Milnor K-theory and zero-cycles on algebraic varieties

by

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Vita

Reza Akhtar was born on January 4, 1973, in Vancouver, Canada. He received an A.B. degree *magna cum laude* in Mathematics and an S.M. degree in Applied Mathematics from Harvard University in 1995. He came to Brown University in the Fall of 1995, and received an Sc.M. degree in Mathematics in 1997. While at Brown, he has been supported by various fellowships and assistantships, including postgraduate scholarships from the Natural Sciences and Engineering Research Council of Canada. He is a member of Phi Beta Kappa and Sigma Xi.

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Quid retribuam Domino pro omnibus quae retribuit mihi?
Calicem salutaris accipiam et nomen Domini invocabo
Vota mea Domino reddam coram omni populo eius

Contents

Notation and Terminology

Conventions

- 0 is considered a natural number.
- All rings (unless explicitly mentioned otherwise) are commutative with unit element.
- $\otimes$ with no subscript means tensor product over $\mathbf{Z}$
- Unless otherwise specified, dimension (of a scheme or a ring) means Krull dimension.

General Definitions

In the following, k is a field.

- An *algebraic scheme* over k is a scheme X (often written X/k) together with a morphism $X \longrightarrow \operatorname{Spec} k$ which is separated and of finite type.
- A *variety* over k is an integral algebraic scheme over k.
- Let X be any algebraic scheme over a field k. A *subvariety* of X is a closed subscheme of X which is a variety.
- A *curve* over k is a one-dimensional variety over k; a *surface* is a two-dimensional variety over k.
- A variety X/k is called *smooth* if the structure morphism $X \longrightarrow \operatorname{Spec} k$ is a smooth morphism of schemes.

- A variety over k is *defined over k* if X is geometrically irreducible; that is, if $X \times_k \bar{k}$ is irreducible for an algebraic closure $\bar{k}$ of k.

- Let X be a variety defined over k, A a k-algebra. An *A-rational point* is a k-morphism $\operatorname{Spec} A \longrightarrow X$.

Notation

When two pieces of notation are listed on one line, the second is shorthand for the first and is used if there is no danger of ambiguity.

General Algebra

$G[n]$	kernel of multiplication by n on the abelian group G
G/n	cokernel of multiplication by n on G
$G\{p\}$	for a prime p, the p-primary part of G
R^*	multiplicative group of invertible elements of a ring R
I^e_f, I^e	extension of ideal $I \subseteq R$ with respect to $R \xrightarrow{f} S$
J^c_f, J^c	contraction of ideal $J \subseteq S$ with respect to $R \xrightarrow{f} S$
$\mathfrak{N}(R)$, $\mathfrak{N}$	nilradical of a ring R

Fields

k	a field
$\bar{k}$	an algebraic closure of k
k^{sep}	a separable algebraic closure of k
$\mathcal{T}_r(k)$	the collection of finitely generated extension fields of k of transcendence degree r
$\mathcal{P}(K/k)$	when $K \in \mathcal{T}_1(k)$, the places of K over k
v	a place in $\mathcal{P}(K/k)$
K_v	completion of K with respect to v
K_v^h	henselization of K with respect to v
O_v	valuation ring of K defined by v
$\hat{O}_v$	valuation ring of K_v defined by v
m_v	maximal ideal of O_v
$k(v)$	residue field O_v/m_v of v
$\mathrm{ord}(v)$	order function associated to v
$(\cdot,\cdot)_v$	local symbol associated to v
$w \mapsto v$	w restricts to v

Schemes

$\mathcal{O}_{x,X}$	local ring of a point x on a scheme X
$k_X(x), k(x)$	residue field of a point x on a scheme X
$\mathcal{O}_{Y,X}$	local ring of (the generic point of) a subvariety Y of X
$\bar{x}$	Zariski closure of $\{x\}$ in X
X_i	points of dimension i=$\{x \in X : \dim \bar{x} = i\}$
X^i	points of codimension i=$\{x \in X : \dim(\mathcal{O}_{x,X}) = i\}$
$X \times_Y Z$	fibered product of X and Z over Y
$X \times_A B, X_B$	$(A, B$ are rings) fibered product of X and Spec B over Spec A
$X(A)$	A-rational points of X
$(\mathbf{G}_m)_k, \mathbf{G}_m$	the group scheme Spec $\frac{k[x,y]}{(xy-1)}$
$\mu_n(k)$	group of nth roots of unity in k
$\mathbf{A}^n_R$	affine n-space over R=Spec $R[x_1, \ldots, x_n]$ (x_i are indeterminates)
$\mathbf{P}^n_R$	projective n-space over R=Proj $R[x_1, \ldots, x_n]$ (x_i are indeterminates)

K-theory and Chow groups

$K^M_r(k)$	rth Milnor K-group of a field k
$K^M_r(R)$	rth Milnor K-group of a ring R
$K_r(X)$	rth Quillen K-group of a scheme X
$CH_r(X)$	Chow group of dimension r cycles on an algebraic scheme X
$CH^r(X)$	Chow group of codimension r cycles on X
$CH^r(X, \cdot)$	Higher Chow groups

Chapter 1

Introduction

The subject of Milnor K-theory has been the object of intense study since its inception in the late 1960s. Historically, the development of Milnor K-theory grew out of a search for the "algebraic K-theory" of a ring, in analogy with the topological K-theory of vector bundles developed by Atiyah and Hirzebruch in 1961. By the end of the 1970, functors K_0, K_1 and K_2 from the category of rings to the category of abelian groups had been constructed. The definitions of $K_0(R)$ and $K_1(R)$ were fairly easy to understand; however, the definition of $K_2(R)$ was much more complicated. In 1968, H. Matsumoto proved that if one restricted attention to fields, the group $K_2(F)$ had an elegant presentation in terms of generators and relations. In his 1970 paper [Mil 70], Milnor defined groups $K_n^M(F)$ for all $n \geq 0$ in imitation of Matsumoto's presentation for $K_2(F)$, as follows:

$$K_0^M(F) = \mathbf{Z}, \;\; K_1^M(F) = F^*, \;\; K_r^M(F) = \frac{\bigotimes_{i=1}^r F^*}{I_r} \text{ for } r \geq 2,$$

where $I_r \subseteq \bigotimes_{i=1}^r F^*$ is the subgroup generated by elements of the form $a_1 \otimes \ldots \otimes a_r$ such that $a_i + a_j = 1$ for some $i < j$. Milnor stated that "for $n \geq 3$, the definition is purely ad hoc". In hindsight, Milnor made a very good guess; although his definition did not turn out to be the long-sought algebraic K-theory, it independently spawned the development of a rich theory now known as Milnor K-theory.

The definition of the Milnor K-groups $K_n^M(F)$ looks simple, but in general it is quite hard to compute $K_n^M(F)$ for $n \geq 2$. Moreover, it is not at all clear, based on this definition alone, that the defining relations have any intuitive meaning. Nevertheless, Milnor K-

groups have some nice functorial properties (see [BT 73] I.4, I.5 for details):

- The graded group $\bigoplus_{n\geq 0} K_n^M(F)$ is naturally endowed with a ring structure: $\cdot : K_r^M(F) \otimes K_s^M(F) \longrightarrow K_{r+s}^M(F)$ for all $r, s \geq 0$.
- If $E \overset{\phi}{\hookrightarrow} F$ is an extension of fields, there is a naturally induced homomorphism: $K_n^M(E) \xrightarrow{\phi_*} K_n^M(F)$ for each $n \geq 0$.
- If $E \overset{\phi}{\hookrightarrow} F$ is a *finite* extension of fields, there is a homomorphism: $K_n^M(F) \xrightarrow{\phi^*} K_n^M(E)$ for each $n \geq 0$.
- Suppose F is a field and K is a finitely generated extension field of transcendence degree 1 over k equipped with a discrete valuation v satisfying $v(F) = 0$. Let $F(v)$ denote the associated residue field. Then for all $n \geq 0$, there is a "boundary" homomorphism: $K_{n+1}^M(K) \xrightarrow{\partial_v} K_n^M(F(v))$

Moreover, we have the following results relating the maps mentioned above:

- (Projection formula) Let $\phi : E \hookrightarrow F$ be a finite extension of fields. For all $x \in K_n^M(E)$ and $y \in K_m^M(F)$, $\phi^*(\phi_*(x) \cdot y) = x \cdot \phi^*(y)$.
- (Reciprocity law) If $\mathcal{P}(K/F)$ denotes the set of places of K over F, and $\phi_v : F \hookrightarrow F(v)$ the canonical inclusion, then $\sum_{v\in\mathcal{P}(K/F)} \phi_v^*(\partial_v) = 0$.

Examination of the properties listed above raises two questions:

- Can a more functorial definition of Milnor K-theory be formulated?
- Can the definition of Milnor K-theory be extended in a meaningful way to a category larger than the category of fields?

