Asymptotic determination of edge-bandwidth of multidimensional grids and Hamming graphs

Reza Akhtar * Tao Jiang † Zevi Miller ‡

May 18, 2006

Abstract

The edge-bandwidth $B'(G)$ of a graph G is the bandwidth of the line graph of G. More specifically, for any bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$, let $B'(f, G) = \max\{|f(e_1) - f(e_2)| : e_1$ and e_2 are incident edges of $G\}$, and let $B'(G) = \min_f B'(f, G)$. We determine asymptotically the edge-bandwidth of d-dimensional grids P_n^d and of the Hamming graph K_n^d, the d-fold Cartesian product of K_n. Our results are as follows.
(1) For fixed d and $n \to \infty$, $B'(P_n^d) = c(d)dn^{d-1} + O(n^{d-\frac{3}{2}})$, where $c(d)$ is a constant depending on d, which we determine explicitly.
(2) For fixed even n and $d \to \infty$, $B'(K_n^d) = (1 + o(1))\frac{\sqrt{d}}{\sqrt{2\pi}}n^d(n-1)$.

Our results extend recent results by Balogh et al. [5] who determined $B'(P_n^2)$ asymptotically as a function of n and $B'(K_2^d)$ asymptotically as a function of d.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph on n vertices. A *labeling* f is a bijection of $V(G)$ to $\{1, \ldots, n\}$. When there is no ambiguity, we will simply write $B(f)$ for $B(f, G)$. The *bandwidth* of f is

$$B(f, G) := \max\{|f(u) - f(v)| : uv \in E(G)\}.$$

The *bandwidth* $B(G)$ of G is

$$B(G) := \min_f\{B(f, G)\}.$$

The notion was introduced by Harper in his influential paper [12] in which he determined the bandwidth of the d-dimensional hypercube by solving the corresponding vertex isoperimetric problem in the hypercube. There are several motivations for studying the bandwidth

*Dept. of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA, reza@calico.mth.muohio.edu.

†Dept. of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA, jiangt@muohio.edu.

‡Dept. of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA, millerz@muohio.edu

problem: sparse matrix computations, representing data structures by linear arrays, VLSI layouts, mutual simulations of interconnection networks and minimizing the effects of noise in the multi-channel communication of data (see [9, 10, 21, 6]). The bandwidth problem is NP-hard and is inappromaximable by any multiplicative constant even for trees [24]. Bandwidths are known only for a few families of graphs including hypercubes [12], mutlidimensional grids [7], complete trees [15], and various mesh-like graphs (see [15, 16, 19]).

The edge-bandwidth was introduced by Hwang and Lagarias [17]. Here we label the edges instead of the vertices, and the *bandwidth* of an edge-labeling f of a graph G is

$$B'(f, G) := \max\{|f(uv) - f(vw)| : uv, vw \in E(G)\}.$$

In other words, it is the maximum difference of labels between a pair of incident edges. When there is no ambiguity, we will write $B'(f)$ for $B(f, G)$. The edge-bandwidth of a graph G is

$$B'(G) := \min_f B'(f, G).$$

Naturally, $B'(G) = B(L(G))$, where $L(G)$ is the line graph of G. In [18], Jiang et al. re-introduced the notion of edge-bandwidth and studied the relationship between $B(G)$ and $B'(G)$. They determined the edge-bandwidth of caterpillars, the complete graph K_n, and the balanced complete bipartite graph $K_{n,n}$. A. Gupta [11] pointed out that the inequality $B(T) \leq B'(T) \leq 2B(T)$ for a tree T obtained in [18] together with Unger's inappromixation result [24] for bandwidth imply that determining the edge-bandwidth is also NP-hard.

Recently, there has been an increase of interest in the study of edge-bandwidth. Calamoneri et al [8] obtained tight bounds on the edge-bandwidth of complete k-ary trees and bounds on the edge-bandwidth of the hypercube and butterfly graphs. Balogh et al [5] subsequently obtained asymptotically tight bounds on the edge-bandwidth of two dimensional grids and tori, the Cartesian product of two cliques and the hypercube. Sharpening the result of Balogh et al [5] on the two dimensional grids and tori while confirming a conjecture of Calamoneri et al [8], Pikhurko and Wojciechowski [23] showed that the edge-bandwidth of an m by n grid, where $m \geq n$, is $2n-1$. They also showed that the edge-bandwidth of an m by n torus, where $m \geq n$, is between $4n-5$ and $4n-1$. In an unpublished manuscript, Akhtar, Jiang, and Pritikin have independently shown that the edge-bandwidth of an m by n grid, where $m \geq n$, is between $2n-2$ and $2n-1$, and that the edge-bandwidth of an m by n torus, where $m \geq n$, is between $4n-5$ and $4n-1$.

In this paper, we determine the edge-bandwidth of the d-dimensional grids P_n^d asymptotically when d is fixed and $n \to \infty$, and we obtain lower and upper bounds on the edge-bandwidth of the Hamming graph K_n^d. When n is a fixed positive even integer and $d \to \infty$ our lower and upper bounds match asymptotically.

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G\Box H) = \{(u, v) : u \in V(G), v \in V(H)\}$ specified by putting (u, v) adjacent to (u', v') if and only (1) $u = u'$ and $vv' \in e(H)$, or (2) $v = v'$ and $uu' \in E(G)$. The d-fold Cartesian product $G\Box G\Box \cdots \Box G$ is denoted by G^d. Let P_n denote a path on n vertices. The d-fold Cartesian product P_n, denoted by P_n^d, is also known as the d-dimensional grid. Equivalently, we can view P_n^d as a graph whose vertices are vectors $\langle u_1, u_2, \ldots, u_d\rangle$ of length d where $\forall i, u_i \in \{0, 1, \ldots, n-1\}$ and any two vertices $\langle x_1, x_2, \ldots, x_d\rangle$ and $\langle y_1, y_2, \ldots, y_d\rangle$ are adjacent if and only if there exists $i \in \{1, 2, \ldots, d\}$ such that $|x_i - y_i| = 1$ and $x_j = y_j$ for all $j \neq i$. Given a vertex $x = \langle x_1, x_2, \ldots, x_d \rangle$, we define the *weight* of x by $wt(x) = x_1 + x_2 + \cdots + x_d$. For each $r \in \{1, 2, \ldots, n\}$, let $L(n, d, r) = \{x \in V(P_n^d) : wt(x) = r\}$. Let $l(n, d, r) = |L(n, d, r)|$ and $l^*(n, d) = \max\{l(n, d, r) : 0 \leq r \leq (n-1)d\}$. We determine $B'(P_n^d)$ asymptotically as a function of n.

Theorem 1.1 *Let d be a fixed positive integer. We have*

$$B'(P_n^d) = (1 + o(1))dl^*(n, d) = c(d)dn^{d-1} + O(n^{d-\frac{3}{2}}),$$

where $c(d)$ is a constant depending on d, given by $c(d) = \Sigma_{j=0}^{\lfloor d/2 \rfloor} (-1)^j \binom{d}{j} \frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!}$. Furthermore, $\frac{1}{2\sqrt{d}} \leq c(d) \leq \frac{2\sqrt{11}}{\sqrt{d}}$.

A simple calculation shows that $c(2) = 1$. Hence $B'(P_n^2) = (1 + o(1))2n$, which was obtained by Balogh et al[5]. For another example, $c(3) = \frac{3}{4}$, yielding $B'(P_n^3) = (1 + o(1))\frac{9}{4}n^2$. Theorem 1.1 determines asymptotically the edge-bandwidth of grids of arbitrary dimension.

The d-fold Cartesian product of the complete graph K_n, denoted by K_n^d, is also called the Hamming graph. We obtain lower and upper bounds on $B'(K_n^d)$ for fixed n as a function of d. When n is even, our lower and upper bounds match asymptotically, yielding

Theorem 1.2 *Let n be a fixed positive even integer. We have*

$$B'(K_n^d) = (1 + o(1))\frac{\sqrt{d}}{\sqrt{2\pi}}\, n^d\, (n-1).$$

2 General bounds

The standard techniques for obtaining lower bounds on bandwidth use isoperimetric inequalities. Many vertex and edge isoperimetric problems have been considered in the literature. Given a graph G and a set $S \subseteq V(G)$, let

$$\partial(S) = \{v \in V(G) - S : \exists u \in S\; s.t. uv \in E(G)\}.$$

We call $\partial(S)$ the (vertex) *boundary* of S. In other words, $\partial(S) = N_G(S) - S$. Given an optimal numbering f of $V(G)$, let S be the set of vertices receiving labels $1, 2, \cdots, k$. Then the highest label assigned to a vertex in $\partial(S)$ is at least $k + |\partial(S)|$. Let v be the vertex with the highest label in $\partial(S)$. It has a neighbor u in S, whose label is at most k. So $|f(v) - f(u)| \geq |\partial(S)|$, which implies $B(G) = B(f, G) \geq |\partial(S)|$. Similarly, consider a vertex x in $\partial(V - S)$ with the smallest label. Its label is at most $k - |\partial(V - S)| + 1$. It has a neighbor y in $V - S$, whose assigned label is at least $k + 1$. So $|f(y) - f(x)| \geq |\partial(V - S)|$. Thus, $B(G) = B(f, G) \geq |\partial(V - S)|$. Therefore, we have $B(G) \geq \max\{|\partial(S)|, |\partial(V - S)|\}$. This yields the following

Proposition 2.1 *[12] Let G be a graph and k an integer, where $0 \leq k \leq |V(G)|$. Then*

$$B(G) \geq \min_{S \subseteq V(G), |S| = k} \max\{|\partial(S)|, |\partial(V - S)|\}.$$

For each k, let $L_k(G) = \min_{S \subseteq V(G), |S|=k} |\partial(S)|$. By Proposition 2.1, $B(G) \geq L_k(G)$. Since this holds for each k with $0 \leq k \leq |V(G)|$, we have $B(G) \geq \max_k L_k(G)$. This lower bound $\max_k L_k(G)$ for $B(G)$ is often referred to as the *Harper bound.* In general, the Harper bound needs not be sharp and calculating it is difficult (NP-hard).

When the Harper bound is not very useful, it is sometimes useful to consider the iterated boundary (shadow) instead. Given a nonnegative integer q, let

$$\partial^{(\leq q)}(S) = \{v \in V(G) - S : \ v \text{ is at distance at most } q \text{ from some vertex in } S\}.$$

Hence, in particular, $\partial(S) = \partial^{(\leq 1)}(S)$. Consider an optimal numbering f of $V(G)$. Let S be the set of vertices receiving labels $1, 2, \ldots, k$. Let q be any integer such that $1 \leq q \leq n$. Let v be a vertex in $\partial^{\leq q}(S)$ with the highest label. Then $f(v) \geq k + |\partial^{\leq q}(S)|$. Vertex v is at distance at most q from some vertex u in S. Note that $f(u) \leq k$. Hence for some edge on a shortest u, v-path, the difference between the f-labels of its two endpoints is at least $|f(u) - f(v)|/q \geq |\partial^{\leq q}(S)|/q$. This yields

Proposition 2.2 *Let G be a graph and k an integer, where $0 \leq k \leq |V(G)|$. Then*

$$B(G) \geq \min_{S \subseteq V(G), |S|=k} \max_{1 \leq q \leq n} |\frac{\partial^{(\leq q)}(S)}{q}|.$$

Our discussions above apply similarly to a set of edges. We define the boundary of a set $S' \subseteq E(G)$ of edges in G by

$$\partial(S') = \{e \in E(G) - S' : \exists e' \in S' \text{ such that } e \text{ and } e' \text{ are incident}\}.$$

The iterated boundary for S' is then given, for $q \geq 1$, by

$$\partial^{(\leq q)}(S') = \{e \in E(G) - S' : \ e \text{ is at distance at most } q \text{ from some edge of } S'\}.$$

We see that $\partial(S') = \partial^{(\leq 1)}(S)$ and $\partial^{(\leq q)}(S') = \partial^{(\leq q-1)}(S) \cup \partial(\partial^{(\leq q-1)}(S'))$. The edge analogues of Propositions 2.1 and 2.2 are then obtained by replacing $B(G)$ by $B'(G)$ and $V(G)$ by $E(G)$.

3 The weight function in multidimensional grids

Recall that $V(P_n^d) = \{\langle x_1, x_2, \ldots, x_d \rangle : \forall i \, x_i \in \{0, 1, \ldots, n-1\}\}$. Two vertices $\langle x_1, x_2, \ldots, x_d \rangle$ and $\langle y_1, y_2, \ldots, y_d \rangle$ are adjacent if they differ by 1 in one coordinate and agree in all other coordinates. Again, the weight $wt(x)$ of a vertex $x = \langle x_1, x_2, \ldots, x_d \rangle$ is defined by $wt(x) = x_1 + x_2 + \cdots + x_d$. Given positive integers n, d and an integer r with $0 \leq r \leq (n-1)d$, let $L(n, d, r) = \{x \in V(P_n^d) : wt(x) = r\}$. Let $l(n, d, r) = |L(n, d, r)|$. Let $l^*(n, d) = \max\{l(n, d, r) : 0 \leq r \leq (n-1)d\}$. It is easy to see that $l(n, d, r)$ is the number of integer solutions to the equation $x_1 + x_2 + \cdots + x_d = r$, where $0 \leq x_i \leq n-1$ for each $i \in [d]$. By considering the value of x_1 one can easily derive the following recurrence relation on $l(n, d, r)$.

