordered* $(n+1)$-tuples. Unfortunately, the definition of the face maps *does* depend on the ordering within each such tuple, so we resort instead to the following device: for each $n \geq 0$, let $R_n$ denote the subgroup of $Z_n$ generated elements of the form:

$$[I_0, \ldots, I_n] - (-1)^{sgn\ \sigma}[I_{\sigma(0)}, \ldots, I_{\sigma(n)}]$$

where $\sigma$ in an element of the symmetric group $\mathfrak{S}_{n+1}$ (viewed as permutations of the set $\{0, \ldots, n\}$) and $[I_0, \ldots, I_n]$ is a basis element of $Z_n$. Set $T_n = Z_n/R_n$.

We claim that $\delta_n(R_n) \subseteq R_{n-1}$. Thus we must show

$$\delta_n([I_0, \ldots, I_n]) \equiv (-1)^{sgn\ \sigma}\delta_n([I_{\sigma(0)}, \ldots, I_{\sigma(n)}])(\text{mod } R_{n-1}).$$

Since every permutation may be written as a product of transpositions, we may reduce to the case that $\sigma$ is the transposition which exchanges $r$ and $s$, where $0 \leq r < s \leq n$. In this case,

$$(-1)^{sgn\ \sigma}\delta_n([I_{\sigma(0)}, \ldots, I_{\sigma(n)}]) = -\sum_{i=0}^{n}(-1)^i[I_{\sigma(0)}, \ldots, \hat{I}_{\sigma(i)}, \ldots, I_{\sigma(n)}]$$

$$= \sum_{i \neq r,s}(-1)^{i+1}[I_0, \ldots, I_{r-1}, I_s, I_{r+1}, \ldots, \hat{I}_i, \ldots, I_{s-1}, I_r, I_{s+1}, \ldots, I_n]$$
$$+(-1)^{r+1}[I_0, \ldots, I_{r-1}, I_{r+1}, \ldots, I_{s-1}, I_r, I_{s+1}, \ldots, I_n]$$
$$+(-1)^{s+1}[I_0, \ldots, I_{r-1}, I_s, I_{r+1}, \ldots, I_{s-1}, I_{s+1}, \ldots, I_n]$$

$$\equiv \sum_{i \neq r,s}(-1)^i[I_0, \ldots, I_{r-1}, I_r, I_{r+1}, \ldots, \hat{I}_i, \ldots, I_{s-1}, I_s, I_{s+1}, \ldots, I_n]$$
$$+(-1)^s[I_0, \ldots, I_{r-1}, I_r, I_{r+1}, \ldots, I_{s-1}, I_{s+1}, \ldots, I_n]$$
$$+(-1)^{2s-r}[I_0, \ldots, I_{r-1}, I_{r+1}, \ldots, I_{s-1}, I_s, I_{s+1}, \ldots, I_n](\text{mod } R_{n-1})$$

$$\equiv \sum_{i=0}^{n}(-1)^i[I_0,\ldots,\hat{I}_i,\ldots,I_n](\text{mod } R_{n-1}) \equiv \delta_n([I_0,\ldots,I_n])(\text{mod } R_{n-1})$$

Thus $\delta_n(R_n) \subseteq R_{n-1}$ for all $n \geq 1$, and hence $\mathbf{C}_{.}(R)$ factors through a complex:

$$\bar{\mathbf{C}}_{.}(R): \qquad \ldots \xrightarrow{\partial_1} T_1 \xrightarrow{\partial_0} T_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

By abuse of notation, we continue to use the symbol $[I_0,\ldots,I_n]$ to denote the class of $[I_0,\ldots,I_n]$ in $T_n$; hence the formula for $\partial_n$ (on generators) reads: $\partial_n([I_0,\ldots,I_n]) = \sum_{i=0}^{n}(-1)^i[I_0,\ldots,\hat{I}_i,\ldots,I_n]$.

Finally we define the *homology groups*:

$$H_n(R) = \begin{cases} \frac{\text{Ker } (\partial_{n-1})}{\text{Im } (\partial_n)} & \text{if } n > 0 \\ \frac{T_0}{\text{Im } \partial_0} & \text{if } n = 0 \end{cases}$$

If rk $H_n(R)$ is finite for all $n$ and zero for sufficiently large $n$, we define the *Euler characteristic* of $R$:

$$\chi(R) = \sum_{n=0}^{\infty}(-1)^n \text{rk } H_n(R)$$

Since a field has no proper ideals, we immediately have:

**Proposition 2.1.** *Let $F$ be a field. Then $H_n(F) = 0$ for all $n \geq 0$.*

The term "homology" is used somewhat loosely, since neither the complexes $\bar{\mathbf{C}}_{.}(R)$ nor the groups $H_n(R)$ are functorial in $R$. This is not particularly surprising: given a ring homomorphism $f: R \longrightarrow S$, if $[I_0,\ldots,I_n] \in T_n(R)$, it is possible that $I_0 \ldots I_n = 0$ or one of the $f(I_i)$ may be zero, so it does not necessarily follow that $[f(I_0),\ldots,f(I_n))]$ makes sense as an element of $T_n(S)$. Similarly, if $[J_0,\ldots,J_n] \in T_n(S)$, it does not follow that $[f^{-1}(J_0),\ldots,f^{-1}(J_n)]$ defines an element of $T_n(R)$.