Our goal is to provide affirmative answers and solutions to both questions. Fix a base field k and consider the first question raised above. Nesterenko-Suslin ([NeSu 89], Theorem 4.9) and Totaro ([To 92], Theorem 1) have proven independently and by different methods that there exists a natural map inducing an isomorphism

$$K_s^M(k) \xrightarrow{T_k} CH^s(k, s)$$

where the group on the right is Bloch's higher Chow group. This gives some hope that Milnor K-theory might have an equivalent functorial definition as a higher Chow group; however it still remains to be proved that the isomorphism so described respects the functorial properties listed above. We will show that our solution to the second problem takes care of the first. It should be mentioned that the second question has been addressed to a limited degree by defining Milnor K-groups for rings, although the theory only works well for local rings.

Our proposed solution to the second problem is based on the following idea, in which $\mathbf{G}_m$ represents the group scheme Spec $\frac{k[x,y]}{(xy-1)}$. Kato has defined, for semi-abelian varieties $G_1, \ldots, G_s$ defined over k, a complicated group $K(k; G_1, \ldots, G_s)$ in terms of generators and relations (cf. Section 3), the relations of which mimic the statements of the projection formula and the reciprocity law stated above. Somekawa ([Som 90]) has proven that when $G_1 = \ldots = G_s = \mathbf{G}_m$, this group is canonically isomorphic to the Milnor K-group $K_s^M(k)$. Although this new group is much more difficult to define than $K_s^M(k)$, its relations give a much clearer description of the nature of Milnor K-theory than the original definition. By amalgamating Kato's definition with a similar definition due to Raskind and Spiess ([RS 97], Definition 2.1.1), we define, for any collection $X_1, \ldots, X_r$ of smooth projective varieties defined over k and any $s \geq 0$, a "mixed K-group" $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \underbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}_{s})$, which we abbreviate by writing $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \mathbf{G}_m)$. We then define the sth Milnor K-group of a smooth projective variety X to be $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$

With this definition in hand, we prove the following generalization of the Nesterenko-Suslin/Totaro theorem.

Theorem 7.3.2

Let X be a smooth projective variety of dimension d defined over k. Then for any $s \geq 0$ there exists a canonical isomorphism:

$$CH^{d+s}(X, s) \xrightarrow{\cong} K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$$

A consequence of the proof of this theorem (see Section 7.6) is that the isomorphism T_k of the Nesterenko-Suslin-Totaro result respects the functorial properties of Milnor K-theory discussed earlier.

The group $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$ may also be realized as a cohomology group of a sheaf defined on the Zariski site of X. Let $\mathcal{K}_s^M$ denote the Zariski presheaf on a scheme X associated to the sheaf $U \mapsto K_s^M(\mathcal{O}_X(U))$; the Milnor K-theory of a ring is defined similarly to that of a field, (cf. Definition 6.1.1). We show:

Theorem 6.2.1

Let X be a smooth projective variety of dimension d defined over k. Then for any $s \geq 0$, there exists a canonical isomorphism:

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \xrightarrow{\cong} K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$$

(Actually, we work with a slightly modified group on the right, but later we show that this group coincides with $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$.)

Combining the two results quoted above, we obtain:

Theorem 7.5.5

Let X be a smooth projective variety of dimension d defined over a field k. Then for any $s \geq 0$, there exists a canonical isomorphism:

$$CH^{d+s}(X, s) \xrightarrow{\cong} H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

Theorem 7.5.5 is also valid for smooth quasi-projective nX; we prove this using a different method in Chapter 10.

We see from the above that the proposed definition for the Milnor K-theory of a smooth projective variety is at least somewhat meaningful. It is not difficult to show that these groups satisfy functoriality and naturality properties analogous to those of Milnor K-theory. Moreover, analogues of deeper theorems also hold; for example, the isomorphism $CH^{d+s}(X, s) \cong K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$ above, combined with Bloch's work [Bl 86] implies that $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$ is isomorphic modulo torsion to the highest graded piece of the gamma filtration on the (Quillen) K-group $K_s(X)$ (cf. [Sou 85]).

We observe, then, that there are three ways of describing the groups of zero-cycles under investigation: the symbolic description $K_s(K; \mathcal{CH}_0(X), \mathbf{G}_m)$, the cohomological description $H^d_{Zar}(X, \mathcal{K}^M_{d+s})$ and the geometric description $CH^{d+s}(X, s)$. As we will see, each carries its advantages and disadvantages from the various vantage points of functoriality and amenability to computation; perhaps the most satisfying results are those obtained by playing one definition off against the others.

Armed with the above results, we compute the higher Chow groups of zero-cycles of varieties defined over finite fields. By using the above results together with some techniques of Kahn (cf. [Ka 92]) to control symbols in the groups $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$, we are able to prove the following theorem:

Theorem 9.3.1

Let k be a finite field or an algebraic extension of a finite field and X a smooth projective variety defined over k. Let $d = \dim X$. Then

$$CH^{d+s}(X, s) \cong \begin{cases} \mathbf{Z} \oplus A_0(X) & \text{if } s = 0 \\ k^* & \text{if } s = 1 \\ 0 & \text{if } s \geq 2 \end{cases}$$

Over global fields, we can prove the following weaker statement:

Proposition 11.2.1

Let k be a global field and X a smooth projective variety of dimension d defined over k. Then

$$CH^{d+s}(X, s) \cong \begin{cases} \text{torsion} & \text{if } s = 2 \\ 2-\text{torsion} & \text{if } s \geq 3 \text{ and } k \text{ is a number field} \\ 0 & \text{if } s \geq 3 \text{ and } k \text{ is a function field} \end{cases}$$

We begin in Chapter 2 with a survey of Milnor K-theory, focusing on the functoriality and naturality properties of homomorphisms between Milnor K-groups. Chapter 3 introduces the Somekawa K-groups; we state and prove some fundamental properties of these groups, and conclude with Somekawa's theorem which realizes Milnor K-theory as a group of this type. Chapter 4 is an investigation of the Somekawa groups associated to a single semi-abelian variety, and as such stands apart from the rest of the thesis. Chapter 5 introduces mixed K-groups and includes a description of their functorial behavior. Chapter 6 and 7 are devoted to proving Theorems 6.2.1 and 7.3.2 respectively. Chapter 8 is a digression consisting of some isolated results on certain Somekawa K-groups and a description of the Albanese variety in preparation for the following chapter. The higher Chow groups of zero-cycles on varieties over finite fields are the focus of Chapter 9, the climax of which is Theorem 9.3.1. Chapter 10 is an investigation of certain higher Chow groups of cycles of dimension greater than zero on varieties defined over finite fields. Although the results here are not nearly as comprehensive as those of Chapter 9, we make some progress by working modulo torsion and drawing upon results of Quillen, Bloch, and Soulé. Finally, Chapter 11 is concerned with the higher Chow groups and Somekawa groups in the context of an infinite base field.

Chapter 2

Milnor K-theory

2.1 Basic Definitions

Let k be a field, and r an integer.

Definition 2.1.1. *The Milnor K-groups $K_r^M(k)$ are defined as follows:*

$$K_r^M(k) = 0 \text{ for } r < 0,\ K_0^M(k) = \mathbf{Z},\ K_1^M(k) = k^*$$

For $r \geq 2$,

$$K_r^M(k) = \frac{\bigotimes_{i=1}^r k^*}{I_r}$$

where $I_r \subseteq \bigotimes_{i=1}^r k^*$ is the subgroup generated by elements of the form $a_1 \otimes \ldots \otimes a_r$ such that $a_i + a_j = 1$ for some $i < j$. The class of $a_1 \otimes \ldots \otimes a_r$ in $K_r^M(k)$ is typically denoted $\{a_1, \ldots, a_r\}$.

Definition 2.1.2. *Consider the tensor algebra (cf. [Lang 93], p.633)*

$$T(k^*) = \bigoplus_{r \in \mathbf{N}} T^r(k^*)$$

of the $\mathbf{Z}$-module k^*. Denoting the natural isomorphism $k^* \longrightarrow T^1(k^*)$ by $a \mapsto [a]$, we may consider the two-sided ideal I generated by elements of the form $[a] \otimes [1-a] \in T^2(F^*)$ where a runs through elements of $k^* - \{1\}$. The ideal I is also graded, and so we define *the Milnor ring, denoted $K_*(k)$ by*

$$K_*(k) = T(k^*)/I$$

In practice, one usually interprets the Milnor ring as the graded abelian group in which the product of $x = \{x_1, \ldots, x_n\} \in K_n^M(k)$ and $y = \{y_1, \ldots, y_m\} \in K_m^M(k)$, written $x \cdot y$, is given by $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. Note the presence of the dot between the two elements: juxtaposition (with no dot) of two elements of $K_n^M(k)$ denotes their *sum* in the abelian group $K_n^M(k)$.

Remark.