Proposition 3.1 *Let n, d be positive integers and r an integer. We have $l(n, 1, r) = 1$ if $0 \leq r \leq n-1$ and $l(n, 1, r) = 0$ otherwise. For all d and r with $d \geq 2$ and $0 \leq r \leq d(n-1)$,*

$$l(n, d, r) = \Sigma_{j=0}^{n-1} l(n, d-1, r-j),$$

and for other values of r we have $l(n, d, r) = 0$.

The function $l(n,d,r)$ is well-studied in the theory of posets as the *rank number* in the poset of divisors of a number. Consider a positive integer m with prime factorization $m = p_1^{k_1} p_2^{k_2} \ldots p_d^{k_d}$, where the p_i are distinct primes and $k_i \geq 1$ for each i. The *rank* of m is $K = k_1 + \cdots k_d$ and $N_r(m)$ is the number of divisors of m of rank r. It is easy to see that in the case $k_1 = k_2 = \cdots = k_d = n-1$, $N_r(m)$ is precisely $l(n,d,r)$. Chapter 4 of Anderson [3] gives a detailed discussion about the function $N_r(m)$. We summarize related results in terms of $l(n,d,r)$ in the following proposition.

Proposition 3.2 *([3] Chapter 4) Let $K = (n-1)d$. Then*

1. *$l(n,d,i) = l(n,d,K-i)$ for each i with $0 \leq i \leq K$.*
2. *For fixed n and d, $l(n,d,i)$ is strictly increasing in i for $i \leq \lfloor \frac{K}{2} \rfloor$ and strictly decreasing in i for $i \geq \lceil \frac{K}{2} \rceil$.*

By Proposition 3.2, for fixed n and d, $l(n,d,r)$ is a symmetric and unimodal function of r on $\{0, 1, \ldots, (n-1)d\}$ with maximum value at $r = \lfloor \frac{(n-1)d}{2} \rfloor$ and at $r = \lceil \frac{(n-1)d}{2} \rceil$. It is not hard to find a formula for $l(n,d,r)$ using generating functions. By Proposition 3.2, $l^*(n,d) = l(n,d,\lfloor \frac{(n-1)d}{2} \rfloor)$. Thus, setting $r = \lfloor \frac{(n-1)d}{2} \rfloor$ in the formula for $l(n,d,r)$ will give us $l^*(n,d)$. Unfortunately, this formula of $l(n,d,r)$ from generating functions does not have a closed form; thus neither does the formula for $l^*(n,d)$. Anderson [2] obtained some useful estimates of $l^*(n,d)$ as follows.

Theorem 3.3 *([2]) Let $k_1, k_2, \cdots, k_d$ be nonnegative integers. Let $K = k_1 + \cdots + k_d$. Let s denote the number of integer solutions to the equation*

$$x_1 + x_2 + \cdots + x_d = \frac{K}{2}, \text{ where } \forall i \; 0 \leq x_i \leq k_i.$$

Let $A = \frac{1}{3}\Sigma_{i=1}^d k_i(k_i+2)$ and $\tau = \Pi_{i=1}^d (1+k_i)$. Then there exist positive constants C_1 and C_2 such that

$$C_1 \frac{\tau}{\sqrt{A}} \leq s \leq C_2 \frac{\tau}{\sqrt{A}}.$$

Furthermore, we can take $C_2 = \sqrt{11}$ and for any small $\epsilon > 0$ we can take $C_1 = \frac{1}{\sqrt{3}} - \epsilon$ when K is sufficiently large.

Corollary 3.4 *Let n, d be positive integers, where $n \geq 2$. There exist positive constants C_1 and C_2 such that*

$$C_1 \frac{n^{d-1}}{\sqrt{d}} \leq l^*(n,d) \leq C_2 \frac{n^{d-1}}{\sqrt{d}}.$$

Furthermore, we can take $C_2 = 2\sqrt{11}$ and for any small $\epsilon > 0$ we can take $C_1 = 1 - \epsilon$ when n is sufficiently large.

Proof. By our earlier discussion, $l^*(n,d) = l(n,d,\lfloor\frac{(n-1)d}{2}\rfloor)$ is the number of integer solutions to the equation $x_1 + x_2 + \cdots + x_d = \lfloor\frac{(n-1)d}{2}\rfloor$, where $0 \le x_i \le n-1$ for each i. We apply Theorem 3.3. Here, $k_i = n-1$, for each $i \in [d]$. So $A = \frac{d}{3}(n^2-1)$ and $\tau = n^d$. Also, $s = l^*(n,d)$. The claim follows readily. ∎

We now give the exact formula for $l^*(n,d)$ together with an asymptotic formula for it. Though the leading coefficient of the asymptotic formula does not have a closed form expression, we can bound it using Corollary 3.4. We need the following routine estimation of binomial coefficients. Recall that if x is a real number and k is an integer then $\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!}$.

Lemma 3.5 *Let k be a positive integer and N, m real numbers such that $N > \max\{k^2, mk\}$. We have $\frac{N^k}{k!}(1-\frac{k^2}{N}) \le \binom{N+m}{k} \le \frac{N^k}{k!}(1+\frac{2mk}{N})$.*

Proof. For the lower bound, we have $\binom{N+m}{k}/\frac{N^k}{k!} > \frac{(N+m-k)^k}{k!}/\frac{N^k}{k!} > \frac{(N-k)^k}{k!}/\frac{N^k}{k!} = (1-\frac{k}{N})^k$. Since $N > k^2$, it is straightforward to verify that the terms in the expansion of $(1-\frac{k}{N})^k$ are decreasing in absolute value. So $\binom{N+m}{k}/\frac{N^k}{k!} > 1 - k(\frac{k}{N}) = 1 - \frac{k^2}{N}$.

For the upper bound, we have $\binom{N+m}{k}/\frac{N^k}{k!} < \frac{(N+m)^k}{k!}/\frac{N^k}{k!} = (\frac{N+m}{N})^k = (1+\frac{m}{N})^k$. Since $(1+\frac{1}{x})^x < e$ when $x > 0$, we have $(1+\frac{m}{N})^{\frac{N}{m}} < e$ and hence $(1+\frac{m}{N})^k < e^{\frac{mk}{N}}$. Since $N > mk$, $\frac{mk}{N} < 1$. Since $e^x < 1+2x$ when $0 < x < 1$, we have $e^{\frac{mk}{N}} < 1 + \frac{2mk}{N}$. Therefore, $\binom{N+m}{k}/\frac{N^k}{k!} < 1 + \frac{2mk}{N}$. ∎

Theorem 3.6 *Let n, d, r be positive integers. We have $l(n,d,r) = \Sigma_{j=0}^{\infty}(-1)^j\binom{d}{j}\binom{r-jn+d-1}{d-1}$. Hence,*

$$l^*(n,d) = \Sigma_{j=0}^{\lfloor\frac{d}{2}\rfloor}(-1)^j\binom{d}{j}\binom{\lfloor\frac{(d-2j)n}{2}+\frac{d}{2}\rfloor - 1}{d-1}.$$

We have $l^*(n,1) = 1, l^*(n,2) = 2$. For fixed $d \ge 3$, as $n \to \infty$, we have

$$l^*(n,d) = c(d)n^{d-1} + O(n^{d-2}),$$

where $c(d)$ is a constant depending on d, given by

$$c(d) = \Sigma_{j=0}^{d/2}(-1)^j\binom{d}{j}\frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!}.$$

Also, $\frac{1}{2\sqrt{d}} \le c(d) \le \frac{2\sqrt{11}}{\sqrt{d}}$.

Proof. By prior discussion, $l(n,d,r)$ is the number of integer solutions to $x_1+x_2+\cdots+x_d = r$ with $0 \le x_i \le n-1$ for each $i \in [d]$. For fixed n, d, the generating function for $l(n,d,r)$ is

$$g(x) = (1+\cdots+x^{n-1})^d = \left(\frac{1-x^n}{1-x}\right)^d = (1-x^n)^d(\frac{1}{1-x})^d.$$

As $l(n,d,r)$ equals coefficient of x^r in the above expansion, we have

$$l(n,d,r) = \Sigma_{j=0}^{d}(-1)^j \binom{d}{j}\binom{r-jn+d-1}{d-1}. \tag{1}$$

By Proposition 3.2, $l^*(n,d) = l(n,d,\lfloor\frac{(n-1)d}{2}\rfloor)$. Setting $r = \lfloor\frac{(n-1)d}{2}\rfloor$ in the expression above for $l(n,d,r)$, we have

$$l^*(n,d) = \Sigma_{j=0}^{d}(-1)^j \binom{d}{j}\binom{\lfloor\frac{(n-1)d}{2}\rfloor - jn + d - 1}{d-1} = \Sigma_{j=0}^{\lfloor\frac{d}{2}\rfloor}(-1)^j \binom{d}{j}\binom{\lfloor\frac{(d-2j)n}{2} + \frac{d}{2}\rfloor - 1}{d-1}. \tag{2}$$

Now, for fixed d, we study the asymptotic behavior of $l^*(n,d)$ as a function of n. Fix j with $d-2j>0$. Let $N = \frac{(d-2j)n}{2}$ and $m = (\lfloor\frac{(d-2j)n}{2} + \frac{d}{2}\rfloor - 1) - N$. Then $m \le d/2$. For sufficiently large n, since d is fixed, we have $N > \max\{(d-1)^2, m(d-1)\}$. Applying Lemma 3.5, we have

$$\left|\binom{N+m}{d-1} - \frac{N^{d-1}}{(d-1)!}\right| \le c_j N^{d-2} = c_j' n^{d-2}, \tag{3}$$

where c_j and c_j' are constants depending only on d. Since

$$\frac{N^{d-1}}{(d-1)!} = \frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!} \cdot n^{d-1},$$

we can rewrite (3) as

$$\left|\binom{\lfloor\frac{(d-2j)n}{2} + \frac{d}{2}\rfloor - 1}{d-1} - \frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!} \cdot n^{d-1}\right| \le c_j' n^{d-2}.$$

So,

$$\begin{aligned} l^* &= \Sigma_{j=0}^{d}(-1)^j \binom{d}{j}\binom{\lfloor\frac{(dn-2jn}{2} + \frac{d}{2}\rfloor - 1}{d-1} \\ &= \Sigma_{j=0}^{\lfloor d/2\rfloor}(-1)^j \binom{d}{j} \cdot \frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!} \cdot n^{d-1} + O(n^{d-2}). \end{aligned}$$

Let $c(d) = \Sigma_{j=0}^{\lfloor d/2\rfloor}(-1)^j\binom{d}{j}\frac{(d-2j)^{d-1}}{2^{d-1}(d-1)!}$. We have $l^*(n,d) = c(d)n^{d-1} + O(n^{d-2})$. Note that $c(d)$ is a constant depending only on d and by Corollary 3.4, $\frac{1}{2\sqrt{d}} \le c(d) \le \frac{2\sqrt{11}}{\sqrt{d}}$. ■

Next, we prove a fact about the function $l(n,d,r)$ that is crucial to establishing our lower bound on the edge-bandwidth of P_n^d. We show that for all r relatively close to $\lfloor\frac{(n-1)d}{2}\rfloor$ and $\lceil\frac{(n-1)d}{2}\rceil$, $l(n,d,r)$ is close to the maximum value $l^*(n,d)$.

Lemma 3.7 *Let n,d be positive integers and r an integer. If $|r - \lfloor\frac{(n-1)d}{2}\rfloor| = t$, where $0 \le t \le \lfloor\frac{n}{2}\rfloor - 1$, then $l(n,d,r) \ge (1-\frac{t}{n})l^*(n,d)$.*

Proof. Let $K = (n-1)d$ and $M = \lfloor \frac{K}{2} \rfloor$. By Proposition 3.2, $l^*(n,d) = l(n,d,M)$. For convenience, let $f(j) = l(n, d-1, j)$. By Proposition 3.1,

$$l^*(n,d) = l(n,d,M) = \Sigma_{i=M-(n-1)}^{M} f(i).$$

Let $K' = (n-1)(d-1)$, $M' = \lfloor \frac{K'}{2} \rfloor$ and $M'' = \lceil \frac{K'}{2} \rceil$. We have $M'' = M'$ if K' is even and $M'' = M' + 1$ if K' is odd. By Proposition 3.2 $f(i)$ is unimodal and symmetric on $\{0, 1, \ldots, K'\}$ and has its maximum value at $i = M'$ and at $i = M''$. By the symmetry of $f(i)$ we have $f(M' - x) = f(M'' + x)$ for any $x \in \{0, 1, \cdots, M'\}$. Furthermore, we have for any $x, y \in \{0, 1, \cdots, M'\}$ with $x \geq y$, $f(M'-x) \leq f(M'-y)$. Note that $M - M' = \lfloor \frac{n-1}{2} \rfloor$ or $\lceil \frac{n-1}{2} \rceil$. So M' is at the center of the interval $[M-(n-1), M]$ of integers. Fix t with $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Our goal is to show that $l(n,d,M+t) \geq (1 - \frac{t}{n}) \cdot l^*(n,d)$ and $l(n,d,M-t) \geq (1-\frac{t}{n}) \cdot l^*(n,d)$. Let I_1 denote the interval $[M-(n-1), M-(n-1)+(t-1)]$ of integers, and I_2 the interval $[M-(n-1)+t, M-t]$ of integers. Let I_3 denote the interval $[M-t+1, M]$. We have $l^*(n,d) = l(n,d,M) = \Sigma_{i \in I_1 \cup I_2 \cup I_3} f(i)$.