The following well-known device is often useful in computing the Euler characteristic:

**Proposition 2.2.** *Suppose rk $T_n$ is finite for all $n$ and $T_n = 0$ for $n >> 0$. Then*

$$\chi(R) = \sum_{n=0}^{\infty}(-1)^n rk\ T_n$$

**Proof.**
By definition of $H_0(R)$, there is an exact sequence:

$$0 \to \text{Im } \partial_0 \to T_0 \to H_0(R) \to 0$$

and
for each $n \geq 1$, there is a short exact sequence:

$$0 \to \text{Im } \partial_n \to \text{Ker } \partial_{n-1} \to H_n(R) \to 0$$

Since the rank is additive across exact sequences, we have:

$$\chi(R) = \sum_{n=0}^{\infty}(-1)^n \text{rk } H_n = \text{rk } T_0 - \text{rk Im } \partial_0 + \sum_{n=1}^{\infty}(-1)^n(\text{rk Ker } \partial_{n-1} - \text{rk Im } \partial_n)$$

Furthermore for any $n \geq 0$, rk Im $\partial_n =$ rk $T_{n+1} -$ rk Ker $\partial_n$, so the above expression for $\chi(R)$ becomes:

$$\chi(R) = \text{rk } T_0 - \text{rk } T_1 + \text{rk Ker } (\partial_0) + \sum_{n=1}^{\infty}(-1)^n(\text{rk Ker } \partial_{n-1} - \text{rk } T_{n+1} + \text{rk Ker } \partial_n)$$

$$= \text{rk } T_0 - \text{rk } T_1 + \sum_{n=1}^{\infty}(-1)^n \text{rk } T_{n+1} = \sum_{n=0}^{\infty}(-1)^n \text{rk } T_n$$

# 3 The group $H_0(R)$

Let $R$ be a commutative ring with unity. In order to analyze $H_0(R)$, we recall the construction of the so-called *ideal graph* $\mathcal{I}_R$. This is a (simple) graph whose vertices are the proper ideals of $R$, with $\{I, J\}$ being an edge if and only if $IJ = 0$. We will be more interested in the *complement* graph $\bar{\mathcal{I}_R}$, whose vertices are the same as $\mathcal{I}_R$, but in which $\{I, J\}$ is an edge if and only if $IJ \neq 0$.

If $\sum_{i=1}^n [I_i] \in T_0$ is an element whose class in $H_0(R)$ is zero, this means that $\sum_{i=1}^n [I_i] = \partial_0(\sum_{j=1}^m c_j[A_j, B_j])$ for some integers $c_j$ and proper ideals $A_j$, $B_j$. Without loss of generality we may assume $c_j = \pm 1$. Equality still holds if we replace $[A_j, B_j]$ by $-[B_j, A_j]$, so we may always write $\sum_{i=1}^n [I_i] = \partial_0(\sum_{k=1}^r [C_k, D_k])$ for some proper ideals $C_k, D_k$.

**Proposition 3.1.** *Let $I$ and $J$ be distinct proper ideals of $R$. Then $[I]$ and $[J]$ have the same class in $H_0(R)$ if and only if $I$ and $J$ lie in the same connected component of the graph $\bar{\mathcal{I}}_R$.*

**Proof.**
If $I$ and $J$ are in the same connected component of $\bar{\mathcal{I}}_R$, then there is some path $I = A_0 - A_1 - \ldots - A_n = J$ connecting $I$ and $J$, where the $A_i$ are ideals such that for each $i = 0, \ldots, n-1$, $A_iA_{i+1} \neq 0$. This directly implies that $\sum_{i=0}^{n-1}[A_i, A_{i+1}]$ is an element of $T_1$, and by direct calculation we see that

$$\partial_0(\sum_{i=0}^{n-1}[A_i, A_{i+1}]) = [A_0] - [A_n] = [I] - [J]$$

Hence $[I] = [J]$ in $H_0(R)$.
Conversely, suppose $[I]$ and $[J]$ define the same class in $H_0(R)$. Then $[I] - [J] = \partial_0(\sum_{i=0}^{n}[A_i, B_i]) = \sum_{i=0}^{n}[A_i] - [B_i]$ where $A_i$, $B_i$ are distinct proper ideals of $R$ and $A_iB_i \neq \emptyset$. Let $n$ be the smallest integer for which this is possible. We prove by induction on $n$ that, after suitable reordering of the $A_i$ and $B_i$, there is a path in $\bar{\mathcal{I}}_R$ from $I$ to $J$.

We may assume without loss of generality that $A_0 = I$ and $B_n = J$. If $B_0 = J$, then $IJ \neq 0$ and we are done. Otherwise, assume $B_0 \neq J$; that is, $n > 0$. Since

$$[I] - [J] = [I] - [B_0] + [A_1] - [B_1] + \ldots + [A_n] - [B_n]$$

is a relation in a free abelian group, we may assume without loss of generality that $A_1 = B_0$. Then, adding $[B_0] - [I]$ to both sides of this equation, we get

$$[B_0] - [J] = [A_1] - [B_1] + \ldots + [A_n] - [B_n]$$

so by induction there is a path in $\bar{\mathcal{I}}_R$ from $B_0$ to $J$. Since $A_0B_0 \neq 0$, this means that $\{A_0, B_0\}$ is an edge in $\bar{\mathcal{I}}_R$, and hence that there is a path from $A_0 = I$ to $J$.

**Proposition 3.2.** *Let $I_1, \ldots, I_n$ be distinct proper ideals of $R$ lying in mutually distinct connected components of $\bar{\mathcal{I}}_R$. Then the classes of $[I_1], \ldots, [I_n]$ are linearly independent in $H_0(R)$.*

**Proof.**
If $R$ is a field, the assertion is trivial. Otherwise, let $C_1, \ldots, C_r$ be the components of $\bar{\mathcal{I}}_R$. Suppose the class of $\sum_{i=1}^{n} c_i[I_i]$ in $H_0(R)$ is 0. We may assume that each $I_i$ lies in component $C_i$ of $\bar{\mathcal{I}}_R$. Now

$$\sum_{i=1}^{n} c_i[I_i] = \partial_0(\sum_{j=1}^{m}[A_j, B_j])$$

for some distinct proper ideals $A_j$, $B_j$ such that $A_jB_j \neq 0$. Since $[A_j, B_j] \in T_1$, $A_j$ and $B_j$ must lie in the same component of $\bar{\mathcal{I}}_R$. For each $k$, $1 \leq k \leq r$, let $\mathcal{J}_k = \{j : 1 \leq j \leq m : A_j \in C_k\}$. Then it follows from the above equation that

$$c_k[I_k] = \partial_0(\sum_{j \in \mathcal{J}_k}[A_j] - [B_j])$$

Applying $\varepsilon$ to both sides of this equation, we have $c_k = 0$ for all $k$.