One may deduce several other relations in the Milnor ring:

Since

$$-a = \frac{1-a}{1-a^{-1}}$$

we have

$$0 = \{a^{-1}, 1-a^{-1}\} = \{a^{-1}, \frac{1-a}{-a}\} = \{a^{-1}, 1-a\}\{a^{-1}, -a^{-1}\}$$

$$= \{a, 1-a\}^{-1}\{a, -a\} = \{a, -a\}$$

thus

$$\{a_1, \ldots, a_r\} = 0 \text{ when } a_i + a_j = 0 \text{ for some } i \neq j$$

Consequently, for any $x \in k^*$, we have:

$$\{x, x\} = \{x, -x\}\{x, -1\} = \{x, -1\}$$

Remark.

The calculation

$$\{ab, -ab\} = \{a, -a\}\{b, -b\}\{a, b\}\{b, a\}$$

combined with the previous remark shows that

$$\{a, b\} = -\{b, a\}$$

Thus the multiplication operation in the Milnor ring is in fact *anticommutative*.

The following proposition is an important consequence of the relations described above:

Proposition 2.1.3. *Suppose K is a field equipped with a discrete valuation v. Then for $r \geq 1$, every element $x \in K_r^M(K)$ may be written (noncanonically) as a product of elements of the form $\{a_1, b_2, \dots, b_r\}$, where $v(b_2) = \dots = v(b_r) = 0$.*

Proof.

We prove the result by induction on r. For $r = 1$ the statement is clear, so assume the induction hypothesis and fix a choice of uniformizer π for v. Given $x = \{x_1, \dots, x_r\} \in K_r^M(K)$, write $x = \{x_1\} \cdot \{x_2, \dots, x_r\}$ and use the induction hypothesis to write the second factor as a product of elements of the desired form. Then x is expressible as a product of elements of the form $\{y_1, y_2, \dots, y_r\}$, where $y_3, \dots, y_r \in O_v^*$, so it suffices to prove the statement for $\{y_1, y_2\}$. Write $y_1 = u_1\pi^a$, $y_2 = u_2\pi^b$, where $u_1, u_2 \in O_v^*$. Then, by multilinearity,

$$\{y_1, y_2\} = \{u_1, u_2\}\{u_1, \pi^b\}\{\pi^a, u_2\}\{\pi^a, \pi^b\}$$

$$= \{u_1, u_2\}\{\pi^{-b}, u_1\}\{\pi^a, u_2\}\{\pi, \pi\}^{ab}$$

$$= \{u_1, u_2\}\{\pi^{-b}, u_1\}\{\pi^a, u_2\}\{\pi, -1\}^{ab}$$

which is of the desired form.

Milnor K-theory is generally difficult to calculate explicitly, but a number of maps possessing desirable functorial properties may be defined between various Milnor K-groups, intimating that the Milnor K-theory of a field may be a useful invariant. We begin with a summary of these maps.

2.2 Functoriality

2.2.1 Covariant functoriality

The first observation is that $K_*^M(-)$ may be interpreted as a covariant functor from the category of fields to the category of graded abelian groups. Given an inclusion $\phi : E \hookrightarrow F$

of fields, there is a naturally induced homomorphism of degree 0 of graded groups

$$\phi_* : K_*^M(E) \longrightarrow K_*^M(F)$$

given by the identity map $\mathbf{Z} \longrightarrow \mathbf{Z}$ in degree 0 and by the formula $\phi_*(\{e_1, \ldots, e_r\}) = \{\phi(e_1), \ldots, \phi(e_r)\}$ in degree $r \geq 1$. In fact, ϕ_* is even a homomorphism of graded rings. It is easy to verify that $K_*^M(-)$ is a functor, so that we have

$$(id_E)_* = id_{K_*(E)}$$

and for any homomorphisms $\phi : E \hookrightarrow F$ and $\psi : F \hookrightarrow G$ of fields,

$$(\psi \circ \phi)_* = \psi_* \circ \phi_*$$

2.2.2 Contravariant functoriality

If we only consider finite morphisms in the category of fields (that is, finite field extensions), we may interpret $K_*^M(-)$ as a contravariant functor as follows: given a finite extension $\phi : E \hookrightarrow F$ of fields, there exists a well-defined homomorphism

$$\phi^* : K_*^M(F) \longrightarrow K_*^M(E)$$

of degree 0 of graded groups which satisfies

$$(id_E)^* = id_{K_*(E)}$$

and for any two finite extensions $\phi : E \hookrightarrow F$ and $\psi : F \hookrightarrow G$ of fields,

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*$$

In deference to convention, we often write $N_{F/E}$ in place of ϕ_*, and refer to it as the *norm* map on Milnor K-theory.

There is, in general, no simple formula for the map $N_{F/E}$. A construction was given by Bass and Tate ([BT 73], I.5) for monogenic extensions, and Kato ([Kato 83],1.7) showed that their construction could be extended (in a well-defined manner) to arbitrary extensions. However, we can give explicit formulas in low degrees: in degree 0, we always have

$$N_{F/E}(n) = [F : E]n \text{ for any } n \in K_0^M(F) \cong \mathbf{Z}$$

and in degree 1 we have

$$N_{F/E}(x) = \mathrm{Norm}_{F/E}(x) \text{ for any } x \in K_1^M(F) \cong F^*$$

where $\mathrm{Norm}_{F/E} : F^* \longrightarrow E^*$ is the usual norm map on fields.

If $\phi : E \hookrightarrow F$ is a finite extension of fields, one may describe explicitly the action of the map $\phi^* = N_{F/E}$ on certain elements. In particular, if $a_1, \ldots, a_r \in E^*$ and $b_1, \ldots, b_s \in F^*$, then we have the following identity, known as the *projection formula.*

Proposition 2.2.1.

$$N_{F/E}\{\phi_*(a_1), \ldots, \phi_*(a_r), b_1, \ldots, b_s\} = \{a_1, \ldots, a_r\} \cdot N_{F/E}\{b_1, \ldots, b_s\}$$

Proof.

Since both sides are functorial in the field extension, we may reduce to the case that F/E is monogenic. The proof then follows from [BT 73], I.(5.3).

As a special case of the projection formula, we have

$$N_{F/E}(\phi_*(x)) = [F : E] \cdot x \text{ for all } x \in K_*^M(E)$$

The projection formula is especially useful when dealing with the specific classes of fields described below.

Definition 2.2.2. *Let p be a prime number. A field k has property E_p if every finite extension of k is of degree p^r for some $r \in \mathbf{N}$.*

Our interest in fields with property E_p stems from the following description of the Milnor K-groups associated to these fields.

Proposition 2.2.3. *Let k be a field with property E_p and $r \geq 1$ an integer. Let $\phi : k \hookrightarrow F$ be an extension of degree p. Then $K_r^M(F)$ is generated by elements of the form $\{a_1, \phi_*(b_2), \ldots, \phi_*(b_r)\}$, where $a_1 \in F^*$ and $b_2, \ldots, b_r \in k^*$.*

Proof.

[BT 73], I. Cor. 5.3.

The preceding proposition is useful (in the context of fields with property E_p) in proving various formulas involving the norm map $N_{F/k}$, insofar as it characterizes $K_r^M(F)$ as generated by elements whose norms are readily computable by means of the projection formula. Evidently the vast majority of fields do not have property E_p for any p; thus, in proving statements about the norm maps, one typically reduces to the case of a field with property E_p.

One typically makes that reduction by means of the following:

Proposition 2.2.4. *Given any field k and any prime number p, there exists an algebraic extension $k(p)/k$ such that $k(p)$ has property E_p.*

Proof.

Set $L = \begin{cases} k^{sep} & \text{if char } k = p \\ \bar{k} & \text{if char } k \neq p \end{cases}$

We know that there exists a p-Sylow subgroup $G(p) \subseteq$ Gal (L/k) ([Shtz1 86], 2.Thm. 4); it is clear that its fixed field $k(p)$ is an algebraic extension of k possessing property E_p.

We will need the following result in subsequent chapters:

Proposition 2.2.5. *Let M/L be a finite extension of fields such that $[M' : L] = p^r$, where M' is the Galois closure of M/L, p is a prime number, and $r \geq 1$ is an integer. Then there exist fields $F_0, \ldots, F_r$ such that*

$$M = F_0 \supseteq F_1 \supseteq \ldots \supseteq F_r = L$$

and $[F_i : F_{i+1}] = p$ for all $i = 0, \ldots, r-1$.

Remark.

If L has property E_p, then the hypotheses of Proposition 2.2.5 are satisfied.

Proof.

In the case $r = 1$, there is nothing to prove. If $r > 1$, then by induction it suffices to prove that there exists a field F lying strictly between L and M.

If $M' = M$, i.e. M/L is Galois, simply take F to the fixed field of the subgroup generated by some element of order p in Gal (M/L).