Claim 1. For each integer $i \in I_1$ and integer $j \in I_2$, $f(i) \leq f(j)$.

Proof of Claim 1. By the symmetry and unimodality of $f(i)$, it suffices to show that $M' - i \geq j - M''$, or equivalently $M' + M'' \geq i + j$. We have by the definition of M' and M'' that $M' + M'' = K' = (n-1)(d-1)$. On the other hand, $i + j \leq M - (n-1) + (t-1) + M - t = 2M - n \leq (n-1)d - n < K'$. So, $M' + M'' \geq i + j$ clearly holds. ■

Claim 2. For each j with $0 \leq j \leq t-1$, $f(M-(n-1)+j) \leq f(M-j)$.

Proof of Claim 2. It suffices to show that $M' - [M-(n-1)+j] \geq M - j - M''$ or equivalently, $M' + M'' \geq 2M - (n-1)$. By the definitions of M, M' and M'', this inequality is equivalent to $(n-1)(d-1) \geq 2\lfloor \frac{(n-1)d}{2} \rfloor - (n-1)$, which clearly holds. ■

The interval I_1 contains t integers, the interval I_2 contains $n-2t$ integers, and the interval I_3 contains t integers. Let $A = \Sigma_{i \in I_1} f(i), B = \Sigma_{i \in I_2} f(i)$, and $C = \Sigma_{i \in I_3} f(i)$. Then $A + B + C = l^*(n,d)$. By Claim 2, we have $A \leq C$. By Claim 1, we have $A \leq \frac{t}{n-2t} B$. Hence $A \leq \frac{t}{n}(A+B+C)$, or $(B+C) \geq (1-\frac{t}{n})(A+B+C)$. That is,

$$\Sigma_{i=M-(n-1)+t}^{M} f(i) \geq (1 - \frac{t}{n}) \cdot l^*(n,d).$$

Now, we have

$$l(n,d,M+t) = \Sigma_{i=M-(n-1)+t}^{M+t} f(i) \geq \Sigma_{i=M-(n-1)+t}^{M} f(i) \geq (1 - \frac{t}{n}) \cdot l^*(n,d).$$

By a similar argument, one can show that $l(n,d,M-t) \geq (1-\frac{t}{n}) \cdot l^*(n,d)$. This completes the proof. ■

4 The edge-bandwidth of a multidimensional grid

Bollobás and Leader [7] solved the vertex isoperimetric problem in grids. We will use their result to obtain asymptotically tight bounds on the edge-bandwidth of a multidimensional grids. Our result extends that of Balogh et al. on 2-dimensional grids to grids of any dimension.

Definition 4.1 The *simplicial order* on $V(P_n^d)$ is defined by $x < y$ if either $wt(x) < wt(y)$ or $wt(x) = wt(y)$ and $x_s > y_s$, where $s = \min\{t : x_t \neq y_t\}$.

Bollobás and Leader [7] showed that for any k, with $0 \leq k \leq |V(P_n^d)|$, the initial segment of length k in the simplicial order has the smallest boundary among all sets of k vertices. For each r with $0 \leq r \leq (n-1)d$, let $B(n,d,r) = \{x \in V(P_n^d) : wt(x) \leq r\}$. In other words, $B(n,d,r) = \bigcup_{j=0}^{r} L(n,d,j)$. Note that each edge in P_n^d has one endpoint in $L(n,d,r)$ and the other endpoint in $L(n,d,r+1)$ for some r.

Theorem 4.2 *(*Vertex isoperimetric inequality in the grid*) [7] Let $A \subseteq V(P_n^d)$ and let C be the initial segment of length $|A|$ in the simplicial order on $V(P_n^d)$. Then $|N(A)| \geq |N(C)|$. In particular, if $|A| \geq |B(n,d,r)|$ then $|N(A)| \geq |B(n,d,r+1)|$. Also, for all $q \geq 1$ we have $|N^{(\leq q)}(A)| \geq |B(n,d,r+q)|$.*

Definition 4.3 Let G be a graph on n vertices. Let $\sigma : x_1 < x_2 < \cdots < x_n$ be a linear order on $V(G)$. The labeling f of $V(G)$ satisfying $f(x_i) = i$ is the vertex labeling of $V(G)$ *induced by* σ.

Lemma 4.4 *Let $G = P_n^d$. Let f be the labeling of $V(G)$ induced by the simplicial order on G. For every $uv \in E(G)$, we have $|f(u) - f(v)| \leq l^*(n,d) + 2l^*(n,d-1) - 1$.*

Proof. Without loss of generality, suppose $wt(u) = r$ and $wt(v) = r+1$. Let $A = \{x \in L(n,d,r) : f(x) > f(u)\}$, $B = \{x \in L(n,d,r+1) : f(x) < f(v)\}$ and $C = \{x \in L(n,d,r+1) : f(x) > f(v)\}$. Let $a = |A|, b = |B|, c = |C|$. Then

$$|f(u) - f(v)| = a + b + 1 \quad \text{and} \quad b + c + 1 = l(n,d,r+1).$$

Suppose $u = \langle u_1, u_2, \ldots, u_d \rangle$ and $v = \langle v_1, v_2, \ldots, v_d \rangle$. Since $uv \in E(G)$ and $wt(v) = wt(u) + 1$, there exists $j \in [d]$ such that $v_j = u_j + 1$ and $v_i = u_i$ for all $i \in [d] - j$. By our definition of A, vertices in A all have their first coordinate at most u_1. Let $A_1 = \{x \in A : x_1 = u_1 \text{ or } u_1 - 1\}$ and let $A_2 = \{x \in A : x_1 \leq u_1 - 2\}$. We have $A = A_1 \cup A_2$. Since there are at most $l^*(n,d-1) - 1$ vertices other than u that have u_1 in the first coordinate and there are at most $l^*(n,d-1)$ vertices that have $u_1 - 1$ in the first coordinate, $|A_1| \leq 2l^*(n,d-1) - 1$.

For each $x \in A_2$, let $g(x) = \langle x_1 + 1, x_2, \ldots, x_d \rangle$. It is easy to see that g is an injection of A_2 into $L(n,d,r+1)$. Furthermore, for each $x \in A_2$, since $x_1 + 1 < u_1 \leq v_1$, we have $f(g(x)) > f(v)$ by the definition of the simplicial order. So $g(x) \in C$. So, g is an injection of A_2 into C. Thus, $|A_2| \leq c$ and $a = |A| = |A_1| + |A_2| \leq 2l^*(n,d-1) - 1 + c$. Now, we have

$$\begin{aligned}
&|f(u) - f(v)| = a + b + 1 \leq 2l^*(n,d-1) - 1 + c + b + 1 \\
&= 2l^*(n,d-1) - 1 + l(n,d,r+1) \leq l^*(n,d) + 2l^*(n,d-1) - 1.
\end{aligned}$$

■

Theorem 4.5 *Let d be a fixed positive integer. For all positive integers n we have*

$$B'(P_n^d) \leq d[l^*(n,d) + 2l^*(n,d-1)] = c(d)dn^{d-1} + O(n^{d-2}),$$

where $c(d)$ is defined as in Theorem 3.6.

Proof. First, we define a digraph H from $G = P_n^d$ by orienting each edge xy from x to y if $x < y$ in the simplicial order. For each vertex x, let $E^+(x)$ denote the set of out-edges from x. Note that $|E^+(x)| \leq d$ for each x. We define a labeling g of $E(H)$ (and of $E(G)$) using $1, 2, \ldots, |E(H)|$ as follows. Suppose the vertices are $u_1, u_2, \ldots$ where $u_1 < u_2 < \cdots$ in the simplicial order. Starting with 1 we assign the first $|E^+(u_1)|$ consecutive labels to $E^+(u_1)$, then the next $|E^+(u_2)|$ consecutive labels to $E^+(u_2)$, and so on. Let $e = u_iu_j$ and $e' = u_ju_k$ be two incident edges in G at the vertex u_j. We consider three cases depending on how e and e' are oriented.

Case 1. $i < j < k$ or $i > j > k$

By symmetry, we may assume $i < j < k$. Then we have $wt(u_i) = r-1, wt(u_j) = r, wt(u_k) = r+1$ for some r. By Lemma 4.4, $j - i \leq l^*(n,d) + 2l^*(n,d-1) - 1$. Note that $e \in E^+(u_i)$ and $e' \in E^+(u_j)$. By our definition of g, we have $|g(e') - g(e)| \leq |\bigcup_{t=i}^{j} E^+(u_t)| \leq d(j-i+1) \leq d(l^*(n,d) + 2l^*(n,d-1))$.

Case 2. $i < j$ and $j > k$

In this case, we have $wt(u_i) = wt(u_k) = r-1$ and $wt(u_j) = r$ for some r. In particular, $|k-i| \leq l(n,d,r-1) - 1 \leq l^*(n,d) - 1$. Also, $e \in E^+(u_i)$ and $e' \in E^+(u_k)$. Without loss of generality, suppose $i < k$. By our definition of g, we have $|g(e') - g(e)| \leq |\bigcup_{t=i}^{k} E^+(u_t)| \leq d|k-i+1| \leq dl^*(n,d) \leq d(l^*(n,d) + 2l^*(n,d-1))$.

Case 3. $i > j$ and $k > j$

In this case, $e, e' \in E^+(u_j)$, and $|g(e) - g(e')| \leq d < d(l^*(n,d) + 2l^*(n,d-1))$.

We have shown that $|g(e) - g(e')| \leq d(l^*(n,d) + 2l^*(n,d-1))$ for every pair of incident edges e and e' in G. This yields $B'(G) \leq B'(g,G) \leq d(l^*(n,d) + 2l^*(n,d-1))$. Using $l^*(n,d) = c(d)n^{d-1} + O(n^{d-2})$, we get $B'(G) \leq c(d)dn^{d-1} + O(n^{d-2})$.

■

We now derive a lower bound on $B'(P_n^d)$ that matches the upper bound in Theorem 4.5 asymptotically when d is fixed and $n \to \infty$. Our proof is based on the method used by Calamoneri et al. [8] and Balogh et al. [5]. We need an easy lemma.

Lemma 4.6 *Let n, d be positive integers. Let $A \subseteq V(P_n^d)$. Then there are at least $d|A| - dn^{d-1}$ edges incident to A in P_n^d. Also, $|E(P_n^d)| = dn^d - dn^{d-1}$.*

Proof. The graph P_n^d is a spanning subgraph of C_n^d, the d-fold cartesian product of C_n. We can think of obtaining C_n^d from P_n^d by adding edges of the form uv, where $u = \langle u_1, u_2, \ldots, u_d \rangle$ and $v = \langle v_1, v_2, \ldots, v_d \rangle$ satisfy that $u_i = 0, v_i = n-1$ or $u_i = n-1, v_i = 0$ for some $i \in [d]$ and $u_j = v_j$ for all $j \in [d] - \{i\}$. It is easy to see that there are dn^{d-1} such edges. Since C_n^d is $2d$-regular. We have $E(P_n^d) = E(C_n^d) - dn^{d-1} = dn^d - dn^{d-1}$.

In C_n^d, since each vertex has degree $2d$, there are at least $2d|A|/2 = d|A|$ edges incident to A. So in P_n^d, there are at least $d|A| - dn^{d-1}$ edges incident to A. ■

Theorem 4.7 *Let $d \geq 2$ be a fixed positive integer. Then*

$$B'(P_n^d) \geq dl^*(n,d)(1-o(1)) = c(d)dn^{d-1} + \Omega(n^{d-\frac{3}{2}}).$$

Proof. Throughout the proof, whenever necessary, we assume that n is sufficiently large. Let g be an edge-labeling of $G = P_n^d$ with $B'(g,G) = B'(G)$. Let S denote the set of edges receiving labels $1, 2\ldots, |E(G)|/2$. We color the edges in S red and the rest of the edges white.

Let us call a vertex *red* if all of its incident edges are red, a vertex *white* if all of its incident edges are white, a vertex *mixed* if it is incident to both red edges and white edges. Let R denote the set of red vertices, W the set of white vertices, and M the set of mixed vertices. We consider two cases. For convenience, let $l^* = l^*(n,d)$.