Combining the previous two propositions, we have:

**Corollary 3.3.** *Let $R$ be a ring, and $r$ the number of connected components of $\bar{\mathcal{I}}_R$. Then*

$$H_0(R) \cong \mathbb{Z}^r.$$

Corollary 3.3 is a useful tool for calculating $H_0(R)$ in particular cases; nevertheless, using only elementary facts about ideals, one can prove even more. We begin with an elementary lemma:

**Lemma 3.4.** *Let $R$ be a ring and $\mathfrak{m}_1$, $\mathfrak{m}_2$ distinct maximal ideals of $R$. If $\mathfrak{m}_1\mathfrak{m}_2 = 0$, then $R$ is isomorphic to a product of two fields.*

**Proof.**
Let $\mathfrak{p}$ be a prime ideal of $R$. Then $\mathfrak{p} \supseteq \mathfrak{m}_1\mathfrak{m}_2 = 0$, so $\mathfrak{p} \supseteq \mathfrak{m}_1$ or $\mathfrak{p} \supseteq \mathfrak{m}_2$; i.e. $\mathfrak{p} = \mathfrak{m}_1$ or $\mathfrak{p} = \mathfrak{m}_2$. Hence $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only prime ideals of $R$ and so $R$ is an Artin ring with two maximal ideals. By the structure theorem for Artin rings, $R \cong R_1 \times R_2$, where $R_1$, $R_2$ are Artin local rings with respective maximal ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$. Then without loss of generality, $\mathfrak{m}_1 = \mathfrak{n}_1 \times R_2$ and $\mathfrak{m}_2 = R_1 \times \mathfrak{n}_2$. Thus, $0 = \mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{n}_1 \times \mathfrak{n}_2$ so $\mathfrak{n}_1 = 0$, $\mathfrak{n}_2 = 0$ and so $R_1$, $R_2$ are fields.

**Proposition 3.5.** *Let $R$ be a nonlocal ring which is not isomorphic to the product of two fields. Then $H_0(R) \cong \mathbb{Z}$.*

**Proof.**
By Corollary 3.3 it suffices to prove that $\bar{\mathcal{I}}_R$ is connected. Indeed, let $\mathfrak{m}_1$, $\mathfrak{m}_2$ be distinct maximal ideals of $R$. If $I$ is any other proper ideal of $R$, then $\operatorname{ann}(I)$ is a proper ideal of $R$, so $\operatorname{ann}(I)$ does not contain both $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Hence for each such $I$, at least one of $\{I, \mathfrak{m}_1\}$, $\{I, \mathfrak{m}_2\}$ is an edge in $\bar{\mathcal{I}}_R$. If $\mathfrak{m}_1\mathfrak{m}_2 = 0$, then it follows from

Lemma 3.4 that $R$ is isomorphic to a product of two fields. Thus $\mathfrak{m}_1\mathfrak{m}_2 \neq 0$, $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ is an edge of $\bar{\mathcal{I}}_R$, and it follows that $\bar{\mathcal{I}}_R$ is connected.

We have seen that $H_0(F) = 0$ when $F$ is a field and $H_0(R) \cong \mathbb{Z}$ for a large class of rings. Direct computation shows that if $F_1$ and $F_2$ are fields, then $H_0(F_1 \times F_2) \cong \mathbb{Z}^2$ and $H_n(F_1 \times F_2) = 0$ for all $n > 0$. A natural question that arises is: given any integer $s \geq 0$, is there a ring $R$ such that $H_0(R) \cong \mathbb{Z}^s$? The discussion above shows that when $s \geq 3$, any such $R$ must necessarily be local. Following an idea supplied to us by Dennis Keeler, we show below that the rank of $H_0(R)$ may be arbitrarily large.

Let $k$ be a field and $x_1, \ldots, x_s$ independent indeterminates. Let $S$ be the localization of $k[x_1, \ldots, x_s]$ with respect to the maximal ideal $(x_1, \ldots, x_s)$. Now let $I$ be the ideal of $k[x_1, \ldots, x_s]$ generated by all products $x_i x_j$, where $i \leq j$. Since $I \subseteq (x_1, \ldots, x_s)$, $I$ corresponds, in the usual manner, to an ideal $\tilde{I} \subseteq S$. Now let $R = S/\tilde{I}$. Observe now that the proper ideals of $R$ correspond bijectively to ideals $(x_{i_1}, \ldots, x_{i_\nu}) \subseteq k[x_1, \ldots, x_s]$, where $1 \leq \nu \leq s$ and $1 \leq i_1 < \ldots < i_\nu \leq s$. Furthermore, each such ideal (of $R$), when multiplied by any other, yields 0. Thus $\bar{\mathcal{I}}_R$ is a completely disconnected graph on $2^s - 2$ vertices, and so $H_0(R) \cong \mathbb{Z}^{2^s-2}$.

# 4 Calculation of $H_1(\mathbb{Z}/p^r\mathbb{Z})$

In this section, we compute the group $H_1(\mathbb{Z}/p^r\mathbb{Z})$ where $p$ is a prime number and $r \geq 1$ an integer. It is easy to see by direct calculation that if $r \leq 3$, then $H_1(\mathbb{Z}/p^r\mathbb{Z}) = 0$. We assume henceforth that $r \geq 4$.