If $M' \neq M$, let M' be the Galois closure of of M/L and let N be the normalizer of Gal (M'/M) in Gal (M'/L). Now Gal (M'/L) is a p-group by assumption; also, every p-group is nilpotent by the Burnside Theorem ([MB 88], XIII. Theorem 9). By the Lemma to Theorem 15 ([MB 88], p.474), Gal (M'/M), being a subgroup of the nilpotent group Gal (M'/L), is a proper subgroup of its normalizer N. Since M/L is not Galois, Gal (M'/M) is not a normal subgroup of Gal (M'/L); therefore $N \neq$ Gal (M'/L). Thus N lies strictly between Gal (M'/M) and Gal (M'/L), and the fixed field F of N lies strictly between L and M, as desired.

2.2.3 Boundary Maps

Let K be a field, complete with respect to a discrete valuation v; let $k(v)$ be the associated residue field. One may define a degree -1 homomorphism

$$\partial_v : K_*^M(K) \longrightarrow K_*^M(k(v))$$

of graded groups, called the *boundary homomorphism.* Several constructions have been given for this map (cf. [Mi2 86] for details); the following is the most widely used and was initially proposed by Serre. We omit verification of the details, and instead refer the interested reader to [BT 73], I.4.

Fix a choice of uniformizer $\pi \in K^*$ for v. Adjoin a symbol Π of degree 1 to the graded ring $K_*^M(k(v))$; we stipulate that it satisfies (only) the relations $\Pi\{x\} = (\{x\}\Pi)^{-1}$ and $\Pi\Pi = \Pi\{-1\}$ for all $x \in k(v)^*$.

Now define a correspondence $K^* \longrightarrow K_*^M(k(v))[\Pi]$ as follows: given $x \in K^*$, write $x = \pi^i u$, where $v(u) = 0$, and send

$$\{\pi^i u\} \mapsto \{\bar{u}\} + i\Pi$$

(The overhead bar denotes reduction modulo m_v)

One then checks that the rule described above extends uniquely to a ring homomorphism

$$\theta_\pi : K_*^M(K) \longrightarrow K_*^M(k(v))[\Pi]$$

Since the ring $K_*^M(k(v))[\Pi]$ is a free $K_*^M(k(v))$-module of rank 2 with basis $\{1, \Pi\}$, one may write the homomorphism as

$$\theta_\pi = \theta_\pi^0 + \Pi\theta_\pi^1$$

One shows that $\theta_\pi^0 : K_*^M(K) \longrightarrow K_*^M(k(v))$ is a ring homomorphism and $\theta_\pi^1 : K_*^M(K) \longrightarrow K_*^M(k(v))$ is a group homomorphism. Finally, one checks that $\theta^1 = \theta_\pi^1$ is independent of the choice of uniformizer π; this enables us to define

$$\partial_v = \theta^1$$

The boundary homomorphism ∂_v is characterized by the following properties (cf. [BT 73], I.4; [Mi2 86]):

- ∂_v is surjective
- Ker ∂_v is the subring of $K_*^M(K)$ generated by elements of the form $\{u\}$ where $u \in O_v^*$.
- If $u_1, \ldots, u_{r-1} \in O_v^*$, $a \in K^*$, then $\partial_v(\{u_1, \ldots, u_{r-1}, a\}) = \text{ord}_v(a)\{\overline{u_1}, \ldots, \overline{u_{r-1}}\}$
- In degree 1, $\partial_v(\{a\}) = \text{ord}_v(a)$
- In degree 2, $\partial_v(\{a, b\}) = \{\bar{c}\}$ where $c = (-1)^{\text{ord}_v(a)\text{ord}_v(b)} a^{\text{ord}_v(b)} b^{-\text{ord}_v(a)}$

There is in general no known canonical formula for $\partial_v(x)$. However, having fixed a choice of uniformizer $\pi_v \in K^*$ for v, it is possible to write down an explicit formula for ∂_v; see [BT 73], I.4.6 for details.

Now fix a natural number r, and let k be any field, $K \in \mathcal{T}_1(k)$. Then for every $v \in \mathcal{P}(K/k)$, the boundary map $\partial_v : K_r^M(K) \longrightarrow K_{r-1}^M(k(v))$ may be composed with the norm $N_{k(v)/k}$:

$K^M_{r-1}(k(v)) \longrightarrow K^M_{r-1}(k)$ to obtain

$$N_{k(v)/k} \circ \partial_v : K^M_r(K) \longrightarrow K^M_{r-1}(k)$$

The following statement concerning these maps is known as the *reciprocity law*:

Proposition 2.2.6.

$$\sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k} \circ \partial_v = 0$$

Note that when $r = 1$, this reduces to the formula

$$\sum_{v \in \mathcal{P}(K/k)} [k(v) : k]\mathrm{ord}_v(x) = 0 \text{ for } x \in K^*$$

We defer the proof of the reciprocity law until the end of Section 2.3.3.

2.3 Naturality

In this section, we present several propositions concerning naturality relationships among the maps defined above, all of which are important tools in the proofs of later assertions.

2.3.1 Boundary and covariant maps

Let E be a field and v a discrete valuation on E; furthermore, let $\phi : E \hookrightarrow F$ be a field extension of E, equipped with a discrete valuation w extending v. Choose uniformizers π_v, π_w for v, w respectively, and set $e(w/v) = w(\phi(\pi_v))$. Then ϕ naturally induces a homomorphism $k(v) \stackrel{(\phi)_{w/v}}{\longrightarrow} k(w)$ on residue fields.

Proposition 2.3.1. *Under the above hypotheses, the following diagram commutes:*

$$\begin{array}{ccc} K^M_*(E) & \xrightarrow{\phi_*} & K^M_*(F) \\ \downarrow{\scriptstyle \partial_v} & & \downarrow{\scriptstyle \partial_w} \\ K^M_*(k(v)) & \xrightarrow{e(w/v)\cdot(\phi_{w/v})_*} & K^M_*(k(w)) \end{array}$$

Proof.

[BT 73], Proposition I.4.8

2.3.2 Covariant and contravariant maps

Now suppose $k(\alpha)$ is a monogenic extension of k; choose an irreducible polynomial $h(x) \in k[x]$ for α. Let $\phi : k \hookrightarrow L$ be any *algebraic* extension; let $h(x) = \prod_{i=1}^n h_i^{e_i}$ be the decomposition into distinct irreducible factors of $h(x)$ in $L[x]$, and set $L(\alpha_i) = L(x)/(h_i)$. Then the Chinese remainder theorem implies that $L \otimes_k k(\alpha)$ modulo its nilradical is isomorphic to $\prod_{i=1}^n L(\alpha_i)$. Let $\phi_i : k(\alpha) \hookrightarrow L_i(\alpha)$ denote the inclusion map.

Proposition 2.3.2. *The following diagram commutes:*

$$\begin{array}{ccc} \bigoplus_{i=1}^n K_*^M(L(\alpha_i)) & \xrightarrow{N_{L(\alpha_i)/L}} & K_*^M(L) \\ \uparrow {\scriptstyle e_i \cdot (\phi_i)_*} & & \uparrow {\scriptstyle \phi_*} \\ K_*^M(k(\alpha)) & \xrightarrow{N_{k(\alpha)/k}} & K_*^M(k) \end{array}$$

Proof.

[BT 73], I. 5.9, diagram 15

2.3.3 Boundary and contravariant maps

Suppose k is a field, and $K, L \in \mathcal{T}_1(k)$ such that L/K is a finite extension. Fix $v \in \mathcal{P}(K/k)$ and consider all possible extensions $w_1, \ldots, w_n$ of v to places in $\mathcal{P}(L/k)$. Our objective is to prove:

Proposition 2.3.3. *The following diagram commutes:*

$$\begin{array}{ccc} K_*^M(L) & \xrightarrow{(\partial_{w_i})} & \bigoplus_{i=1}^n K_*^M(k(w_i)) \\ \downarrow {\scriptstyle N_{L/K}} & & \downarrow {\scriptstyle (N_{k(w_i)/k(v)})} \\ K_*^M(K) & \xrightarrow{\partial_v} & K_*^M(k(v)) \end{array}$$

Remark.

Proposition 2.3.3 was first proved by Suslin in [Su 79]. The proof given here was developed independently by the author prior to having located Suslin's result; we include it here for the sake of completeness, as it employs techniques to be used in later results.

Before proving the semi-local statement of Proposition 2.3.3, we prove a local version of it:

Lemma 2.3.4. *Let k be a field and E the henselization of some $K \in \mathcal{T}_1(k)$ with respect to some $v \in \mathcal{P}(K/k)$. Let $\eta : E \hookrightarrow F$ be a finite extension and w the (unique) extension of v to F. Then the following diagram commutes:*

$$\begin{array}{ccc} K_*^M(F) & \xrightarrow{\partial_w} & K_*^M(k(w)) \\ \Big\downarrow{\scriptstyle N_{F/E}} & & \Big\downarrow{\scriptstyle N_{k(w)/k(v)}} \\ K_*^M(E) & \xrightarrow{\partial_v} & K_*^M(k(v)) \end{array}$$

Remark.