Case 1. $|M| \geq 5l^*$

Call a vertex v *bad* if $d(v) \leq 2d-3$. Let D denote the set of bad vertices in G. If $v = \langle v_1, v_2, \ldots, v_d \rangle$ is bad, then it has a 0 or $n-1$ in at least three of the d coordinates. So $|D| \leq \binom{d}{3} 2^3 n^{d-3} < 2d^3 n^{d-3}$. By Corollary 3.4, when n is sufficiently large, we have $l^* \geq \frac{1}{2\sqrt{d}} n^{d-1} > 2d^3 n^{d-3} > |D|$. Hence, $|M - D| \geq 5l^* - l^* = 4l^*$. Each vertex in $M - D$ has degree at least $2d-2$. So the total number of edges incident to M is at least $(2d-2)|M-D|/2 = (d-1)|M-D| \geq 4(d-1)l^* \geq 2dl^*$, since $d \geq 2$.

Let $E(M)$ denote the set of edges in G incident to M. We have $|E(M)| \geq 2dl^*$. So, either $|E(M) \cap (E-S)| \geq dl^*$ or $|E(M) \cap S| \geq dl^*$. Note that $E(M) \cap (E-S) \subseteq \partial(S)$ and $E(M) \cap S \subseteq \partial(E-S)$. Hence, we have either $|\partial(S)| \geq dl^*$ or $|\partial(E-S)| \geq dl^*$. By Proposition 6.8, we have $B'(g,G) \geq dl^* = c(d)dn^{d-1} + O(n^{d-2})$.

Case 2. $|M| < 5l^*$

We have $|R| + |M| + |W| = n^d$. Hence, either $|R| < n^d/2$ or $|W| < n^d/2$. Without loss of generality, we may assume that $|R| < n^d/2$; otherwise we switch S with $E-S$ and hence R with W. Let $H = G[R \cup M]$ denote the subgraph of G induced by R and M. Note that $S \subseteq E(H)$. Hence, $|S| \leq |E(H)| \leq 2d(|R| + |M|)/2 = d(|R| + |M|)$. On the other hand, $|S| = |E(G)|/2 \geq (1/2)(dn^d - dn^{d-1})$. So, we have

$$d(|R| + |M|) \geq |S| \geq \frac{1}{2}dn^d - \frac{1}{2}dn^{d-1}.$$

From this we get

$$|R| + |M| \geq \frac{1}{2}n^d - \frac{1}{2}n^{d-1}.$$

So,

$$|R| \geq \frac{1}{2}n^d - \frac{1}{2}n^{d-1} - |M| \geq \frac{1}{2}n^d - \frac{1}{2}n^{d-1} - 5l^*.$$

By Corollary 3.4, for sufficiently large n we have $l^* \geq \frac{1}{2\sqrt{d}} n^{d-1}$. Hence, $\frac{1}{2}n^{d-1} \leq \sqrt{d}l^*$ and $|R| \geq \frac{1}{2}n^d - \sqrt{d}l^* - 5l^* \geq \frac{1}{2}n^d - (d+5)l^*$. We have

$$\frac{1}{2}n^{d-1} - (d+5)l^* < |R| < \frac{1}{2}n^{d-1} \tag{4}$$

Since $l(n,d,j)$ is symmetric on $[0, d(n-1)]$ and $\Sigma_{j=0}^{d(n-1)} l(n,d,j) = |V(G)| = n^d$, we have

$$\Sigma_{j=0}^{\lfloor \frac{(n-1)d}{2} \rfloor - 1} l(n,d,j) \leq \frac{1}{2} n^d \leq \Sigma_{j=0}^{\lfloor \frac{(n-1)d}{2} \rfloor} l(n,d,j). \tag{5}$$

For convenience, let $M = \lfloor \frac{(n-1)d}{2} \rfloor$. By Lemma 3.7, $l(n,d,M-t) \geq (1-\frac{t}{n}) l^*$, for $t \in [0, \lfloor \frac{n}{2} \rfloor - 1]$. Let n be sufficiently large so that $n > \binom{d+7}{2}$. We have

$$\Sigma_{t=1}^{d+6} l(n,d,M-t) \geq \Sigma_{t=1}^{d+6} (1 - \frac{t}{n}) l^* = (d+6) l^* - \frac{\binom{d+7}{2}}{n} l^* \geq (d+5) l^*. \tag{6}$$

By (5) and (6), we have $\Sigma_{j=0}^{M-d-7} l(n,d,j) \leq |R| \leq \Sigma_{j=0}^{M} l(n,d,j)$. In other words,

$$|B(n,d,M-d-7) < |R| < |B(n,d,M+1)|. \tag{7}$$

Therefore, we have $|B(n,d,r)| \leq |R| < |B(n,d,r+1)|$ for some $r \in [M-d-7, M+1]$. Let $q = \lfloor \sqrt{n} \rfloor - 1$. We may assume that n is sufficiently large so that $q \geq d+7$. Let A be a subset of R with $|A| = |B(n,d,r)|$. By Theorem 4.2, $|N^{(\leq q)}(A)| \geq |B(n,d,r+q)|$. Since $N^{(\leq q)}(A) \subseteq N^{(\leq q)}(R)$, we have $|N^{(\leq q)}(R)| \geq |B(n,d,r+q)|$. Hence,

$$|\partial^{(\leq q)}(R)| = |N^{(\leq q)}(R)| - |R| \geq |B(n,d,r+q)| - |B(n,d,r+1)| = \Sigma_{j=2}^{q} l(n,d,r+j). \tag{8}$$

For each $j \in [r+2, r+q]$, we have $j \in [M-d-7, M+q+1]$. Since $|j-M| \leq q+1$, by Lemma 3.7,

$$l(n,d,j) \geq (1 - \frac{q+1}{n}) l^* \geq (1 - \frac{\lfloor \sqrt{n} \rfloor}{n}) l^* \geq (1 - \frac{1}{\sqrt{n}}) l^*.$$

Hence,

$$|\partial^{(\leq q)}(R)| \geq \Sigma_{j=2}^{q} l(n,d,r+j) \geq (q-1)(1 - \frac{1}{\sqrt{n}}) l^*.$$

Let $E(\partial^{(\leq q)}(R))$ denote the set of edges incident to $\partial^{(\leq q)}(R)$. By Lemma 4.6,

$$|E(\partial^{(\leq q)}(R))| \geq d |\partial^{(\leq q)}(R)| - d n^{d-1}.$$

Finally, note that $E(\partial^{(\leq q)}(R)) \subseteq \partial^{(\leq q+1)}(S)$. Applying Proposition 2.2 to the line graph $L(G)$, we have

$$\begin{aligned}
B'(g,G) &\geq |\partial^{(\leq q+1)}(S)|/(q+1) \\
&\geq |E(\partial^{(\leq q)}(R))|/(q+1) \\
&\geq \left(d|\partial^{(q)}(R)| - dn^{d-1} \right)/(q+1) \\
&\geq d \left[(q-1)(1 - \frac{1}{\sqrt{n}}) \right] l^*/(q+1) - dn^{d-1}/(q+1) \\
&= dl^* \left(\frac{\sqrt{n}-2}{\sqrt{n}} \right) \left(1 - \frac{1}{\sqrt{n}} \right) + O(n^{d-\frac{3}{2}}) \quad (\text{using } q = \sqrt{n} - 1) \\
&= c(d) d n^{d-1} + O(n^{d-\frac{3}{2}}). \quad (\text{by Theorem 3.6})
\end{aligned}$$

■

5 Edge-bandwidth of the Hamming graph I: upper bound

In this section we derive an upper bound on $B'(K_n^d)$. We view any vertex x in K_n^d as an n-ary d-string $x = \langle x_1, x_2, \ldots, x_d \rangle$, where $\forall i \;\; x_i \in \{0, 1, \ldots, n-1\}$. Two vertices $x = \langle x_1, x_2, \ldots, x_d \rangle$ and $y = \langle y_1, y_2, \ldots, y_d \rangle$ are adjacent in K_n^d if and only if the two strings differ in precisely one coordinate. As in Section 4, for each $x = \langle x_1, x_2, \ldots, x_d \rangle$, we define the *weight* of x as $wt(x) = x_1 + x_2 + \ldots + x_d$. Note that when $n = 2$, K_n^d is just the d-dimensional hypercube Q_d and for each vertex x, $wt(x)$ is the number of 1's in the binary string that represents x. The edge-bandwidth $B'(Q_d)$ was asymptotically determined by Balogh, Mubayi, and Pluhár [5] while the vertex bandwidth $B(Q_d)$ was completely determined by Harper in his paper [12]. We will combine the labelings used in these results to design a labeling that yields an upper bound on $B'(K_n^d)$. Let us recall the labelings used in [5] and [12].

Definition 5.1 *The vertex-Hales numbering of* $V(Q_d)$ *is a bijection* $h : V(Q_d) \to \{1, 2, \ldots, 2^d\}$ *such that* $h(x) < h(y)$ *if either* $wt(x) < wt(y)$ *or* $wt(x) = wt(y)$ *and* $x_s > y_s$*, where* $s = \min\{t : x_t \neq y_t\}$.

Note that in Section 4, we called a similar ordering on $V(P_n^d)$ the simplicial order. Harper showed that the vertex-Hales numbering achieves the vertex bandwidth for Q_d.

Theorem 5.2 *[12] Let* h *be the vertex-Hales numbering of* Q_d*. Then*

$$B(Q_d) = B(h) = \Sigma_{k=0}^{d-1} \binom{k}{\lfloor k/2 \rfloor} = (1 + o(1)) \binom{d}{\lfloor d/2 \rfloor}.$$

The last equality used standard estimates of binomial coefficients. Next we describe the labeling used by Balogh, Mubayi, and Pluhár in establishing an asymptotically tight upper bound on $B'(Q_d)$. For convenience, we will call this numbering the *edge-Hales numbering.* Our description of the numbering is equivalent to the one used in [5].

Definition 5.3 *[5] The edge-Hales numbering of* Q_d *is a bijection* $f : E(Q_d) \to \{1, 2, \ldots, d2^{d-1}\}$ *such that for any two edges* vw *and* xy *where* $wt(w) = wt(v) + 1$ *and* $wt(y) = wt(x) + 1$ *we have* $f(vw) < f(xy)$ *if either (1)* $h(v) < h(x)$ *or (2)* $v = x$ *and* $h(w) < h(y)$.

Theorem 5.4 *[5] Let* f *denote the edge-Hales labeling of* Q_d*. Then*

$$\left(\frac{d}{2} + o(d)\right) \binom{d}{\lfloor d/2 \rfloor} \leq B'(Q_d) \leq B'(f) \leq \left(\frac{d}{2} + o(d)\right) \binom{d}{\lfloor d/2 \rfloor}.$$

Now, we combine the two numberings mentioned above to obtain a total numbering on $V(Q_d) \cup E(Q_d)$, which we will call the *mixed-Hales numbering* of Q_d. This numbering is produced by the following algorithm.

Figure 1. The mixed-Hales numbering of Q_3.

Algorithm 5.5 *(The mixed-Hales numbering m of Q_d)*
Input: The d-dimensional hypercube Q_d.
Output: A bijection m: $V(Q_d) \cup E(Q_d) \to \{1, 2, \ldots, 2^d + d2^{d-1}\}$.
Initialization:

(1) Denote the edges of Q_d by e_i, $1 \leq i \leq d2^{d-1}$, according to the edge-Hales numbering f of Q_d. That is, e_i is the edge e with $f(e) = i$.

(2) Let $m(0^d) = 1$, where 0^d denotes the all 0 string of length d.

(3) For all i, $1 \leq i \leq d2^{d-1}$, let $m(e_i) = i + 1$.

(4) Set $i = 1$.

Iteration:

(5) Suppose $e_i = xy$, where $wt(y) = wt(x) + 1$. If $m(y)$ is not yet defined, then

(5a) Let $m(y) = m(e_i) + 1$;

(5b) For all $j > i$, let $m(e_j) = m(e_j) + 1$.

(6) Let $i = i + 1$.

(7) If $i = d2^{d-1} + 1$, terminate; otherwise go to step (5).

Intuitively speaking, to obtain the mixed-Hales numbering we process the edges one by one in increasing order of edge-Hales label. The algorithm gives a vertex y an m-label at the earliest opportunity, as soon as we process the first edge in the edge-Hales numbering incident to y. See Figure 1 for the mixed-Hales labeling of Q_3.

Next, we summarize some useful facts about mixed-Hales numbering in the following proposition. In particular, we see that the ordering on $V(Q_d)$ and the ordering on $E(Q_d)$ inherited from the mixed-Hales numbering m of Q_d are precisely the vertex-Hales numbering h and the edge-Hales numbering f of Q_d, respectively.