Recall first that

$$H_1(R) = \frac{\text{Ker } (\partial_0 : T_1 \to T_0)}{\text{Im } (\partial_1 : T_2 \to T_1)}$$

where

$$\partial_0(\sum_j [A_j, B_j]) = \sum_j [A_j] - [B_j]$$

and

$$\partial_1(\sum_j [A_j, B_j, C_j]) = \sum_j [B_j, C_j] - \sum_j [A_j, C_j] + \sum_j [A_j, B_j]$$

**Definition 4.1.** *Let $n \geq 0$ be an integer. An element $\alpha \in T_1$ is called an $n$-circuit (or simply a* circuit*) if there exist proper ideals $I_1, \ldots, I_n$ of $R$ such that*

$$\alpha = [I_1, I_2] + \ldots + [I_{n-1}, I_n] + [I_n, I_1]$$

*A 3-circuit is called a* triangle*.*

Clearly the definition has been chosen to reflect the fact that in the above context, $I_1 - I_2 - \ldots I_n - I_1$ is a circuit in the graph $\bar{I}_{\mathbb{Z}/p^r\mathbb{Z}}$. The analysis of Ker $\partial_0$ proceeds by a sequence of lemmas.

**Lemma 4.2.** *Every element $\beta \in$ Ker $\partial_0$ may be written*

$$\beta = \sum_{k=1}^{m} \alpha_k$$

*where each $\alpha_k$ is a circuit.*

**Proof.**
The proof is by induction on the number of symbols in $\beta$. If $\beta = 0$, the claim is clear. Otherwise, let $\beta = \sum_{j=1}^{r} [A_j, B_j]$ with $r$ chosen to be as small as possible. We may assume that there is no pair of integers $(j_1, j_2)$, $1 \leq j_1 < j_2 \leq r$ such that $A_{j_1} = B_{j_2}$ and $A_{j_2} = B_{j_1}$, for then we may use the relation $[I, J] = -[J, I]$ in $T_1$ to simplify the expression for $\beta$ and obtain a relation with smaller $r$.
Since $\beta \in$ Ker $\partial_0$, we have:

$$0 = \partial_0(\beta) = \partial_0(\sum_{j=1}^{r} [A_j, B_j]) = \sum_{j=1}^{r} [A_j] - [B_j]$$

Since this is a relation in the (free abelian) group $T_0$, it follows that there is some $j$ such that $B_1 = A_j$. Without loss of generality we may assume that $j = 2$. By the previous discussion, it follows that $A_1 \neq B_2$. Now it must be the case that there is some $j$ such that $B_2 = A_j$; without loss of generality, we assume that $j = 3$. Continue this procedure until one reaches $s \leq r$ such that $B_s = A_1$. Then

$$\beta_1 = [A_1, B_1] + [B_1, B_2] + \ldots + [B_{s-2}, B_{s-1}] + [B_{s-1}, A_1]$$

is a circuit in $T_1$. By induction, $\beta - \beta_1$ is a sum of circuits in $T_1$; hence $\beta$ itself is a sum of circuits.

**Lemma 4.3.** *Every nonzero circuit in $T_1 = T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles.*

**Proof.**
Let $\alpha = \sum_{j=1}^{r-1}[A_j, A_{j+1}] + [A_r, A_1]$ be a circuit in $T_1$. If $\alpha$ is a 3-circuit, there is nothing to prove. By induction, it suffices to prove that $\alpha$ has a chord, i.e. there exist distinct integers $i$, $j$, $1 \le i < j \le r$ such that $[A_i, A_j] \in T_1$ and $j - i > 1$. Suppose $\alpha$ is an $n$-circuit, with $n > 3$. For each $k$, $1 \le k \le r-1$, let $I_k$ denote the ideal of $\mathbb{Z}/p^r\mathbb{Z}$ generated by (the class of) $p^k$. Let $\mathcal{S} = \{I_k : 1 \le k < r/2\}$. Observe that if $C, D \in \mathcal{S}$, then $[C, D] \in T_1$. Furthermore, if $[C, D] \in T_1$ and $C \notin \mathcal{S}$, then $D$ must be in $\mathcal{S}$.

If all the $A_i$ appearing in the cycle $\alpha$ are members of $\mathcal{S}$, then by the above observation $[A_1, A_2] + [A_2, A_3] + [A_3, A_1]$ is a triangle. If not, then we may assume without loss of generality that $A_2 \notin \mathcal{S}$. Since $[A_1, A_2] \in T_1$ and $[A_2, A_3] \in T_1$, we must have $A_1 \in \mathcal{S}$, $A_3 \in \mathcal{S}$. This forces $[A_1, A_3] \in T_1$, which completes the proof.

**Lemma 4.4.** *Every triangle in $T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles of the form $\tau_{ij} = [I_1, I_i] + [I_i, I_j] + [I_j, I_1]$, where $1 < i, j < r$.*

**Proof.**
This follows immediately from the formal identity:

$$[I_h, I_i] + [I_i, I_j] + [I_j, I_h] =$$

$$([I_1, I_h] + [I_h, I_i] + [I_i, I_1]) + ([I_1, I_i] + [I_i, I_j] + [I_j, I_1]) + ([I_1, I_j] + [I_j, I_h] + [I_h, I_1])$$

$$= \tau_{hi} + \tau_{ij} + \tau_{jh}$$

**Lemma 4.5.** *The set of triangles $\mathcal{T} = \{\tau_{ij} : 1 < i < j < r\}$ is ($\mathbb{Z}$)-linearly independent in $T_1$.*

**Proof.**
This follows readily from the fact that $\tau_{ij}$ is the only member of $\mathcal{T}$ involving the symbol $[I_i, I_j]$.