From functorial properties of the norm, it is easy to see that if Proposition 2.3.3 or Lemma 2.3.4 holds for extensions G/F and F/E, then it holds for the composite extension G/E.

Once we have proved Lemma 2.3.4, we will use Propositions 2.3.1 and 2.3.2 and the operation of henselization to prove Proposition 2.3.3.

Proof of Lemma 2.3.4

We first prove Lemma 2.3.4 in the case that F/E is prime and then treat the general case. To ease notation, we suppress subscripts where possible.

Note that since every algebraic extension may be decomposed into a separable extension followed by a purely inseparable extension, and every finite purely inseparable extension may be decomposed into a sequence of extensions of prime degree, we may assume that F/E is separable when treating the general case.

Assume that $[F : E] = p$, where p is a prime number. We will show that the difference $N \circ \partial - \partial \circ N$ is prime-to-q torsion for all primes q. We prove this assertion first for $q = p$, then for $q \neq p$.

To this end, let $E(p)$ be an maximal algebraic extension of E of degree prime to p (cf. Proposition 2.2.4); clearly $E(p)$ has property E_p. Write $F = \dfrac{E[x]}{(f)}$ for some irreducible polynomial $f(x) \in E[x]$. Then, since $E(p)$ consists only of elements whose degree over E is prime to p, f remains irreducible in $E(p)[x]$. Define $F(p) = \dfrac{E(p)[x]}{(f)} = E(p) \cdot F$.

Now fix $z \in K_{r+1}^M(F)$ and consider the image $(i_{F(p)/F})_*(z) \in K_{r+1}^M(F(p))$, where $i_{F(p)/F} : F \hookrightarrow F(p)$ is the natural map. By Proposition 2.2.3, we may write $(i_{F(p)/F})_*(z)$ as a

finite product $\prod z_j$ of elements of the form $z_j = \{a_{j_1}, (\eta_p)_*(b_{j_2}), \ldots, (\eta_p)_*(b_{j_{r+1}})\}$, where $a_{j_1} \in F_(p)^*$, $b_{j_2}, \ldots, b_{j_{r+1}} \in E(p)^*$, and $\eta_p : E(p) \hookrightarrow F(p)$ is the natural map.
Choose a finite extension E'/E contained in $E(p)$, large enough so that for all j and l, $b_{j_l} \in (E')^*$, and such that $a_{j_1} \in F' = FE'$ for all j.
Denote by v' and w' the (unique) extensions of v and w to E' and F' respectively.

Proposition 2.3.5. *Let K be the henselization of a field with respect to a valuation u; suppose further that the valuation ring O_u is the henselization of a localization of a finitely generated k-algebra. Let L be a finite extension of K of degree n and z the (unique) extension of u to L. Let e denote the ramification index of z/u and f the degree of the residue field extension $k(z)/k(u)$. Then*

$$ef = n$$

Proof.
By [Neu 92], II. Satz 6.8, we always have $ef \leq n$, with equality if L/K is separable. Since every finite extension may be written as a separable extension followed by a purely inseparable extension, and every finite purely inseparable extension may be decomposed as a sequence of extensions of prime degree, we may assume henceforth that L/K is purely inseparable of degree p, where $p = \text{char } K$. Note that the ramification index and residue degree remain unchanged if we replace K and L by their completions K_u and L_z. Furthermore, since K is henselian with respect to u, it satisfies the property that any $x \in K_u$ which is separably algebraic over K is itself in K (cf. [Neu 92], II.6).

Lemma 2.3.6. $[L_z : K_u] = p$

Proof.
By [Neu 92], we have $L_z = LK_u$, so certainly $[L_z : K_u] \leq p = [L : K]$.
To show the other inequality, choose $\alpha \in L - K$. Considering α as an element of L_z, we claim that $\alpha \notin K_u$. By the theory of inseparable extensions, (cf. [Lang 93],V.6), the irreducible polynomial for α (in $K[x]$) is $x^p - a$ for some $a \in K$. Fix a uniformizer π_u for u and write $a = s\pi_u^c$ for some $s \in O_u^*$ and $c \in \mathbf{Z}$. Now pick a set of coset representatives

$R \subseteq O_u$ for O_u/m_u such that $0, s \in R$. (Of course, 0 and s lie in distinct cosets, since $s \in O_u^*$)

Now, since we are assuming $\alpha \in K_u$, α may be represented as a convergent power series

$$\alpha = \pi_u^m (a_0 + a_1 \pi_u + a_2 \pi_u^2 + \ldots)$$

([Neu 92], II. Satz 4.4), where $a_i \in R$, $a_0 \neq 0$, and $m \in \mathbf{Z}$.

On the other hand, α satisfies $\alpha^p = a$, so the series for a must be

$$a = \pi_u^{mp} (a_0^p + a_1^p \pi_u^p + a_2^p \pi_u^{2p} + \ldots)$$

However, by uniqueness of series representations, this power series must be the same as $s\pi_u^c$, since s was (by construction) among our choices of coset representatives. Hence all but one of the a_i are 0 and so $\alpha \in K$, which is a contradiction. We conclude that $\alpha \notin K_u$.

Recall that $g(x) = x^p - a \in K[x]$ is the irreducible monic polynomial for α; let $h(x) \in K_u[x]$ denote the irreducible monic polynomial for $\alpha \in L \subseteq L_z$. Then $h(x)|g(x)$. The above argument shows that deg $h > 1$; we assert that $h = g$. If not, then h has degree strictly between 1 and p, which implies that h is separable. However, all of the roots of h are roots of g, and the only root of g is α; this implies that h has a single root α of multiplicity between 2 and $p-1$, contradicting the inseparability of h. Thus $h = g$ and α has degree p over K_u. We conclude that $[L_z : K_u] \geq [K_u(\alpha) : K_u] = p$, and hence $[L_z : K_u] = p$.

We now return to the proof of Proposition 2.3.5. The proof of [Neu 92], II. Satz 6.8 that $ef = [L_z : K_u] = p$ does not require the separability hypothesis *per se* and still stands provided one can prove that the valuation ring $O_z \subseteq L_z$ is finitely generated as a O_u-module. However, O_u is a local ring which is the completion of the localization of a finitely generated k-algebra, hence Noetherian ([AM 69], Proposition 10.26); thus O_u is excellent and the desired conclusion follows from [Gro 65], IV. 7.8.

We now resume the proof of Lemma 2.3.4.

Proposition 2.3.7.

$$\partial_{v'} \circ N_{F'/E'} \circ (i_{F'/F})_*(z) = N_{k(w')/k(v')} \circ \partial_{w'} \circ (i_{F'/F})_*(z)$$

Proof.

Since $(i_{F'/F})_*(z)$ may be written as a product $\prod z_j$ as described beforehand, it suffices to prove the proposition in the case that $(i_{F'/F})_*(z)$ is of the form $\{c_1, (i_{F'/E'})_*(d_2) \ldots, (i_{F'/E'})_*(d_{r+1})\}$ with $c_1 \in F'$ and $d_2, \ldots, d_{r+1} \in E'$. By Proposition 2.1.3, we may assume that $v(d_3) = \ldots = v(d_{r+1}) = 0$; it is then evident that we are reduced to proving the proposition for the case $r = 1$; that is, we wish to show that the images of elements of the form $\{a, \phi'_*(b)\} \in K_2^M(E')$ have the same image under both compositions of maps in the diagram below:

$$\begin{array}{ccc} K_2^M(F') & \xrightarrow{\partial_{w'}} & K_1^M(k(w')) \\ \downarrow{\scriptstyle N_{F'/E'}} & & \downarrow{\scriptstyle N_{k(w')/k(v')}} \\ K_2^M(E') & \xrightarrow{\partial_{v'}} & K_1^M(k(v')) \end{array}$$

Case 1. F'/E' is unramified; that is, $e = 1$, $f = p$.