Proposition 5.6 *Let h, f, m denote the vertex-Hales, edge-Hales, and mixed-Hales numberings of Q_d, respectively. Let 0^d denote the all 0 string of length d and 1^d the all 1 string of length d. For each vertex $x \in V(Q^d) - \{0^d, 1^d\}$, let x^- denote the neighbor of x with smallest h label and x^+ the neighbor of x with largest h label. Then*

1. *For each vertex $x \in V(Q_d) - \{0^d, 1^d\}$, the string representing x^- is obtained from the string for x by flipping the rightmost 1 to a 0 and the string representing x^+ is obtained from the string for x by flipping the rightmost 0 to a 1.*

2. *For each vertex $x \in V(Q_d) - \{0^d, 1^d\}$, among the edges incident to x, xx^- has the smallest m label and xx^+ has the largest m label. Hence, in particular, $m(x) = m(xx^-) + 1$.*

3. *For any two edges e, e' in Q_d, if $f(e) < f(e')$ then $m(e) < m(e')$.*

4. *For any two edges vw, xy in Q_d, where $wt(w) = wt(v) + 1$ and $wt(y) = wt(x) + 1$, $m(vw) < m(xy)$ if and only if either $h(v) < h(x)$ or $v = x$ and $h(w) < h(y)$.*

5. *For any two vertices $x, y \in V(Q_d) - \{0^d, 1^d\}$, if $h(x) < h(y)$, then $h(x^-) \leq h(y^-)$ and $h(x^+) \leq h(y^+)$.*

6. *For any two vertices, x, y in Q_d, if $h(x) < h(y)$ then $m(x) < m(y)$.*

Proof. Part 1 follows immediately from the definition of the vertex-Hales numbering (Definition 5.1). Parts 2 and 3 follow immediately from Algorithm 5.5. Part 4 follows from the definition of the edge-Hales numbering f (Definition 5.3) and Part 3.

To prove Part 5, suppose $h(x) < h(y)$. Since $h(x) < h(y)$, we have $wt(x) \leq wt(y)$. If $wt(x) < wt(y)$, then $wt(x^-) < wt(y^-)$ and $h(x^-) < h(y^-)$ hold trivially. So we may assume that $wt(x) = wt(y)$. By Part 1, the string representing x^- is obtained from the string representing x by flipping the rightmost 1 in x to 0 and the string for y^- is obtained from the string for y by flipping the rightmost 1 in y to 0. Let j denote the smallest coordinate in which x and y differ. Since $h(x) < h(y)$, we have $x_j = 1$ and $y_j = 0$. Since $wt(x) = wt(y)$, if x has k many 1's in coordinates $j+1, j+2, \ldots, d$ then y should have exactly $k+1$ many 1's in coordinates $j+1, j+2, \ldots, d$. If $k \geq 1$, then clearly $h(x^-) < h(y^-)$. If $k = 0$, then $x^- = y^-$ and hence $h(x^-) = h(y^-)$. By a very similar argument, we have $h(x^+) \leq h(y^+)$.

To prove Part 6, we may assume without loss of generality that $x, y \notin \{0^1, 1^d\}$. Suppose $h(x) < h(y)$. Since $m(x) = m(xx^-) + 1$ and $m(y) = m(yy^-) + 1$, to prove $m(x) < m(y)$ it suffices to prove that $m(xx^-) < m(yy^-)$. By Part 5, $h(x^-) \leq h(y^-)$. Thus, we have either $h(x^-) < h(y^-)$ or $x^- = y^-$ and $h(x) < h(y)$. By Part 4, we have $m(xx^-) < m(yy^-)$. This completes the proof. ∎

In the next proposition, we bound the number of vertices and the number of edges whose m-labels lie between the m-labels of two incident edges.

Proposition 5.7 *Let h, f, m denote the vertex-Hales, edge-Hales, and mixed-Hales numberings of Q_d, respectively. Let e, e' be two incident edges in Q_d, where $m(e) < m(e')$. Then there are at most $B(h) + 1$ vertices z with $m(e) \leq m(z) \leq m(e')$ and there are at most $B'(f) + 1$ edges e^* with $m(e) \leq m(e^*) \leq m(e')$.*

Proof. By Proposition 5.6 part 2, an edge e^* satisfies $m(e) \leq m(e^*) \leq m(e')$ if and only if $f(e) \leq f(e^*) \leq f(e')$. Hence, since e and e' are incident, there are at most $|f(e') - f(e)| + 1 \leq B'(f) + 1$ such edges e^*. Next, suppose e and e' are both incident to x. Let z be a vertex with $m(e) \leq m(z) \leq m(e')$. If $x = 0^d$ or 1^d, then it is easy to see that there are at most $d < B(h)$ such vertices z. Hence, we may assume that $x \notin \{0^d, 1^d\}$. By Proposition 5.6 Part 2, $m(xx^-) \leq m(z) \leq m(xx^+)$. Since $m(x) = m(xx^-) + 1$ and $m(z) > m(xx^-)$, we have $m(z) \geq m(x)$. We show that also $m(z) \leq m(x^+)$. Suppose first that $h(z^-) > h(x)$. Then by Proposition 5.6 Part 4, $m(zz^-) > m(xx^+)$, and hence $m(z) = m(zz^-) + 1 > m(xx^+)$, a contradiction. So, we must have $h(z^-) \leq h(x)$. By Proposition 5.6 Part 5, $h((z^-)^+) \leq h(x^+)$. Since $h(z) \leq h((z^-)^+)$, we have $h(z) \leq h(x^+)$ and thus $m(z) \leq m(x^+)$. So, any vertex z satisfying $m(e) \leq m(z) \leq m(e')$ must satisfy $h(x) \leq h(z) \leq h(x^+)$. There are at most $|h(x^+) - h(x)| + 1 \leq B(h) + 1$ such vertices z.

■

Lemma 5.8 *Let p, q, t be positive integers. Let G be a graph obtained from Q_d by replacing each vertex x of Q_d by a t-vertex graph G_x having p edges and each edge xy of Q_d by a set of q cross edges between $V(G_x)$ and $V(G_y)$. Then $B'(G) \leq p(B(h) + 1) + q(B'(f) + 1)$, where h, f denote the vertex-Hales and edge-Hales numberings of Q_d, respectively.*

Proof. Apply Algorithm 5.5 to obtain the mixed Hales numbering m of Q_d. List elements of $V(Q_d) \cup E(Q_d)$ in the order determined by m, call this list L. We produce a labeling g of $E(G)$ as follows. Start with label 1. As we scan L, each time we encounter a vertex x, we allocate the next p consecutive labels to the edges of G_x, and each time we encounter an edge $e = xy$, we allocate the next q consecutive labels to the set of q cross edges between $V(G_x)$ and $V(G_y)$.

Consider any pair of incident edges e and e' in G with $g(e) < g(e')$. Suppose both are incident to vertex w, and w lies in G_x. If $x \in \{0^d, 1^d\}$, then it is easy to see that $|g(e) - g(e')| \leq p + dq < p(B(h) + 1) + q(B'(f) + 1)$. Hence we may assume that $x \notin \{0^d, 1^d\}$. By our labeling scheme $|g(e) - g(e')|$ is maximum when e is among the set of q edges of G associated with edge xx^- in Q_d and e' is among the set of q edges of G associated with edge xx^+ in Q_d. By Proposition 5.7 and the definition of g, we have $|g(e) - g(e')| \leq p(B(h) + 1) + q(B'(f) + 1)$.
■

Now, we apply Lemma 5.8 to get an upper bound on $B'(K_n^d)$. For convenience, we consider only even n. For odd n, we can upper bound $B'(K_n^d)$ by $B'(K_{n+1}^d)$. We can view K_n^d as being obtained from Q_d by replacing each vertex of Q_d with a copy of $K_{n/2}^d$ and replacing each edge of Q_d by the set of edges between two neighboring copies of $K_{n/2}^d$ in K_n^d. More specifically, for each $x = \langle x_1, \ldots, x_d \rangle \in V(Q_d)$, let $O(x)$ denote the subgraph of K_n^d induced by the set of vertices $\{w = \langle w_1, \dots, w_d \rangle : 0 \le w_i \le n/2 - 1 \text{ if } x_i = 0 \text{ , } n/2 \le w_i \le n-1 \text{ if } x_i = 1\}$. Then each $O(x)$ is a copy of $K^d_{n/2}$. For each edge $xy \in E(Q_d)$, let $E(O(x), O(y))$ denote the set of edges in K^d_n having on endpoint in $O(x)$ and the other endpoint in $O(y)$. It is easy to see that $|E(O(x), O(y))| = (\frac{n}{2})^2 (\frac{n}{2})^{d-1} = (\frac{n}{2})^{d+1}$ for all $xy \in E(Q_d)$. We denote this quantity by $q(n, d)$. Applying Lemma 5.8 with $p = e(K^d_{n/2})$ and $q = q(n, d)$, we have the following.

Lemma 5.9 *Let d be a positive integer and n a positive even integer. We have*

$$B'(K^d_n) \le e(K^d_n)(B(h) + 1) + q(n, d)(B'(f) + 1).$$

Theorem 5.10 *Let n be a fixed positive even integer. Let d be a positive integer. We have*

$$B'(K^d_n) \le (d + o(d)) \binom{d}{\lfloor d/2 \rfloor} \left(\frac{n}{2}\right)^d (n-1) = (1 + o(1)) \frac{\sqrt{d}}{\sqrt{2\pi}} n^d (n-1),$$

as $d \to \infty$.

Proof. Using Lemma 5.9, Theorem 5.2, and Theorem 5.4, $e(K^d_{n/2}) = d(\frac{n}{2})^{d-1}\binom{n/2}{2}$, $q(n, d) = (\frac{n}{2})^{d+1}$, and $\Sigma_{k=0}^{d-1}\binom{k}{\lfloor k/2 \rfloor} = (1 + o(1))\binom{d}{\lfloor d/2 \rfloor} = (1 + o(1))\frac{\sqrt{2}}{\sqrt{\pi d}} \cdot 2^d$, we have

$$\begin{aligned}
B'(K^d_n) &\le \left[d\left(\frac{n}{2}\right)^{d-1}\binom{n/2}{2}\right] \cdot \left[1 + \Sigma_{k=0}^{d-1}\binom{k}{\lfloor k/2 \rfloor}\right] + \left(\frac{n}{2}\right)^{d+1} \cdot \left[\left(\frac{d}{2} + o(d)\right)\binom{d}{\lfloor d/2 \rfloor}\right] \\
&= (1 + o(1))\binom{d}{\lfloor d/2 \rfloor}\left[d\left(\frac{n}{2}\right)^{d-1} \cdot \binom{n/2}{2}\right] + (1 + o(1)) \cdot \frac{d}{2} \cdot \binom{d}{\lfloor d/2 \rfloor} \cdot \left(\frac{n}{2}\right)^{d+1} \\
&= (1 + o(1))\binom{d}{\lfloor d/2 \rfloor} \cdot d \cdot \left(\frac{n}{2}\right)^{d-1} \cdot \left(\binom{n/2}{2} + \frac{1}{2}\left(\frac{n}{2}\right)^2\right) \\
&= (1 + o(1))\binom{d}{\lfloor d/2 \rfloor} \cdot \left(\frac{1}{2}\right)^{d+1} d\, n^d\, (n-1) \\
&= (1 + o(1))\frac{\sqrt{d}}{\sqrt{2\pi}} n^d (n-1)
\end{aligned}$$

■

On a historic note, the idea of blowing up each vertex of Q_d to an "orthant" subgraph $O(x)$ of K^d_n was used by Harper in [14] in his constructive upper bound for $B(K^d_n)$. It was independently discovered by one of us while supervising an M.A. thesis [4]. To develop a constructive upper bound on $B'(K^d_n)$, it was natural to use this idea by labeling edges internal to orthants in blocks in the order of the vertex-Hales numbering of Q_d which achieved $B(Q_d)$ [12]. It would also be natural to label cross edges between neighboring orthants in blocks using the edge-Hales labeling of Q_d achieving $B'(Q_d)$ asymptotically, described earlier and originating in [5]. The difficulty lies in merging these two edge labelings. The key idea here is to notice that the edge-Hales labeling of Q_d is in some natural sense induced by the vertex-Hales numbering. This allowed us to optimally merge these two labelings using our mixed Hales labeling m of Q_d.

6 Edge-bandwidth of the Hamming graph II: lower bound

In this section we establish a lower bound for $B'(K_n^d)$ which matches the upper bound of the previous section asymptotically when n is even. Our technique employs a theorem of Harper giving a solution to the isoperimetric problem in $[0,1]^d$.

The approach is to look at K_n^d geometrically as a d-dimensional box having side length 1 and so containing n^d many d-dimensional cells of length $1/n^d$ in each dimension. More specifically, we consider the following mapping from $V(K_n^d)$ to $[0,1]^d$. For each $i \in \{0,1,\ldots,n-1\}$, let I_i denote the interval $[\frac{i}{n}, \frac{i+1}{n}]$ of real numbers. For each vertex $x = \langle x_1, x_2, \cdots, x_d\rangle$ in K_n^d, where each $x_i \in \{0,1,\ldots,n-1\}$, let $g(x) = I_{x_1} \times I_{x_2} \times \cdots I_{x_d}$. Thus $g(x)$ is a d-dimensional cell of length $1/n^d$ in each dimension. Under this mapping, a subset W of k vertices in $V(K_n^d)$ then corresponds to a collection S of k of these cells. S is a set of measure k/n^d in $[0,1]^d$, the d-fold product of the unit interval, equipped with the Lebesque measure.