It follows from the sequence of lemmas above that:

**Corollary 4.6.** *The group Ker $\partial_0$ is a free abelian group with basis $\mathcal{T}$.*

In fact, $\tau_{ij} \in \mathcal{T}$ if and only if $i + j < r$, so an elementary counting argument gives:

**Corollary 4.7.** *The rank of Ker $\partial_0$ is $\frac{(r-4)^2}{4}$ if $r$ is even or $\frac{(r-4)^2-1}{4}$ if $r$ is odd.*

We now examine the group Im $\partial_1$. Observe that:

$$\gamma = \partial_1([I_i, I_j, I_k]) = [I_i, I_j] - [I_i, I_k] + [I_j, I_k] = [I_i, I_j] + [I_j, I_k] + [I_k, I_i]$$

is a triangle of $T_1$.
Since $I_iI_jI_k \neq 0$ and $I_1$ contains $I_i$, $I_j$ and $I_k$, it follows readily that each of the symbols $[I_1, I_i, I_j]$, $[I_1, I_i, I_k]$ and $[I_1, I_j, I_k]$ are in $T_2$; furthermore,

$$\gamma = \partial_1([I_i, I_j, I_k]) = \partial_1([I_1, I_i, I_j]) + \partial_1([I_1, I_j, I_k]) + \partial_1([I_1, I_k, I_i]) = \tau_{ij} + \tau_{jk} + \tau_{ki}$$

so in fact Im $\partial_1$ is generated by those elements $\tau_{ij} \in \mathcal{T}$ such that $1 + i + j < r$, i.e. $i + j < r - 1$.

By the same computation as used to derive Corollary 4.7, we obtain:

**Corollary 4.8.** *The group Im $\partial_1$ is a free abelian group of rank $\frac{(r-5)^2-1}{4}$ if $r$ is even or $\frac{(r-5)^2}{4}$ if $r$ is odd.*

In particular, we observe that the basis elements $\tau_{ij}$ for Im $(\partial_1)$ identified in the previous discussion are a subset of those identified as a basis for Ker $(\partial_0)$. Thus, we have:

**Corollary 4.9.** *Suppose $r \geq 4$. Then $H_1(\mathbb{Z}/p^r\mathbb{Z})$ is a free abelian group of rank $\frac{r-4}{2}$ if $r$ is even or $\frac{r-5}{2}$ if $r$ is odd.*

# 5 Acyclicity

In this section, we make a general study of the higher homology groups $H_n(R)$, $n > 0$; in particular, we give various conditions sufficient for these groups to be zero.

Towards this end, it is convenient to introduce some notation: if $I_{j_0}, \ldots, I_{j_m}$ ($j = 1 \ldots r$) and $J_0, \ldots, J_n$ are mutually distinct ideals of a ring $R$ such that $[I_{j_0}, \ldots, I_{j_m}] \in T_m(R)$ for each $j$ and $[J_0, \ldots, J_n] \in T_n(R)$, and also $I_{j_0} \ldots I_{j_m} J_0 \ldots J_n \neq 0$, for each $j$, we write:

$$\sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}] \times [J_0, \ldots, J_n] = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}, J_0, \ldots, J_n]$$

**Lemma 5.1.** *(Acyclicity Lemma) Suppose* $n > 0$ *and* $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in$ *Ker* $(\partial_{n-1})$*. If there exists an ideal* $J \notin \{I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n\}$ *such that* $JI_{j_0} \ldots I_{j_n} \neq 0$ *for all* $j$*,* $1 \leq j \leq r$*, then* $\alpha \in$ *Im* $(\partial_n)$*. Thus the class of* $\alpha$ *in* $H_n(R)$ *is zero.*

**Proof.**
If such $J$ exists, then

$$\partial_n((-1)^{n+1} \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \times [J]) = (-1)^{n+1} \sum_{i=0}^{n} \sum_{j=1}^{r} (-1)^n [I_{j_0}, \ldots, \hat{I}_{j_i}, \ldots, I_{j_n}, J] + \alpha$$

$$= -\partial_{n-1}(\alpha) \times [J] + \alpha = \alpha$$

So indeed $\alpha \in \text{Im } (\partial_n)$, as desired.

**Theorem 5.2.** *Let* $R$ *be a ring satisfying at least one of the following conditions:*

- *There exists a nonzero element* $x \in R$ *which is neither a unit nor a zero-divisor.*
- $R$ *has infinitely many maximal ideals.*
- $R$ *is reduced, Noetherian, and of positive (Krull) dimension.*

*Then* $H_n(R) = 0$ *for all* $n > 0$*.*

**Proof.**
First, suppose $x \in R$ is a nonzero element which is neither a unit nor a zero-divisor. Then it is easy to see that $x^i$ and $x^j$ are associate if and only if $i = j$. Thus,

$$(x) \supset (x^2) \supset (x^3) \supset \ldots$$

is a descending chain of distinct ideals. Furthermore, if $I$ is a nonzero ideal, then $(x^i)I \neq 0$, for any $i \geq 1$ because $x$ (and hence $x^i$) is not a zero-divisor. Given any $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \text{Ker } (\partial_{n-1})$ as in Lemma 5.1, choose $m$ such that $(x^m) \neq I_{j_k}$ for all $j, k$. Then $J = (x^m)$ satisfies the hypotheses of the Lemma and the assertion follows.