In this situation, the desired equality is proved by writing out both sides explicitly using the formulas for the norm on K_1^M and the tame symbol (∂) on K_2^M. The following formulas are needed for the proof:

- $N_{F'/E'}(\{a, (i_{F'/E'})_*(b)\}) = \{N_{F'/E'}(a), b\}$ (Projection formula)
- $(N_{k(w')/k(v')}(\bar{x}))^e = \overline{N_{F'/E'}(x)}$
- $\mathrm{ord}_v(N_{F'/E'}(a)) = f \cdot \mathrm{ord}_w(a)$
- $\mathrm{ord}_w((i_{F'/E'})_*(b)) = e \cdot \mathrm{ord}_v(b)$

We calculate:

$$N \circ \partial(\{a, (i_{F'/E'})_*(b)\})$$

$$N\{((-1)^{\mathrm{ord}_w(a)\mathrm{ord}_w((i_{F'/E'})_*(b))} a^{\mathrm{ord}_w(i_{F'/E'})_*(b)} b^{-\mathrm{ord}_w(a)})\}$$

$$= \{(-1)^{f\cdot(\mathrm{ord}_v(N(a))/f)\cdot \mathrm{ord}_v(b)} (N(a))^{\mathrm{ord}_v(b)} b^{-\mathrm{ord}_v(N(a))}\}$$

Similarly,

$$\partial \circ N(\{a, (i_{F'/E'})_*(b)\})$$

$$= \partial(\{N(a), b\})$$

$$= \{(-1)^{\mathrm{ord}_v(N(a))\mathrm{ord}_v(b)} N(a)^{\mathrm{ord}_v(b)} b^{-\mathrm{ord}_v(N(a))}\}$$

proving the desired equality in this case.

Case 2. F'/E' is totally ramified; that is, $e = p$, $f = 1$.

In this case, we may assume without loss of generality that $N_{F'/E'}(\pi_w) = \pi_v$. Also note that $\overline{\sigma(x)} = \overline{x}$ for all $x \in F'$ and all $\sigma \in \mathrm{Gal}\ (F'/E')$.

We remark that all such elements $\{a, (i_{F'/E'})_*(b)\}$ may be written as products of elements of the form $\{u_w, (i_{F'/E'})_*(u_v)\}$, $\{\pi_w, (i_{F'/E'})_*(u_v)\}$, or $\{u_w, (i_{F'/E'})_*(\pi_v)\}$, where $u_v \in O_v^*$ and $u_w \in O_w^*$, so it suffices to check commutativity for elements of the above form.

For elements $\{u_w, (i_{F'/E'})_*(u_v)\}$, both compositions are obviously zero.

For elements $\{\pi_w, (i_{F'/E'})_*(u_v)\}$ we calculate:

$$N \circ \partial(\{\pi_w, (i_{F'/E'})_*(u_v)\})$$

$$= N(\{\overline{(i_{F'/E'})_*(u_v)}\}^{-1}) \text{ (Definition of tame symbol)}$$

$$= \{\overline{u_v}^{-1}\} \text{ (since } f = 1)$$

On the other hand

$$\partial \circ N(\{\pi_w, (i_{F'/E'})_*(u_v)\})$$

$$= \partial(\{\pi_v, u_v\}^{-1})$$

$$= \{\overline{u_v}\}^{-1}$$

For elements $\{u_w, (i_{F'/E'})_*(\pi_v)\}$, write $(i_{F'/E'})_*(\pi_v) = y\pi_w^e$ where $y \in O_w^*$. Then:

$$N \circ \partial(\{u_w, y\pi_w^e\})$$

$$= N(\partial\{\pi_w, u_w\}^e \partial\{y, u_w\})$$

$$= N(\{\overline{u_w}^e\})$$

$$= \{\overline{u_w}\}^p$$

Likewise,

$$\partial \circ N(\{u_w, y\pi_w^e\})$$

$$= \partial(\{N(u_w), \pi_v\}) \text{ (Projection Formula)}$$

$$= \{\overline{N(u_w)}\}$$

$$= \{\overline{u_w}^p\}$$

This concludes the proof of Proposition 2.3.7.

We now "base extend" the maps in the statement of Lemma 2.9 by the field E' and calculate:

$$N_{k(v')/k(v)} \circ \partial_{v'} \circ N_{F'/E'} \circ (i_{F'/F})_*$$

$$= N_{k(v')/k(v)} \circ \partial_{v'} \circ (i_{E'/E})_* \circ N_{F/E} \text{ (Proposition 2.3.2)}$$

$$= N_{k(v')/k(v)} \circ e(v'/v) \cdot (i_{k(v')/k(v)})_* \circ \partial_v \circ N_{F/E} \text{ (Proposition 2.3.1)}$$

$$= [E' : E]\ \partial_v \circ N_{F/E}$$

By Proposition 2.3.7, this is the same as

$$N_{k(v')/k(v)} \circ N_{k(w')/k(v')} \circ \partial_{w'} \circ (i_{F'/F})_*$$

$$= N_{k(w)/k(v)} \circ N_{k(w')/k(w)} \circ \partial_{w'} \circ (i_{F'/F})_*$$

$$= N_{k(w)/k(v)} \circ N_{k(w')/k(w)} \circ e(w'/w) \cdot (i_{k(w')/k(w)})_* \circ \partial_w \text{ (Proposition 2.3.1)}$$

$$= [F' : F]\ N_{k(w)/k(v)} \circ \partial_w \text{ (Proposition 2.3.2)}$$

$$= [E' : E]\ N_{k(w)/k(v)} \circ \partial_w$$

Thus

$$[E' : E](N \circ \partial - \partial \circ N) = 0$$

By construction, $([E' : E], p) = 1$, so the statement above implies that $N \circ \partial - \partial \circ N$ is prime-to-p torsion.

Given a prime $q \neq p$, choose (as above) an algebraic extension $E(q)$ of E which satisfies the property that every finite extension of $F(q)$ is of degree q^r for some natural number r. Note, however, that if we write $F = \dfrac{E[x]}{(f)}$ for some irreducible polynomial $f(x) \in E[x]$, then every root of f has degree p (which is prime to q), and hence is contained in $E(q)$. Thus f splits completely in $E(q)$, and $F \otimes_E E(q)$ modulo its radical is either equal to the direct sum of p copies of $E(q)$ (if F/E is separable) or one copy of $E(q)$ (if F/E is purely inseparable)

Now fix $z \in K^M_{r+1}(F)$ and choose a finite extension E'' as before; note that we may choose E'' large enough so that all roots of f are contained in E''. Set F'' equal to $F \otimes_E E''$ modulo its radical. We have $F'' = \bigoplus_{i=1}^{p^\nu} E''$ where

$$\nu = \begin{cases} 1 & \text{if } F/E \text{ is separable} \\ 0 & \text{if } F/E \text{ is inseparable} \end{cases}$$

Commutativity of the square

$$\begin{array}{ccc} \bigoplus_{j=1}^{p^\nu} K^M_{r+1}(E'') & \xrightarrow{\partial_{v''}} & \bigoplus_{j=1}^{p^\nu} K^M_r(k(v'')) \\ \downarrow{\scriptstyle N_{E''/E''}=id} & & \downarrow{\scriptstyle N_{k(v'')/k(v'')}=id} \\ K^M_{r+1}(E'') & \xrightarrow{\partial_{v''}} & K^M_r(k(v'')) \end{array}$$

is trivial and is equivalent to the statement that $\partial_{v''}$ is a homomorphism.

Calculating as before,

$$\partial_{v''} \circ \sum_j (N_{E''/E''}) \circ p^{1-\nu} \cdot (i_{E''_j/F})_*$$

$$= \partial_{v''} \circ (i_{E''/E})_* \circ N_{F/E} \text{ (Proposition 2.3.2)}$$

$$= e(v''/v) \cdot (i_{k(v'')/k(v)})_* \circ \partial_v \circ N_{F/E} \text{ (Proposition 2.3.1)}$$

On the other hand, the previous commutative square shows that the first expression above is also equal to:

$$\sum_j (N_{k(v'')/k(v'')})_j \circ (\partial_{v''})_j \circ p^{1-\nu} \cdot (i_{E''/F})_*$$

$$= \sum_j p^{1-\nu} \cdot e(v''/w) \cdot (i_{k(v'')/k(w)})_* \circ \partial_w$$

$$= p \cdot e(v''/w)(i_{k(v'')/k(w)})_* \circ \partial_w$$

To switch the order of covariant and contravariant maps above, we wish to apply Proposition 2.3.2. We split the calculation into two cases, depending on whether F/E is unramified or totally ramified:

If $e(w/v) = 1$, then the diagram

$$\begin{array}{ccc} \bigoplus_{j=1}^{p^\nu} K_r^M(k(v'')) & \xrightarrow{N=\sum} & K_r^M(k(v'')) \\ {\scriptstyle p^{1-\nu}(i_{k(v'')_j/k(w)})_*} \uparrow & & \uparrow {\scriptstyle (i_{k(v'')/k(v)})_*} \\ K_r^M(k(w)) & \xrightarrow{N} & K_r^M(k(v)) \end{array}$$

commutes, so continuing the calculation above, we obtain

$$= e(v''/w)(i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w \text{ (Proposition 2.3.2)}$$

$$= e(v''/w)e(w/v)(i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w$$

$$= e(v''/v)(i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w$$

If $e(w/v) = p$, then the diagram

$$\begin{array}{ccc} \bigoplus_{j=1}^{p^\nu} K_r^M(k(v'')) & \xrightarrow{N=\sum} & K_r^M(k(v'')) \\ \uparrow {\scriptstyle p^{1-\nu}(i_{k(v'')_j/k(w)})_*} & & \uparrow {\scriptstyle (i_{k(v'')/k(v)})_*} \\ K_r^M(k(w)) & \xrightarrow{p \cdot N = p} & K_r^M(k(v)) \end{array}$$

commutes, so we obtain

$$= e(v''/w) \cdot p \cdot (i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w \text{ (Proposition 2.3.1)}$$

$$= e(v''/w)e(w/v) \cdot (i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w \text{ (Proposition 2.3.1)}$$

$$= e(v''/v) \cdot (i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w$$

Thus, comparing our calculations, we have:

$$e(v''/v) \cdot (i_{k(v'')/k(v)})_* \circ \partial_v \circ N_{F/E} = e(v''/v) \cdot (i_{k(v'')/k(v)})_* \circ N_{k(w)/k(v)} \circ \partial_w$$

Applying $N_{k(v'')/k(v)}$ to both sides, we obtain

$$= [E'' : E] \cdot \partial_v \circ N_{E/F} = [E'' : E] \cdot N_{k(w)/k(v)} \circ \partial_w$$

and since $q \not| [E'' : E]$ by construction, this implies $N \circ \partial - \partial \circ N$ is prime-to-q torsion for $q \neq p$.