Given two points $x = \langle x_1, x_2, \ldots, x_d\rangle$ and $y = \langle y_1, y_2, \ldots, y_d\rangle$ in $[0,1]^d$, we say that x and y are *neighbors* if there exists $k \in \{1,2,\ldots,d\}$ such that $x_k \neq y_k$ but $x_i = y_i$ for all $i \neq k$. We write $x \leftrightarrow y$ if x and y are neighbors. Given a measurable subset S of $[0,1]^d$, we define the shadow $\Phi(S)$ to be

$$\Phi(S) = \{y \in [0,1]^d - S : \exists x \in S \ x \leftrightarrow y\}.$$

We define iterated shadows of S recursively as following. Let $\Phi^{(1)}(S) = \Phi(S)$. For each $l > 1$, suppose we have defined $\Phi^{(l)}(S)$, we define $\Phi^{(l+1)}(S) = \Phi(\Phi^{(l)}(S))$. Let $\Phi^{(\leq l)}(S) = \Phi^{(1)}(S) \cup \Phi^{(2)}(S) \cup \cdots \cup \Phi^{(l)}(S)$.

For a measurable subset S of $[0,1]$, let $|S|$ denote its measure. The following proposition is clear from our definitions and discussions above.

Proposition 6.1 *Let g be the mapping from $V(K_n^d)$ to $[0,1]^d$ defined above. Let W be a subset of $V(K_n^d)$ and l a positive integer with $1 \leq l \leq d$. We have $g(\partial^{(\leq l)}(W)) = \Phi^{(\leq l)}(g(W))$. Hence, $|\partial^{(\leq l)}(W)| = |\Phi^{(\leq l)}(g(W)| \cdot n^d$.*

To establish a lower bound on $B'(K_n^d)$ we will need to establish a lower bound on $\partial(W)$ for any subset W of $V(K_n^d)$ of a given size k. By Proposition 6.1, it suffices to consider the problem of minimizing $|\Phi(S)|$ over all measurable subsets S of $[0,1]^d$ of a given measure. In [13], Harper showed that the problem of minimizing $|\Phi(S)|$ over all measurable subsets S of $[0,1]^d$ of a given measure reduces to that of minimizing $|\Phi(S)|$ over all suitably "compressed" such subsets. He then showed using variational methods that for any given v, $0 \leq v \leq 1$, the smallest value of $|\Phi(S)|$ over all such subsets of measure v is achieved by a "Hamming ball", which we define as follows.

Definition 6.2 *Let t be real number with $0 \leq t \leq 1$. For any binary d-tuple x of Q_d, let $K(x,t)$ be the subset of $[0,1]^d$ defined by*

$$K(x,t) = \{y \in [0,1]^d : \ 0 \leq y_i \leq t \text{ if } x_i = 1, \text{ and } t < y_i \leq 1 \text{ if } x_i = 0, 1 \leq i \leq d \ \}.$$

Define the subset $HB(d,r,t)$ of $[0,1]^d$ called a Hamming ball by

$$HB(d,r,t) = \bigcup_{wt(x) \leq r} K(x,t).$$

Note that if x has weight k, then the measure of $K(x,t)$ is $t^k(1-t)^{d-k}$. So the measure of $HB(d,r,t)$ is $v = v(d,r,t) = \Sigma_{k=0}^{r}\binom{d}{k}t^k(1-t)^{d-k}$ and the size of its shadow is $|\Phi(HB(d,r,t))| = \binom{d}{r+1}t^{r+1}(1-t)^{d-r-1}$. We can now state Harper's theorem.

Theorem 6.3 ([13] Theorem 1) *For any given v with $0 \le v \le 1$, the minimum of $|\Phi(S)|$ over all subsets S of $[0,1]^d$ of measure v is achieved by some $HB(d,r,t)$ for suitable r and t.*

To make effective use of Theorem 6.3, we need to analyze $|\Phi(d,r,t)|$, motivating the definition which follows.

Definition 6.4 *Let d, k be positive integers such that $k \le d$. Let t be a real number such that $0 \le t \le 1$. Let $\alpha(d,k,t) = \binom{d}{k}t^k(1-t)^k$.*

Note that for positive integers $r \le d-1$, $|\Phi(d,r,t)| = \alpha(d,r+1,t)$. Note also that if X is a random variable drawn from the binomial distribution $BIN(d,t)$, then $v(d,k,t) = |HB(d,k,t)| = Pr(X \le k)$ and $\alpha(d,k,t) = Pr(X = k)$.

Our general approach is again the one used in [8], [5], and the proof of Theorem 4.7. Given an optimal edge labeling of $E(K_n^d)$, we consider the set $S' \subseteq E(K_n^d)$ of size about $|E(K_n^d)|/2$ receiving the smallest labels. We then lower bound $|\partial(S')|$ or $|\partial^{(\le q)}(S)|$ for an adequate q. By an argument similar to the one used the proof of Theorem 4.7, lower bounding $|\partial(S')|$ is reduced to lower bounding $|\partial(S)|$ for a corresponding set $S \subseteq V(K_n^d)$ with $|S|$ near $\frac{1}{2}n^d$. This then reduces to lower bounding $|\Phi(S)|$ where $S \subseteq [0,1]^d$, with S having measure near $\frac{1}{2}$. By Theorem 6.3, we need to estimate $|\Phi(HB(d,r,t))| = \alpha(d,r+1,t)$ when $v(d,r,t)$ is near $\frac{1}{2}$. In light of Lemma 7.1 [20] given in the Appendix, this means estimating $\alpha(d,k,t)$ when k is near dt, the expected value of the associated binomial random variable.

We would like to stress that here we will use a self-contained and completely combinatorial approach in establishing our estimates, which we think is of independent interest. We need some technical lemmas. In some places, we need a couple of lemmas from [20]. Those lemmas are listed in the Appendix.

Our first lemma is useful in later estimates, while the second lower bounds $\alpha(d,k,t)$ when k is exactly dt.

Lemma 6.5 *If z be a real number with $0 < z < \frac{1}{2}$, then $1 - z \ge e^{-2z}$. Let x, y, a be positive real numbers such that $x, y \ge 2a$. Then*

$$(1+\frac{a}{x})^x(1-\frac{a}{y})^y \ge e^{-2a} \quad \textit{and} \quad (1+\frac{a}{x})^x(1-\frac{a}{y})^y \ge e^{-\frac{a^2}{x}-\frac{a^2}{y}}.$$

Proof. For any real number w with $0 < w \le \frac{1}{2}$, we have

$$\begin{aligned}
\ln(1+w) &= w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots \ge w - \frac{w^2}{2} > w - w^2 \\
\ln(1-w) &= -w - \frac{w^2}{2} - \frac{w^3}{3} - \frac{w^4}{4} - \cdots \ge -w - w^2.
\end{aligned}$$

If $0 < z < \frac{1}{2}$ then $\ln(1-z) \geq -z - z^2 \geq -2z$. Hence $1 - z \geq e^{-2z}$. Since $x, y \geq 2a$, $0 < \frac{a}{x}, \frac{a}{y} \leq \frac{1}{2}$. We have $(1+\frac{a}{x})^x(1-\frac{a}{y})^y \geq 1 \cdot (1-\frac{a}{y})^y \geq e^{-2(\frac{a}{y})\cdot y} = e^{-2a}$. To get a refined bound, we let $K = (1+\frac{a}{x})^x(1-\frac{a}{y})^y$. We have

$$\ln K = x\ln(1+\frac{a}{x}) + y\ln(1-\frac{a}{y}) \geq x(\frac{a}{x} - \frac{a^2}{x^2}) + y(-\frac{a}{y} - \frac{a^2}{y^2}) = -\frac{a^2}{x} - \frac{a^2}{y}$$

Hence, $K \geq e^{-\frac{a^2}{x} - \frac{a^2}{y}}$. ■

Lemma 6.6 *For each pair of positive integers k, d with $k \leq d$, we have*

$$\alpha(d,k,\frac{k}{d}) \geq \frac{1}{2\sqrt{\pi k}}(1-\frac{1}{8k}), \ \textit{and} \quad \alpha(d,k,\frac{k}{d}) \geq \frac{\sqrt{2}}{\sqrt{\pi d}}e^{-\frac{2}{d}}, \ \textit{as } d \to \infty.$$

Proof. Recall that $\alpha(d,k,\frac{k}{d}) = \binom{d}{k}\left(\frac{k}{d}\right)^k\left(1-\frac{k}{d}\right)^{d-k}$. By Lemma 7.2, $\alpha(d,k,\frac{k}{d}) \geq \binom{2k}{k}/2^{2k+1}$. Using the standard estimate $\binom{2k}{k} \geq \frac{2^{2k}}{\sqrt{\pi k}}(1-\frac{1}{8k})$ (see [22]), we have $\alpha(d,k,\frac{k}{d}) \geq \frac{1}{2\sqrt{\pi k}}(1-\frac{1}{8k})$.

To prove the second inequality, first observe that $\alpha(d,k,\frac{k}{d}) = \alpha(d,d-k,\frac{d-k}{d})$. Hence, we may assume that $1 \leq k \leq d/2$. Assume first that $k \leq \frac{d}{16}$. Then

$$\alpha(d,k,\frac{k}{d}) \geq \frac{1}{2\sqrt{\pi k}}(1-\frac{1}{8k}) \geq \frac{\sqrt{2}}{\sqrt{\pi d}},$$

for sufficiently large d. Hence, we may assume that $\frac{d}{16} \leq k \leq \frac{d}{2}$. We use stirling's formula that for all integers $n \geq 1$,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}}, \text{ where } \theta = \theta(n) \text{ satisfies } 0 < \theta < 1. \ . \tag{9}$$

We have (noting that $-\frac{1}{12k} - \frac{1}{12(d-k)} \geq -\frac{2}{d}$, since $\frac{d}{16} \leq k \leq \frac{d}{2}$)

$$\begin{aligned}
\alpha(d,k,\frac{k}{d}) &= \binom{d}{k}\left(\frac{k}{d}\right)^k\left(1-\frac{k}{d}\right)^{d-k} = \frac{d!}{k!(d-k)!}\cdot\frac{k^k}{d^k}\cdot\frac{(d-k)^{d-k}}{d^{d-k}} = \frac{d!}{d^d}\cdot\frac{k^k}{k!}\cdot\frac{(d-k)^{d-k}}{(d-k)!} \\
&= \frac{\sqrt{2\pi d}}{e^d}\cdot\frac{e^k}{\sqrt{2\pi k}}\cdot\frac{e^{d-k}}{\sqrt{2\pi(d-k)}}\cdot e^{\frac{\theta_1}{12d}-\frac{\theta_2}{12k}-\frac{\theta_2}{12(d-k)}} \qquad (\text{for some } \theta_1,\theta_2,\theta_3 \in (0,1)) \\
&= \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{d}{k(d-k)}}\cdot e^{\frac{\theta_1}{12d}-\frac{\theta_2}{12k}-\frac{\theta_2}{12(d-k)}} \\
&\geq \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{d}{d^2/4}}\cdot e^{-\frac{1}{12k}-\frac{1}{12(d-k)}} \\
&\geq \frac{\sqrt{2}}{\sqrt{\pi d}}\cdot e^{-\frac{2}{d}},
\end{aligned}$$

for sufficiently large d. $\blacksquare$

Throughout the rest of the section, we will be giving estimates valid for a sufficiently large integer dimension d. This condition will be stated as a hypothesis in the lemmas and theorems which follow, but will not be repeated each time it is applied in their proofs. The reader is asked to assume the condition as necessary. In our next lemma, we lower bound $\alpha(d,k,t)$ when k is reasonably near dt.

Lemma 6.7 *Let d be a sufficiently large positive integer. Let t be a real number such that $0 < t < 1$. Let k be a positive integer such that $2 \le k \le d-2$.*

$$\text{If } |k - dt| \le \min\{\sqrt{\ln d}, \frac{k}{2}, \frac{d-k}{2}\}, \text{ then } \alpha(d,k,t) \ge \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Proof. Let $a = dt - k$. Then $t = \frac{k+a}{d}$ and $|a| \le \min\{\sqrt{\ln d}, \frac{k}{2}, \frac{d-k}{2}\}$. Let $A = \alpha(d,k,t)$ and $B = \alpha(d,k,\frac{k}{d})$. We have

$$\begin{aligned}
A &= \alpha(d,k,\frac{k+a}{d}) = \binom{d}{k}\left(\frac{k+a}{d}\right)^k \left(1 - \frac{k+a}{d}\right)^{d-k} \\
B &= \alpha(d,k,\frac{k}{d}) = \binom{d}{k}\left(\frac{k}{d}\right)^k \left(1 - \frac{k}{d}\right)^{d-k} \\
\frac{A}{B} &= \left(\frac{k+a}{k}\right)^k \left(\frac{d-k-a}{d-k}\right)^{d-k} = \left(1 + \frac{a}{k}\right)^k \left(1 - \frac{a}{d-k}\right)^{d-k} \qquad (10)
\end{aligned}$$

By our assumption $k \ge 2|a|$ and $d - k \ge 2|a|$. By Lemma 6.5, regardless of whether a is positive or negative, we have

$$\left(1 + \frac{a}{k}\right)^k \left(1 - \frac{a}{d-k}\right)^{d-k} \ge \max\{\, e^{-2|a|},\, e^{-\frac{a^2}{k} - \frac{a^2}{d-k}}\,\}. \qquad (11)$$

We consider three cases.

Case 1. $k \le d^{\frac{3}{4}}$.