Now suppose $R$ has infinitely many maximal ideals, and suppose $\alpha$ is as above. For each $j$, let $A_j = \text{ann}(I_{j_0} \ldots I_{j_n})$; $A_j$ is a proper ideal of $R$, so choose some maximal ideal $\mathfrak{m}_j$ such that $A_j \subseteq m_j$. For each $j$, $1 \leq j \leq r$ and $k$, $1 \leq k \leq n$, choose a maximal ideal $\mathfrak{m}_{jk}$ such that $I_{j_k} \subseteq \mathfrak{m}_{jk}$. Now let

$$D = \bigcup_{j=1}^{r} \mathfrak{m}_j \cup \bigcup_{j=1}^{r} \bigcup_{k=1}^{n} \mathfrak{m}_{jk}$$

Let $\mathfrak{m}$ be some other maximal ideal of $R$ not equal to any $\mathfrak{m}_j$ or $\mathfrak{m}_{jk}$. By [4], Proposition 1.11, $\mathfrak{m} \not\subseteq D$. Choose $x \in \mathfrak{m} - D$. Evidently, $(x)$ is a proper ideal of $R$. Furthermore, since $x \notin \mathfrak{m}_{jk}$, $(x) \neq I_{j_k}$ for any $j$, $k$. Finally, $x \notin \mathfrak{m}_j \supseteq A_j$ implies that $(x)I_{j_0} \dots I_{j_n} \neq 0$ for all $j$. Thus, $J = (x)$ satisfies the hypotheses of Lemma 5.1, and the assertion is proved.

Last, suppose $R$ is reduced, Noetherian, and $\dim R > 0$. Let $\mathfrak{p}_0$ be a minimal prime ideal of $R$ which is not also maximal. Then $\dim(R/\mathfrak{p}_0) > 0$, so in particular $R/\mathfrak{p}_0$ is not Artinian. Thus, there is a strictly descending sequence of ideals of $R$:

$$R \supsetneq J_1 \supsetneq J_2 \supsetneq \dots$$

each of which strictly contains $\mathfrak{p}_0$.

Let $\mathfrak{p}_0, \dots, \mathfrak{p}_n$ be the minimal prime ideals of $R$; there are only finitely many of them because $R$ is Noetherian ([4], Chapter 6, Exercise 9). It is well-known (cf. [4], Prop. 1.8) that the nilradical of $R$ is the intersection of the prime ideals of $R$ – hence also of the minimal prime ideals of $R$. Thus in our case, $\bigcap_{i=0}^{n} \mathfrak{p}_i = 0$.

We claim that $IJ_m \neq 0$ for any nonzero ideal $I$ and any $m \geq 1$. Suppose to the contrary that $IJ_m = 0$. Since $\bigcap_{i=0}^{n} \mathfrak{p}_i = 0$, this means $\mathfrak{p}_i \supseteq IJ_m$ for each $i$. Since $\mathfrak{p}_i$ is prime, $\mathfrak{p}_i \supseteq I$ or $\mathfrak{p}_i \supseteq J_m$. In the latter case, $\mathfrak{p}_i \supseteq J_m \supseteq \mathfrak{p}_0$, so by minimality of $\mathfrak{p}_i$, we must have $\mathfrak{p}_i = J_m = \mathfrak{p}_0$. However, $J_m$ strictly contains $\mathfrak{p}_0$, so this is impossible. Thus, we must have $\mathfrak{p}_i \supseteq I$ for each $i$; hence, $0 = \bigcap_{i=0}^{n} \mathfrak{p}_i \supseteq I$ and so $I = 0$.

Continuing with the proof of Theorem 5.2, suppose $n > 0$ and $\alpha = \sum_{j=1}^{r} [I_{j_0}, \dots, I_{j_n}] \in$ Ker $(\partial_{n-1})$ as in Lemma 5.1. Choose $m \geq 1$ such that $J_m \notin \{I_{j_k} : 1 \leq j \leq r, 0 \leq k \leq n\}$. Then the previous paragraph shows that for any $j$, $1 \leq j \leq r$, $JI_{j_0} \dots I_{j_n} \neq 0$; thus we may take $J = J_m$ and apply Lemma 5.1 to conclude.

# 6 $\chi$ for finite rings

Theorem 5.2 establishes that the higher homology groups are uninteresting for a large class of rings. Finite rings, on the other hand, satisfy none of the conditions of the theorem; in this section, we examine these rings more closely. While the prospect of computing the actual homology groups seems daunting, the Euler characteristic turns out to be a much more tractable object. In particular, if $R$ is a finite ring – hence having only finitely many ideals – it is clear from the definition that each $T_n(R)$

has finite rank and that $T_n(R) = 0$ for sufficiently large $n$. Hence the hypotheses of Proposition 2.2 are satisfied and we may use it to compute the Euler characteristic. In particular, let $U_n = U_n(R)$ denote the number of *unordered* $(n+1)$-tuples $\{I_0, \ldots, I_n\}$ of distinct ideals whose product is nonzero. Then we have the convenient formula

$$\chi(R) = \sum_{n=0}^{\infty} (-1)^n |U_n|$$

.

Throughout this section, if a set is denoted by an uppercase letter, we will use the corresponding lower case letter for the number of elements in that set. For example, we will write $u_n$ for $|U_n|$ as defined above.

We begin by examining the same rings encountered in Sec. 4, namely those of the form $R = \mathbb{Z}/p^r\mathbb{Z}$ where $p$ is a prime and $r \geq 1$ is some integer. Recall that for each $i$, $1 \leq i \leq r-1$, there is an ideal $I_i$ of $R$ generated by (the class of) $(p^i)$ and that these are all the proper ideals of $R$. In the following, we implicitly identify the ideal $I_i$ with the integer $i$. Since $U_n$ is the set of unordered $(n+1)$-tuples $\{I_0, \ldots, I_n\}$ of distinct proper ideals of $R$, we have

$$u_n = \sum_{k=1}^{r-1} P(k, n+1)$$

where $P(k, n+1)$ represents the number of partitions of $k$ into $(n+1)$ distinct positive integer parts. Hence

$$\chi(R) = \sum_{n=0}^{\infty} (-1)^n s_n = \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{r-1} P(k, n+1) = \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n+1} P(k, n)$$

We may interpret the inner sum $\sum_{n=1}^{\infty} (-1)^{n+1} P(k, n) = -\sum_{n=1}^{\infty} (-1)^n P(k, n)$ as the coefficient of $x^k$ in the power series:

$$-(1-x)(1-x^2)(1-x^3)\ldots$$

By Euler's pentagonal theorem, we have:

$$-(1-x)(1-x^2)(1-x^3)\ldots = -1 + x + x^2 - x^5 - x^7 + x^{12} + x^{15} - x^{22} - x^{26} + \ldots$$

where the pattern of signs on the right (from the second term forth) is + + - - and the exponents alternate between the "pentagonal" numbers of the form $P_m = \dfrac{m(3m-1)}{2}$ and the related numbers $Q_m = \dfrac{m(3m+1)}{2}$, where $m = 1, 2, 3, \ldots$.