The observation that $N \circ \partial - \partial \circ N$ is prime-to-q torsion for *all* primes q implies that the difference is zero, which concludes the proof of Lemma 2.3.4 in the case that $[F : E]$ is prime.

Now suppose F/E is separable and of arbitrary (finite) degree. Write $F = \dfrac{E[x]}{(f)}$ for some irreducible $f \in E[x]$. Fix a prime number p, and let $E(p)$ be a maximal algebraic extension of E of degree prime to p. Suppose f splits into (distinct) irreducible elements as $f = f_1 \dots f_s \in E(p)[x]$. Then $F \otimes_E E(p) = \bigoplus_{j=1}^s L_j$ where $L_j = \dfrac{E(p)[x]}{(f_j)}$. Note that each extension $L_j/E(p)$ is separable.

For each j, let L'_j denote the Galois closure of the extension $L_j/E(p)$. L'_j is certainly the splitting field of f_j, and being an extension of $E(p)$, has degree p^{r_j} for some $r_j \in \mathbf{N}$. Thus every root of each f_j has p-power degree over $E(p)$. Since $E(p)$ is algebraic over E, there is some finite extension E' of E contained in $E(p)$ such that every root of each f_j has prime power degree over E'. Enlarging E' if necessary, we may assume that f splits as $f = f_1 \dots f_s$ in $E'[x]$, with f_i mutually distinct.

Now consider

$$F \otimes_E E' = \frac{E[x]}{(f)} \otimes_E E' \cong \frac{E'[x]}{(f)} \cong \bigoplus_{j=1}^{s} \frac{E'[x]}{(f_j)} = \bigoplus_{j=1}^{s} M_j$$

Since each extension M_j/E' satisfies the hypotheses of Proposition 2.2.5, we may write the extension M_j/E' as a sequence of extensions, each of degree p. By the remark following Lemma 2.3.4, the fact that the conclusion of Lemma 2.3.4 holds in each extension of the sequence implies that it holds for M_j/E'. Thus, if v denotes a valuation on E, w its extension to F, v' its extension to E', and w'_j its extension to M_j, we have, for each j,

$$\partial_{v'} \circ N_{M_j/E'} = N_{k(w'_j)/k(v')} \circ \partial_{w'_j}$$

Now we calculate:

$$N_{k(v')/k(v)} \circ \partial_{v'} \circ \sum_{j=1}^{s} N_{L_j/E'} \circ (i_{L_j/F})_*$$

$$= N_{k(v')/k(v)} \circ \partial_{v'} \circ (i_{E'/E})_* \circ N_{F/E} \text{ (Proposition 2.3.2)}$$

$$= N_{k(v')/k(v)} \circ e(v'/v) \cdot (i_{k(v')/k(v)})_* \circ \partial_v \circ N_{F/E} \text{ (Proposition 2.3.1)}$$

$$= [E' : E]\ \partial_v \circ N_{F/E}$$

By the last statement preceding this calculation, the above is equal to:

$$N_{k(v')/k(v)} \circ \sum_{j=1}^{s} N_{k(w'_j)/k(v')} \circ \partial_{w'_j} \circ (i_{L_j/F})_*$$

$$= N_{k(w)/k(v)} \circ \sum_{j=1}^{s} N_{k(w'_j)/k(w)} \circ \partial_{w'_j} \circ (i_{L_j/F})_*$$

$$= N_{k(w)/k(v)} \circ \sum_{j=1}^{s} (N_{k(w'_j)/k(w)} \circ (e(w'_j/w) \cdot (i_{k(w'_j)/k(w)})_*) \circ \partial_w \text{ (Proposition 2.3.1)}$$

$$= \sum_{j=1}^{s} [L_j : F]\ N_{k(w)/k(v)} \circ \partial_w$$

However, counting degrees, this equals

$$= [E' : E]\ N_{k(w)/k(v)} \circ \partial_w$$

Thus

$$[E' : E](N \circ \partial - \partial \circ N) = 0$$

and $N \circ \partial - \partial \circ N$ is prime-to-p torsion. Since p was arbitrary, this implies

$$N \circ \partial - \partial \circ N = 0$$

and concludes the proof of Lemma 2.3.4.

Proof of Proposition 2.3.3.

Set $L^s = \{x \in L : x \text{ is separable over } K\}$ By decomposing the extension L/K as $L/L^s/K$, it suffices to prove Proposition 2.3.3 in two cases: L/K is purely inseparable, and L/K is separable.

In the purely inseparable case, we may reduce, by the remark following Lemma 2.3.4, to the case $[L : K] = p$. Choose α such that $L = K(\alpha)$. Given any $\beta \in L$ and a place w lying over v, we must have $\beta^p = b \in K$, so

$$p \cdot w(\beta) = w(\beta^p) = w(b) = e(w/v)v(b)$$

hence there is only one extension of v to a place w of L.

Denote by K_v^h the henselization of K with respect to the valuation v, and similarly for L_w^h; let K_v and L_w denote their respective completions. Then the argument of Lemma 2.3.6 implies that $\alpha \in L \subseteq L_w$ has degree p over K_v; since α has degree p over K, it must also have degree p over K_v^h. Hence the irreducible polynomial $f(x) \in K[x]$ for α remains irreducible in K_v^h, and so $L \otimes_K K_v^h = K_v^h[x]/(f) = LK_v^h$ is a field extension of K_v^h of degree p which contained in L_w^h.

We claim that $L \otimes_k K_v^h = LK_v^h$ is in fact equal to L_w^h. To see this, fix $\beta \in L_w^h$. Thus β is an element of L_w which satisfies some separable irreducible polynomial $f \in L[x]$. Examining the power series expansion for $\beta \in L_w$, we note that by raising β to the pth power, we obtain an element of K_v. Thus $\beta \in K_v$ and we see easily that β^p satsifies $g = f^p$, the polynomial obtained from f by raising all of its coefficients to the pth power. Since L/K is purely inseparable of degree p, each of the coefficients of g lies in K; thus $g \in K[x]$. Furthermore, g is separable over K. Therefore $\beta^p \in K_v$ is separably algebraic over K, so we conclude that $\beta^p \in K_v^h$. Clearly β is a solution of $h(x) = x^p - \beta^p \in K_v^h$. If $\beta \in K_v^h$, we are done; if not, then β satisfies both h, viewed as a polynomial of $LK_v^h[x]$ and f, also viewed as a polynomial in $LK_v^h[x]$, and must therefore satisfy the greatest common divisor (in $LK_v^h[x]$) of the two polynomials. Since the only root of h is β and separability of f implies that β appears with multiplicity one as a root of f, we conclude that the greatest common divisor of f and h is $x - \beta$, which implies that $\beta \in LK_v^h$.

Now using Proposition 2.3.2 and Proposition 2.3.1, we have:

$$\partial_v \circ N_{L_w^h/K_v^h} \circ (i_{L_w^h/L})_*$$

$$= \partial_v \circ (i_{K_v^h/K})_* \circ N_{L/K}$$

$$= e(v/v) \cdot (i_{k(v)/k(v)})_* \circ \partial_v \circ N_{L/K}$$

$$= \partial_v \circ N_{L/K}$$

By Lemma 2.3.4 this is the same as

$$N_{k(w)/k(v)} \circ \partial_w \circ (i_{L_w^h/L})_*$$

$$= N_{k(w)/k(v)} \circ e(w/w) \cdot (i_{k(w)/k(w)})_* \circ \partial_w$$

$$= N_{k(w)/k(v)} \circ \partial_w$$

which concludes the proof of Proposition 2.3.3 in this case.