We have

$$\frac{A}{B} \ge e^{-2|a|} \ge e^{-2\sqrt{\ln d}} \ge e^{-\frac{1}{16}\ln d} = d^{-\frac{1}{16}},$$

for d large enough. On the other hand, by Lemma 6.6,

$$B = \alpha(d,k,\frac{k}{d}) \ge \frac{1}{2\sqrt{\pi k}}(1 - \frac{1}{8k}) \ge \frac{1}{4\sqrt{\pi d^{\frac{3}{4}}}}.$$

Hence,

$$\alpha(d,k,t) = A \ge d^{-\frac{1}{16}} \cdot \frac{1}{4\sqrt{\pi d^{\frac{3}{4}}}} \ge \frac{\sqrt{2}}{\sqrt{\pi d}}.$$

Case 2. $d^{\frac{3}{4}} \le k \le d - d^{\frac{3}{4}}$.

Note first that $-\frac{1}{k} - \frac{1}{d-k} \ge -d^{-\frac{3}{4}} - d^{-\frac{3}{4}} = -2d^{-\frac{3}{4}}$. We have

$$\frac{A}{B} \ge e^{a^2(-\frac{1}{k}-\frac{1}{d-k})} \ge e^{-a^2 \cdot 2d^{-\frac{3}{4}}} \ge e^{-2(\ln d)d^{-\frac{3}{4}}} \ge e^{-\frac{1}{2\sqrt{d}}}.$$

On the other hand, by Lemma 6.6,

$$B = \alpha(d, k, \frac{k}{d}) \ge \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{2}{d}}.$$

Hence,

$$\alpha(d, k, t) = A \ge e^{-\frac{1}{2\sqrt{d}}} \cdot \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{2}{d}} \ge \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Case 3. $k \ge d - d^{\frac{3}{4}}$.

Let $m = d - k$. Then $2 \le m \le d^{\frac{3}{4}}$. Observe that $\alpha(d, m, \frac{m}{d}) = \alpha(d, k, \frac{k}{d})$. Hence, the lower bounds on $\frac{A}{B}$ and B from Case 1 both apply, and we conclude as in Case 1.

■

We are now ready for our lower bound on $|\Phi(S)|$, for subsets S of $[0,1]^d$ having measure near $\frac{1}{2}$.

Theorem 6.8 *Let d be a sufficiently large positive integer. Let S be a subset of $[0,1]^d$ with measure $v = |S|$.*

$$\text{If } ||S| - \frac{1}{2}| \le \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}, \text{ then } |\Phi(S)| \ge \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Proof. By Theorem 6.3, $|\Phi(S)|$ is minimum when S is a Hamming ball. Hence, we may assume $S = HB(d, k, t)$ for some k and t, where k is an integer with $1 \le k \le d-1$ and t is a real number with $0 < t < 1$. We have $|S| = v(d, k, t) = \Sigma_{i=0}^{k} \alpha(d, i, t)$ and $|\Phi(S)| = \alpha(d, k+1, t)$. By our assumption,

$$|v(d, k, t) - \frac{1}{2}| \le \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}. \tag{12}$$

By the remarks after Definition 6.4, if X is a random variable drawn from the binomial distribution $BIN(d, t)$, then $v(d, k, t) = Pr(X \le k)$. By Lemma 7.1,

$$v(d, \lfloor dt \rfloor - 1, t) \le \frac{1}{2} \le v(d, \lceil dt \rceil, t) \tag{13}$$

We consider cases.

Case 1. $10\sqrt{\ln d} \le k \le d - 10\sqrt{\ln d}$.

Claim 1. For all positive integers m with $|m - dt| \le \sqrt{\ln d}$, $\alpha(d, m, t) \ge \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}$.

Proof of Claim 1. Recall that if X is a random variable drawn from the binomial distribution $BIN(d, t)$ then $Pr(X \le k) = v(d, k, t)$ and hence $Pr(X \ge k+1) = 1 - v(d, k, t)$. By

(12), $Pr(X \geq k+1) = \frac{1}{2} + O(\frac{\sqrt{\ln d}}{\sqrt{d}})$. Note that $E(X) = dt$. If $k+1 \geq 3dt = 3E(X)$, then by Markov's inequality, $Pr(X \geq k+1) < \frac{1}{3}$, a contradiction for large d. Hence, $3dt > k+1$. Thus $dt > \frac{k}{3} \geq \frac{10}{3}\sqrt{\ln d}$. Since $|m - dt| \leq \sqrt{\ln d}$, we have $m \geq \frac{7}{3}\sqrt{\ln d}$ for large d. Thus, $\frac{m}{2} \geq \frac{7}{6}\sqrt{\ln d} > \sqrt{\ln d}$.

Let $Y = d - X$. Then $E(Y) = d - dt$ and $Pr(Y \geq d-k) = Pr(X \leq k) = v(d,k,t)$. By (12), $Pr(Y \geq d-k) = \frac{1}{2} + O(\frac{\sqrt{\ln d}}{\sqrt{d}})$. If $d - k > 3(d-dt) = 3E(Y)$, then by Markov's inequality, $Pr(Y \geq d-k) \leq \frac{1}{3}$, a contradiction for large d. Hence $3(d-dt) \geq d-k$. Therefore, $d - dt \geq \frac{d-k}{3} \geq \frac{10}{3}\sqrt{\ln d}$. Since $|m-dt| \leq \sqrt{\ln d}$, $d - m \geq \frac{7}{3}\sqrt{\ln d}$ for large d. Thus, $\frac{d-m}{2} \geq \frac{7}{6}\sqrt{\ln d} > \sqrt{\ln d}$.

Since $|m - dt| \leq \sqrt{\ln d}$ and both $\frac{m}{2}$ and $\frac{d-m}{2}$ exceed $\sqrt{\ln d}$, $|m-dt| \leq \min\{\sqrt{\ln d}, \frac{m}{2}, \frac{d-m}{2}\}$. Hence, by Lemma 6.7,

$$\alpha(d,m,t) \geq \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

■

Claim 2. $|k - dt| \leq \sqrt{\ln d} - 3$.

By Claim 1, (12), and (13), for large enough d, we have

$$\begin{aligned} v(d, \lceil dt \rceil + (\sqrt{\ln d} - 4), t) &\geq \frac{1}{2} + (\sqrt{\ln d} - 4) \cdot \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}} \\ &= \frac{1}{2} + (1 - \frac{4}{\sqrt{\ln d}}) \cdot \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}} \geq \frac{1}{2} + e^{-\frac{8}{\sqrt{\ln d}}} \cdot \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}} \\ &> \frac{1}{2} + \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}} \geq v(d,k,t). \end{aligned}$$

Since $v(d,k,t)$ is non-decreasing in k, we have $k \leq \lceil dt \rceil + \sqrt{\ln d} - 4$. Similarly, by Claim 1, (12), and (13), for large enough d, we have

$$v(d, \lfloor dt \rfloor - 1 - (\sqrt{\ln d} - 5), t) \leq \frac{1}{2} - (\sqrt{\ln d} - 5)\frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}} \leq \frac{1}{2} - \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}} \leq v(d,k,t).$$

Hence, by monotonicity, $\lfloor dt \rfloor - \sqrt{\ln d} + 4 \leq k$. Thus, we have $|k - dt| \leq \sqrt{\ln d} - 3$. ■

Now, clearly $|k + 1 - dt| \leq \sqrt{\ln d}$. By Claim 1, we have

$$|\Phi(S)| = \alpha(d, k+1, t) \geq \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Case 2. $k < 10\sqrt{\ln d}$

Claim 3. We have $\frac{1}{4d} \leq t \leq \frac{20\sqrt{\ln d}}{d}$.

Proof of Claim 3. Suppose for contradiction that $t < \frac{1}{4d}$. Then

$$v(d,k,t) \geq v(d,0,t) = (1-t)^d \geq (1 - \frac{1}{4d})^d \geq e^{-2 \cdot \frac{1}{4d} \cdot d} \geq e^{-\frac{1}{2}} > 0.6,$$

contradicting (12) for large d.

Next, suppose for contradiction that $t > \frac{20\sqrt{\ln d}}{d}$. Then $dt \geq 20\sqrt{\ln d} \geq 2k$. Recall that X is a random variable drawn from $BIN(d,t)$, $E(X) = dt$, and $Pr(X \leq k) = v(d,k,t)$. Since $k \leq \frac{dt}{2} = \frac{E(X)}{2}$, we have by Chernoff's inequality that

$$v(d,k,t) = Pr(X \leq k) \leq Pr(|X - E(X)| \geq \frac{1}{2}E(X)|) \leq 2e^{-\frac{1}{12}E(X)} \leq 2e^{-\frac{1}{12}(20\sqrt{\ln d})} \to 0,$$

as $d \to \infty$, contradicting (12) for sufficiently large d. ∎

Let $m = k+1$. Then $m \leq 10\sqrt{\ln d}$. We have

$$\begin{aligned}
|\Phi(S)| &= \alpha(d,k+1,t) = \alpha(d,m,t) = \binom{d}{m} t^m (1-t)^{d-m} \\
&\geq \left(\frac{d}{m}\right)^m \cdot t^m \cdot (1-t)^d \\
&\geq \left(\frac{dt}{m}\right)^m \cdot e^{-2td} \\
&\geq \left(\frac{1}{4m}\right)^m \cdot e^{-40\sqrt{\ln d}} \quad (\text{ since } \tfrac{1}{4} \leq dt \leq 20\sqrt{\ln d})
\end{aligned}$$

Let K denote the righthand side of the last inequality. Since $m \leq 10\sqrt{\ln d}$, we have

$$\ln K = -m \ln 4m - 40\sqrt{\ln d} \gg -\frac{1}{4}\ln d,$$

for large d. Hence

$$|\Phi(S)| \gg e^{-\frac{1}{4}\ln d} = d^{-\frac{1}{4}} \gg \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Case 3. $k > d - 10\sqrt{\ln d}$

Claim 4. We have $\frac{1}{4d} \leq 1 - t \leq \frac{20\sqrt{\ln d}}{d}$

Proof of Claim 4. Suppose for contradiction that $1 - t < \frac{1}{4d}$. Then

$$1 - v(d,k,t) \geq \alpha(d,d,t) = t^d \geq (1 - \frac{1}{4d})^d \geq e^{-2\cdot\frac{1}{4d}\cdot d} \geq e^{-\frac{1}{2}} > 0.6,$$

contradicting (12) for large d.

Next, suppose for contradiction that $1 - t > \frac{20\sqrt{\ln d}}{d}$. Then $d - dt \geq 20\sqrt{\ln d} \geq 2(d-k)$. Recall that X is a random variable drawn from $BIN(d,t)$, $E(X) = dt$, and $Pr(X \leq k) = v(d,k,t)$. Let $Y = d - X$. Then $E(Y) = d - dt$ and $Pr(Y \geq d-k) = 1 - v(d,k,t)$.

Since $d - k \leq \frac{d-dt}{2} = \frac{E(Y)}{2}$, we have by Chernoff's inequality that

$$1 - v(d,k,t) = Pr(Y \geq d-k) \leq Pr(|Y - E(Y)| \geq \frac{1}{2}E(Y)|) \leq 2e^{-\frac{1}{12}E(Y)} \leq 2e^{-\frac{1}{12}(20\sqrt{\ln d})} \to 0,$$

as $d \to \infty$, contradicting (12) for sufficiently large d. ■

Let $m = k + 1$. Then $d - m \leq 10\sqrt{\ln d}$. The lower bound analysis in Case 2 applies here with $d - m$ playing the role of m and $1 - t$ playing the role of t in (14). The rest of the arguments are the same. This completes the proof of Case 3 and the theorem. ■

Corollary 6.9 *Let d be a sufficiently large positive integer. Let S be a subset of $[0,1]^d$ with measure $v = |S|$. Suppose*

$$\frac{1}{2} - \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{\ln d}}} \leq |S| \leq \frac{1}{2}.$$

Then either

$$|\Phi(S)| \geq \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$$

or there exists an integer $l \geq 2$ such that

$$\frac{|\Phi^{(\leq l)}(S) - \Phi(S)|}{l+1} \geq \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

Proof. For each i, let $S_i = \Phi^{(i)}(S)$. In particular, $S_1 = \Phi(S)$. Let l denote the smallest m such that $|S_1 \cup S_2 \cup \cdots \cup S_m| \geq \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$.

Claim 1. $l \leq \sqrt{\ln d} - 4$.