Hence

$$\chi(R) = -\sum_{k=1}^{r-1}\sum_{n=1}^{\infty}(-1)^n P(k,n)$$

is the sum of the coefficients of the terms $x, x^2, \ldots, x^{r-1}$ appearing in the above series. It is clear from the sign pattern that this sum is either 0, 1, or 2, depending on the value of $r$ in relation to the numbers $P_m$ and $Q_m$.

We summarize our findings in the following:

**Theorem 6.1.** *Let $p$ be a prime and $r \geq 1$ an integer. Then $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is equal to 0, 1, or 2, depending on the value of $r$ in relation to the various pentagonal numbers $\frac{m(3m-1)}{2}$ and the associated numbers $\frac{m(3m+1)}{2}$.*

By being careful with counting methods, we can prove the following theorem, whose proof is facilitated by the paucity of ideals in a field.

**Theorem 6.2.** *Let $R$ be a finite ring and $F$ a field. Then*

$$\chi(R \times F) = 2 - \chi(R)$$

**Proof.**
Let $\pi_1$, $\pi_2$ denote the projection maps onto the respective factors of $R \times F$. Recall that for any $n \geq 0$, the typical element $U_n(R \times F)$ is an unordered $(n+1)$-tuple $\{I_0, \ldots, I_n\}$ where $I_0 \ldots I_n \neq 0$. Moreover, each $I_i = A_i \times B_i$, with $A_i = \pi_1(I_i)$ being an ideal of $R$ and $B_i = \pi_2(I_i)$ an ideal of $F$, i.e. $B_i = 0$ or $B_i = F$. In order to have $I_0 \ldots I_n \neq 0$, at least one of $\prod_{i=0}^n A_i \neq 0$ or $\prod_{i=0}^n B_i \neq 0$. Define:

$$U_n^1(R \times F) = \{\{I_0, \ldots, I_n\} \in U_n(R \times F) : \prod_{i=0}^n A_i \neq 0\}$$

$$U_n^2(R\times F) = \{\{I_0, \ldots, I_n\} \in U_n(R\times F) : \prod_{i=0}^n B_i \neq 0\} = \{\{I_0, \ldots, I_n\} \in U_n : B_i = F \text{ for each } i\}$$

$$U_n^3(R \times F) = U_n^1(R \times F) \cap U_n^2(R \times F)$$

$$= \{\{I_0, \ldots, I_n\} \in U_n(R \times F) : B_i = F \text{ for each } i \text{ and } (A_0, \ldots, A_n) \in U_n(R)\}$$

Thus we have $u_n = u_n^1 + u_n^2 - u_n^3$.

It is clear from the above description that $u_n^3(R\times F) = u_n(R)$ and furthermore that if $\{I_0,\ldots,I_n\}\in U_n^2(R\times F)$, then $A_0,\ldots,A_n$ are allowed to be any (mutually distinct) proper ideals of $R$; hence $u_n^2(R\times F) = \binom{\rho}{n+1}$, where $\rho$ is the number of proper ideals in $R$.

The set $U_n^1$ is slightly more difficult to analyze: define

$$U_n^{1,0}(R\times F) = \{\{I_0,\ldots,I_n\}\in U_n^1(R\times F) : I_i \neq R\times 0 \text{ for all } i,\ 0\le i\le n\}$$

$$U_n^{1,1}(R\times F) = U_n^1(R\times F) - U_n^{1,0}(R\times F)$$

Clearly $u_n^{1,0}(R\times F) + u_n^{1,1}(R\times F) = u_n^1(R\times F)$. Somewhat more subtly, there is a natural bijective map $U_n^{1,0}(R\times F)\to U_{n+1}^{1,1}(R\times F)$ sending $\{I_0,\ldots,I_n\}\mapsto\{I_0,\ldots,I_n,R\times 0\}$, so it is also true that $u_n^{1,0}(R\times F) = u_{n+1}^{1,1}(R\times F)$.

Combining all these relations, we have:

$$\chi(R\times F) = \sum_{n=0}^{\infty}(-1)^n u_n(R\times F)$$

$$= \sum_{n=0}^{\infty}(-1)^n(u_n^1(R\times F) + u_n^2(R\times F) - u_n^3(R\times F))$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(u_n^{1,0}(R\times F) + u_n^{1,1}(R\times F) + \binom{\rho}{n+1} - u_n(R)\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n u_n^{1,0}(R\times F) + \sum_{n=0}^{\infty}(-1)^n u_n^{1,1}(R\times F) + \sum_{n=0}^{\infty}(-1)^n\binom{\rho}{n+1} - \sum_{n=0}^{\infty}(-1)^n u_n(R))$$

$$= \sum_{n=0}^{\infty}(-1)^n u_{n+1}^{1,1}(R\times F) + \sum_{n=0}^{\infty}(-1)^n u_n^{1,1}(R\times F) + 1 - \chi(R)$$

$$= u_0^{1,1}(R\times F) + 1 - \chi(R)$$

$$= 2 - \chi(R)$$

**Corollary 6.3.** *Let $F_1,\ldots,F_n$ be fields. Then*

$$\chi(F_1\times\ldots\times F_n) = 1 + (-1)^n$$

We have not yet found a general method for computing $\chi(\mathbb{Z}/n\mathbb{Z})$, where $n > 0$ is an arbitrary integer. However, it is possible to analyze some specific examples using idiosyncratic counting methods:

**Theorem 6.4.** *Let $p$, $q$ be primes and $r \geq 2$ an integer. Then*

$$\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z})$$

**Proof.**
For convenience, set $R = \mathbb{Z}/p^r\mathbb{Z}$ and $S = \mathbb{Z}/q^2\mathbb{Z}$; to ease notation, we denote the unique proper ideal of $S$ by $(q)$. As in Theorem 6.2, let $\pi_1$, $\pi_2$ be the projection maps onto the respective factors of $R \times S$. As before, for any $n \geq 0$, the typical element $U_n(R \times S)$ is an unordered $(n+1)$-tuple $\{I_0, \ldots, I_n\}$ where $I_0 \ldots I_n \neq 0$ and $I_i = A_i \times B_i$, where $A_i = \pi_1(I_i)$ an ideal of $R$ and $B_i = \pi_2(I_i)$ an ideal of $S$. In this situation, $B_i$ may either be 0, $(q)$, or $S$. As before, $\prod_{i=0}^n A_i \neq 0$ or $\prod_{i=0}^n B_i \neq 0$.

$$U_n^1(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n A_i \neq 0\}$$

$$U_n^2(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n B_i \neq 0\}$$

$$= \{\{I_0, \ldots, I_n\} \in U_n : \text{ there exists some } i_0 \text{ such that } B_{i_0} = S \text{ or } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}$$

$$U_n^3(R \times S) = U_n^1(R \times S) \cap U_n^2(R \times S)$$

Now define

$$U_n^{1,0}(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n^1(R \times S) : I_i \neq R \times 0 \text{ for all } i,\ 0 \leq i \leq n\}$$

$$U_n^{1,1}(R \times S) = U_n^1(R \times S) - U_n^{1,0}(R \times S)$$

$$U_n^{3,q}(R \times S) = \{\{I_0, \ldots, I_n\} \in U_n^3(R \times S) : \text{ there exists } i_0 \text{ such that } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\}$$

$$U_n^{3,S}(R\times S)=U_n^3(R\times S)-U_n^{3,q}(R\times S)$$

$$=\{\{I_0,\dots,I_n\}\in U_n^3(R\times S): B_i=S \text{ for all } i,\ 0\le i\le n\}$$

It follows immediately from the above definitions that $u_n(R\times S)=u_n^1(R\times S)+u_n^2(R\times S)-u_n^3(R\times S)$.

The map $U_n^{1,0}(R\times S)\to U_{n+1}^{1,1}(R\times S)$ sending $\{I_0,\dots,I_n\}\mapsto\{I_0,\dots,I_n,R\times 0\}$ establishes a bijection, so $u_n^{1,0}(R\times S)=u_{n+1}^{1,1}(R\times S)$.

Now let $\rho$ denote the number of proper ideals in $R$. Evidently, by the description given above,

$$u_n^2(R\times S)=\rho\binom{\rho}{n}+\binom{\rho}{n+1}.$$

Finally, it is clear that $u_n^{3,S}(R\times S)=u_n(R)$. Observe that given a typical element $\{I_0,\dots,I_n\}$ of $U_n^{3,q}(R\times S)$, we may assume without loss of generality that $B_j=S$ for all $j>0$ and that $B_0=(p^k)\times(q)$ for some $k$, $1\le k\le r-1$. (This is the only place in the proof where we use the fact that $R$ has the form $\mathbb{Z}/p^r\mathbb{Z}$.) Thus, in order to have $\prod_{i=0}^n A_i\neq 0$, we must have $\{A_1,\dots,A_n\}\in U_{n-1}(\mathbb{Z}/p^{r-k}\mathbb{Z})$. Hence, $u_n^{3,q}(R\times S)=\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})$.

Collecting this information together, we have:

$$\chi(R\times S)=\sum_{n=0}^{\infty}(-1)^n u_n(R\times S)$$

$$=\sum_{n=0}^{\infty}(-1)^n(u_n^1(R\times S)+u_n^2(R\times S)-u_n^3(R\times S))$$

$$=\sum_{n=0}^{\infty}(-1)^n(u_n^{1,0}(R\times S)+u_n^{1,1}(R\times S)+\rho\binom{\rho}{n}+\binom{\rho}{n+1}-u_n(R)-\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z}))$$

$$=\sum_{n=0}^{\infty}(-1)^n(u_n^{1,0}(R\times S)+u_{n+1}^{1,1}(R\times S))+\sum_{n=0}^{\infty}(-1)^n(\rho\binom{\rho}{n}+\binom{\rho}{n+1})$$
$$-\sum_{n=0}^{\infty}(-1)^n u_n(R)-\sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z}))$$

$$= u_0^{1,1}(R \times S) + 1 - \chi(R) + \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n-1} u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})$$

$$= 2 - \chi(R) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z})$$

Thus

$$\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z})$$

From Theorem 6.4 and Theorem 6.1, we see that the value of $\chi(\mathbb{Z}/p^r\mathbb{Z})$ may be made arbitrary large by choosing $r$ large enough. By Theorem 6.2, we see that by taking the product with a field, we can make obtain a ring whose Euler characteristic is arbitrary large and negative. Summarizing, we have:

**Corollary 6.5.** *The value of $\chi(R)$ is unbounded in both the positive and negative directions as $R$ ranges over the set of finite rings.*

It is not difficult to develop *ad hoc* counting methods along similar lines to compute $\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^3\mathbb{Z})$, but it is not clear how to generalize this method to compute $\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z})$ for arbitrary $s \geq 1$.

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