Proof of Claim 1. By our choice of l, for each $i \leq l$ we have

$$|S_1 \cup \cdots \cup S_{i-1}| \leq \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

Let $S' = S \cup S_1 \cdots \cup S_{i-1}$. Then $\Phi(S') = S_i$. Furthermore, S' has measure

$$|S'| = |S| + |S_1 \cup \cdots \cup S_{i-1}| \leq \frac{1}{2} + \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

By Theorem 6.8, we have

$$|S_i| = |\Phi(S')| \geq \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Since this holds for all $i \leq l$, we have

$$|S_1 \cup \cdots \cup S_{l-1}| \geq (l-1)\frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Suppose for contradiction that $l \geq \sqrt{\ln d} - 3$. Then

$$|S_1 \cup \cdots \cup S_{l-1}| \geq (\sqrt{\ln d} - 4) \cdot \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}} > \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}},$$

contradicting our choice of l. ■

Now, if $|S_1| = |\Phi(S)| \geq \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$, then we are done. Otherwise, suppose $|S_1| < \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$. Then since $|S_1 \cup S_2 \cup \cdots \cup S_l| \geq \frac{\sqrt{2}\sqrt{\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$ by our choice of l, we have

$$|\Phi^{(\leq l)}(S) - \Phi(S)| = |S_1 \cup \cdots S_l| - |S_1| \geq \frac{\sqrt{2}(\sqrt{\ln d} - 2)}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

Thus since $l + 1 < \sqrt{\ln d} - 2$, we get

$$\frac{|\Phi^{(\leq l)}(S) - \Phi(S)|}{l+1} \geq \frac{\sqrt{2}}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

This completes the proof. ■

The two corollaries which follow are just the equivalents of Theorem 6.8, and Corollary 6.9 for K_n^d, under the correspondence between $[0,1]^d$ and K_n^d described in Proposition 6.1.

Corollary 6.10 *Let n, d be positive integers, where d is sufficiently large. Let S be subset of $V(K_n^d)$.*

$$\text{If } ||S| - \frac{n^d}{2}| \leq \frac{\sqrt{2\ln d} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}, \text{ then } |\Phi(S)| \geq \frac{\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{d}}}.$$

Corollary 6.11 *Let n, d be positive integers, where d is sufficiently large. Let S be a subset of $V(K_n^d)$. Suppose*

$$\left(\frac{1}{2} - \frac{\sqrt{2\ln d}}{\sqrt{\pi d}} \cdot e^{-\frac{1}{\sqrt{\ln d}}}\right) \cdot n^d \leq |S| \leq \frac{1}{2} \cdot n^d.$$

Then either

$$|\partial(S)| \geq \frac{2\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}$$

or there exists an integer $l \geq 2$ such that

$$\frac{|\partial^{(\leq l)}(S) - \partial(S)|}{l+1} \geq \frac{\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}.$$

We remark that Corollary 6.10 yields a proof of the lower bound in a recent result of Harper in [14] on the vertex bandwidth $B(K_n^d)$.

Theorem 6.12 *[14] Let n be fixed positive integer. For all positive integers d we have $B(K_n^d) \geq (1 - o(1))\frac{\sqrt{2}}{\sqrt{\pi d}} n^d$.*

Proof. Consider a labeling f of $V(K_n^d)$ that achieves the bandwidth. Let S denote the set of vertices receiving labels $1, 2, \ldots, \lfloor n^d/2 \rfloor$. By Corollary 6.10, $|\partial(S)| \geq \frac{\sqrt{2}}{\sqrt{\pi d}}(1 - o(1))n^d$. Thus, $B(K_n^d) = B(f) \geq |\partial(S)| \geq (1 - o(1))\frac{\sqrt{2}}{\sqrt{\pi d}} n^d$. ■

Theorem 6.12 is asymptotically best possible as $d \to \infty$ as Harper gave an upper bound in [14] that asymptotically matches the lower bound in Theorem 6.12 as $d \to \infty$.

We now establish a lower bound on $B'(K_n^d)$ that matches the upper bound from the previous section asymptotically when n is even . We follow the approach used in [8] and [5].

Theorem 6.13 *Let n be a fixed positive even integer. We have*

$$B'(K_n^d) \geq (1-o(1))\frac{\sqrt{d}}{\sqrt{2\pi}} n^d(n-1), \text{ as } d \to \infty.$$

Proof. Let f be an optimal labeling of $E(K_n^d)$ using labels $1, \dots, \binom{n}{2} dn^{d-1}$. Let S denote the set of edges receiving the first half of the labels. That is, S is the set of the edges receiving labels $1, 2, \dots, \frac{1}{2}\binom{n}{2} dn^{d-1}$. Let us call the edges in S red and the rest of the edges white.

For a vertex $x \in V(K_n^d)$, let $E(x)$ denote the set of edges incident to x. A vertex x is called *red* if all the edges in $E(x)$ is red, and *white* if all the edges in $E(x)$ are white; otherwise it is called *mixed*. Let R, W, and M denote the set of red, white, and mixed vertices, respectively.

We have

$$|R| + |W| + |M| = n^d. \tag{14}$$

For $x \in M$, let $r(x)$ denote the number of red edges in $E(x)$. Hence $1 \leq r(x) \leq d(n-1)$. By (double) counting the red edges, we have

$$|R| \cdot d(n-1) + \Sigma_{x \in M} r(x) = \binom{n}{2} dn^{d-1} = |W| \cdot d(n-1) + \Sigma_{x \in M}(n - r(x)). \tag{15}$$

It readily follows from (15) that $|R|, |W| < \frac{1}{2}n^d$. Note that the white edges incident to M belong to $\partial(S)$. Similarly, the red edges incident to M belong to $\partial(E(K_n^d) - S)$. Therefore, we have

$$|\partial(S)| \geq \frac{1}{2}\Sigma_{x \in M}(d(n-1) - r(x)) \text{ and } |\partial(E(K_n^d) - S)| \geq \frac{1}{2}\Sigma_{x \in M} r(x) \tag{16}$$

Combining these two inequalities we obtain

$$B'(K_n^d) \geq \max\{\partial(S), \partial(E(K_n^d) - S)\} \geq \frac{|M| \cdot (n-1)d}{4}. \tag{17}$$

If $|M| \geq \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot n^d \cdot e^{-\frac{10}{\sqrt{\ln d}}}$, then by (17), we have

$$B'(K_n^d) \geq \frac{|M| \cdot (n-1)d}{4} \geq \frac{\sqrt{d}}{\sqrt{2\pi}} n^d(n-1) \cdot e^{-\frac{10}{\sqrt{\ln d}}} = (1-o(1))\frac{\sqrt{d}}{\sqrt{2\pi}} n^d(n-1),$$

and we are done. Hence, we may assume that

$$|M| < \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot n^d \cdot e^{-\frac{10}{\sqrt{\ln d}}}. \tag{18}$$

Either

$$\Sigma_{x \in M} r(x) \leq |M| \cdot d(n-1)/2 \text{ or } \Sigma_{x \in M}(d(n-1) - r(x)) \leq |M| \cdot d(n-1)/2.$$

Without loss of generality, let us assume that the first inequality holds, since otherwise we could switch the roles of red and white vertices. Combining this with (15) and (18), we have

$$\binom{n}{2} dn^{d-1} \leq |R| \cdot d(n-1) + |M| \cdot d(n-1)/2 \leq |R| \cdot d(n-1) + \frac{\sqrt{2}}{\sqrt{\pi d}} n^d \cdot d(n-1). \tag{19}$$

This yields the lower bound in

$$\left(\frac{1}{2} - \frac{\sqrt{2}}{\sqrt{\pi d}}\right) n^d \leq |R| < \frac{1}{2} n^d.$$

By Corollary 6.11, we have either

$$|\partial(R)| \geq \frac{2\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}, \tag{20}$$

or there exists an integer $l \geq 2$ such that

$$\frac{|\partial^{(\leq l)}(R) - \partial(R)|}{l+1} \geq \frac{\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}}. \tag{21}$$

Now, clearly $\partial(R) \subseteq M$. So if (20) holds then we have

$$|M| \geq \frac{2\sqrt{2}}{\sqrt{\pi d}} \cdot n^d \cdot e^{-\frac{10}{\sqrt{\ln d}}},$$

contradicting (18).

Hence, we may assume that (21) holds instead. Let $C = \partial^{(\leq l)}(R) - \partial(R)$. Let $E(C)$ denote the set of edges incident to C.

Note that each vertex in C is a white vertex at distance at most l from R. Hence each edge in $E(C)$ is a white edge at distance at most $l+1$ from S (the set of red edges). Thus, $E(C) \subseteq \partial^{(\leq l+1)}(S)$. Note that $|E(C)| \geq |C| \cdot d(n-1)/2$. Applying 2.2 to the line graph $L(K_n^d)$, we have

$$\begin{aligned} B'(K_n^d) \geq B'(f) &\geq \frac{|\partial^{(\leq l+1)}(S)|}{(l+1)} \geq \frac{|E(C)|}{(l+1)} \geq \frac{|C| \cdot d(n-1)}{2(l+1)} \\ &= \frac{|\partial^{(\leq l)}(R) - \partial(R)| \cdot d(n-1)}{2(l+1)} \\ &\geq \frac{\sqrt{2} \cdot n^d}{\sqrt{\pi d}} \cdot e^{-\frac{10}{\sqrt{\ln d}}} \cdot \frac{d(n-1)}{2} \qquad \text{(by (21))} \\ &= (1 - o(1)) \frac{\sqrt{d}}{\sqrt{2\pi}} n^d (n-1). \end{aligned}$$

■

Now, Theorem 5.10 and Theorem 6.13 together imply Theorem 1.2.

7 Appendix

Lemma 7.1 *([20] Lemma B.7.) Let n be a positive integer and p a real number such that $0 \leq p \leq 1$. Let X be a random variable drawn from the binomial distribution $B(n, p)$ (where n is the number of independent trials and p is the probability of success of each trial). Then*

$$Pr(X \leq \lfloor np \rfloor - 1) \leq 1/2 \leq Pr(X \leq \lceil np \rceil).$$

■

Lemma 7.2 *([20] Lemma B.3 and Lemma B.5) For each pair of integers k and n with $0 \leq k \leq n$,*

$$\binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k} \geq \frac{k^k}{e^k k!} \geq \binom{2k}{k}2^{-2k-1}.$$

■

8 Acknowledgements

The authors thank Dan Pritikin for helpful discussions and Doron Zeilberger and Sergei Bezrukov for helpful communications and references.

References

[1] I. Anderson, *On primitive sequences*, J. Lond. math. soc. **42**, 137-148, 1967.

[2] I. Anderson, *A variance method in combinatorial number theory*, Glasgow math. J. **10**, 126-129, 1969.

[3] I. Anderson, *Combinatorics of Finite Sets*, Oxford University Press, New York, 1987.

[4] E. Appelt, *On the bandwidth of a product of complete graphs*, M.A. thesis, Miami University, 200

[5] J. Balogh, D. Mubayi, A. Pluhár, *On the edge-bandwidth of graph products*, Theoret. Comput. Sci., to appear.

[6] T. Berger-Wolf, E.M. Reingold, *Index assignment for multichannel communication under failure*, IEEE Trans. Inform. Theory **48** (2002) 2656-2668.

[7] B. Bollobás, I. Leader, *Compressions and Isoperimetric Inequalities*, J. Combin. Theor. Ser. A **56** (1991) 47-62.

[8] T. Calamoneri, A. Massini, I. Vrto, *New results on edge-bandwidth*, Theoret. Comput. Sci. **307** (2003) 503-513.

[9] P.Z. Chinn, J. Chvátalová, A.K. Dewdney, N.E. Gibbs, *The bandwidth problem for graphs and matrices - a survey*, J. Graph Theory **6** (1982) 223-254.

[10] F.R.K. Chung, *Labeling of graphs*, in L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory, Vol 3, Academic Press, San Diego (1988) 151-168.

[11] A. Gupta, *Personal communication.*

[12] L.H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Combin. Theory **1** (1966) 385-393.

[13] L.H. Harper, *On an isoperimetric problems for Hamming graphs*, Discrete Appl. Math. **95** (1999) 285-309.

[14] L.H. Harper, *On the bandwidth of a Hamming graph*, Theoret. Comput. Sci. **301** (2003) 491-498.

[15] U. Hendrik, M. Stiebitz, *On the bandwidth of graph poroducts*, J. Inform. Process. Cybernet. **28** (1992) 113-125.

[16] R. Hochberg, C. McDiarmid, M. Saks, *On the bandwidth of triangulated triangles*, Discrete Math. **138** (1995) 261-265.

[17] F.K. Hwang, J.C. Lagarias, *Minimum range sequences of all k-subsets of a set*, Discrete Math. **19** (1977) 257-264.

[18] T. Jiang, D. Mubayi, A. Shahstri, D.B. West, *Edge-bandwidth of graphs*, SIAM J. Discrete Math. **12** (1999) 307-316.

[19] P. Lam, W. Shiu, W. Chan, Y. Lin, *On the bandwidth of convex triangulated meshes*, Discrete Math. **173** (1997) 285-289.

[20] T. Leighton, C. G. Plaxton, *Hypercubic sorting networks*, SIAM J. Comput. **27**, No. 1, pp 1-47, 1998.

[21] B. Monien, L.H. Sudborough, *Embedding one interconnection network in another*, Comput. Suppl. **7** (1990) 257-282.

[22] A. Odlyzko, *Asymptotic enumeration methods*, Handbook of Combinatorics, Vol. 1,2, 1063–1229, Elsvier, Amsterdam, 1995.

[23] O. Pikhurko, J. Wojciechowski, *Edge-bandwidth of grids and tori*, submitted.

[24] W. Unger, *The complexity of the approximation of the bandwidth problem*, in Proc. 1998 Ann. Symp. on Fundations of Computer Science, IEEE Press, Baltimore, 1998, 82-91.