Now assume L/K is separable. Let $w_1, \ldots, w_s$ denote the (distinct) extensions of the place $v \in \mathcal{P}(K/k)$ to places of L, and let $\gamma_i : L \hookrightarrow L_{w_i}$, $i = 1, \ldots, s$ be the canonical maps of L into its completions at the places w_i. Let $\delta_i : K_v \hookrightarrow L_{w_i}$ denote the canonical inclusions. [Neu 92], II. Satz 8.3 states that the map

$$\Phi : L \otimes_K K_v \longrightarrow \bigoplus_{i=1}^{s} L_{w_i}$$

$$x \otimes y \mapsto (\gamma_i(x) \cdot \delta_i(y))_{i=1}^{s}$$

is an isomorphism. We assert that the restriction of Φ to the subgroup $L \otimes_K K_v^h$ maps isomorphically onto $\bigoplus_{i=1}^{s} L_{w_i}^h$. We use the characterization of K_v^h (resp. $L_{w_i}^h$) as the subfield of elements in K_v (resp. L_{w_i}) which are separably algebraic over K (resp. L).

Recall that L/K is separable by assumption. If $y \in K_v^h$, then y is separably algebraic over K, hence over L; so for each i, $\Phi(x \otimes y) = \gamma_i(x)\delta_i(y)$ is certainly separably algebraic over

L. Thus the isomorphism Φ defined above restricts to an injective map

$$\tilde{\Phi} : L \otimes_K K_v^h \longrightarrow \bigoplus_{i=1}^{s} L_{w_i}^h$$

To show that Φ is surjective, we calculate the respective dimensions (as K_v^h-vector spaces) of the domain and the target, and show that they are equal. To this end, we note that by [Neu 92], II. Korollar 8.4, we have

$$[L : K] = \dim_{K_v^h} L \otimes_K K_v^h = \sum_{i=1}^{s} [L_{w_i} : K_v]$$

If $L_{w_i}^h / K_v^h$ is purely inseparable of degree p, then by the argument of Lemma 2.3.6, $[L_{w_i} : K_v] = p = [L_{w_i}^h : K_v^h]$; if $L_{w_i}^h / K_v^h$ is separable, then $[L_{w_i} : K_v] = [L_{w_i}^h : K_v^h]$ by [Neu 92], Korollar 8.4. Thus, we always have

$$[L : K] = \dim_{K_v} L \otimes_K K_v = \sum_{i=1}^{s} [L_{w_i} : K_v] = \sum_{i=1}^{s} [L_{w_i}^h : K_v^h] = \dim_{K_v^h} \bigoplus_{i=1}^{s} L_{w_i}^h$$

Counting dimensions, we conclude that $L \otimes_K K_v^h \cong \bigoplus_{i=1}^{s} L_{w_i}^h$
Using Proposition 2.3.2 and Proposition 2.3.1, we calculate:

$$\partial_v \circ \sum_i N_{L_{w_i}^h / K_v^h} \circ \sum_i (i_{L_{w_i}^h / L})_*$$

$$= \partial_v \circ (i_{K_v^h / K})_* \circ N_{L/K}$$

$$= e(v/v) \cdot (i_{k(v)/k(v)})_* \circ \partial_v \circ N_{L/K}$$

$$= \partial_v \circ N_{L/K}$$

By Lemma 2.3.4 this is the same as

$$\sum_i N_{k(w_i)/k(v)} \circ \partial_{w_i} \circ (i_{L_{w_i}^h / L})_*$$

$$= \sum_i N_{k(w_i)/k(v)} \circ e(w_i/w_i) \cdot (i_{k(w_i)/k(w_i)})_* \circ \partial_{w_i}$$

$$= \sum_i N_{k(w_i)/k(v)} \circ \partial_{w_i}$$

which concludes the proof of Proposition 2.3.3.

We now give a proof of the reciprocity law.

Proof of Proposition 2.2.6

We wish to prove that given a field k and $K \in \mathcal{T}_1(k)$, we have

$$\sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k} \circ \partial_v = 0$$

For $K = k(T)$, this is an immediate corollary of the definition of the maps $N_{k(v)/k}$, cf. [BT 73], I.5, Equation 4.

For general K, choose any element $t \in K$ which is transcendental over k. Then we compute:

$$\sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k} \circ \partial_v$$

$$= \sum_{v \in \mathcal{P}(K/k)} \sum_{w \in \mathcal{P}: v \mapsto w} N_{k(w)/k} \circ N_{k(v)/k(w)} \circ \partial_v$$

Using Proposition 2.3.3, we may write the above

$$= \sum_{w \in \mathcal{P}(k(t)/k)} N_{k(w)/k} \circ \partial_w \circ N_{K/k(t)}$$

$$= 0$$

by the result for $k(T)$.

Chapter 3

Somekawa's K-groups

3.1 Introduction

The definition of Milnor K-theory, while explicit, is indeed mysterious. One might wonder, and quite justifiably so, what is so special about the relations $\{a, 1-a\}$. In the following, we give a more complicated, yet more intuitive description of Milnor K-theory: first we define (for each integer $r \geq 1$) a quotient of a free abelian group by imposing several meaningful relations on it; in Section 3.6, we prove that this group is isomorphic to the group $K_r^M(k)$.

Let k be a field. Consider the group

$$F = \bigoplus_{E/k \text{ finite}} \bigotimes_{i=1}^{r} E^*$$

and the subgroup $R \subseteq F$ generated by relations of the following type:

- For every diagram $k \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ of morphisms of finite extensions of k, all $i_0 \in \{1, \ldots, r\}$ and all choices $g_{i_0} \in E_2^*$, $g_i \in E_1^*$ $(i \neq i_0)$, the relation

$$(\phi_*(g_1) \otimes \ldots \otimes g_{i_0} \otimes \ldots \otimes \phi_*(g_r))_{E_2} - (g_1 \otimes \ldots \otimes N_{E_2/E_1}(g_{i_0}) \otimes \ldots \otimes g_r)_{E_1}$$

  (the subscript refers to the direct summand in which the element resides)

- For every $K \in \mathcal{T}_1(k)$ and all $g_1, \ldots, g_r, h \in K^*$, such that for each $v \in \mathcal{P}(K/k)$, there exists $i(v)$ such that $g_i \in O_v^*$ for all $i \neq i(v)$, the relation

$$\sum_{v \in \mathcal{P}(K/k)} (g_1(v) \otimes \ldots, \partial_v(g_{i(v)}, h) \otimes \ldots \otimes g_r(v))_{k(v)}$$

where $g_i(v) \in k(v)^*$ $(i \neq i(v))$ denotes the reduction of $g_i \in O_v$ modulo m_v, and $\partial_v(g, h)$ is the usual tame symbol; that is, the reduction modulo m_v of the element $(-1)^{\mathrm{ord}_v(g) \cdot \mathrm{ord}_v(h)} g^{\mathrm{ord}_v(h)} h^{-\mathrm{ord}_v(g)} \in O_v$.

Note that the first type of relation corresponds, in some sense, to the projection formula (Proposition 2.2.1), and the second type to the reciprocity law (Proposition 2.2.6) of Milnor K-theory.

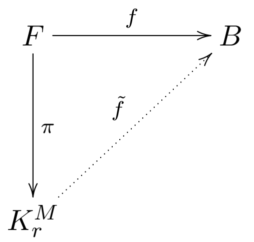
It is a fact, proven in Theorem 3.6.1 below, that for each $r \geq 1$, there exists a surjective homomorphism

$$\pi : F \longrightarrow K_r^M(k)$$

with the following universal property: let B be an abelian group and

$$f : F \longrightarrow B$$

a homomorphism such that $f(R) = 0$. Then there exists a unique homomorphism $\tilde{f} : K_r^M(k) \longrightarrow B$ such that the following diagram commutes:



In fact, one can even prove that $\pi(R) = 0$, so taking $B = F/R$ and f the canonical map, we conclude that

$$K_r^M(k) \cong F/R$$

We will make these remarks precise in Section 3.6.

3.2 Local symbols

The goal of this section is to give a brief introduction to the theory of local symbols, which, as we will soon see, are intimately connected with the boundary maps (cf. Section 2.2.3) of Milnor K-theory. Our main reference for this section is [Se 97].
Throughout this section, k is an algebraically closed field and K some field in $\mathcal{T}_1(k)$. For convenience, set $\mathcal{P} = \mathcal{P}(K/k)$.

Definition 3.2.1. *Let $S \subseteq \mathcal{P}$ be a finite subset. A* modulus $\mathfrak{m}$ *supported on S is a set map*

$$\mathfrak{m} : S \longrightarrow \mathbf{N} - \{0\}$$

We often identify a modulus with a formal sum $\sum_{v \in S} \mathfrak{m}(v) \cdot v$ of places.

If $g \in K$, we write

$$g \equiv 1(\text{mod } \mathfrak{m})$$

if $\text{ord}_v(1 - g) \geq \mathfrak{m}(v)$ for every $v \in S$; intuitively, this says that g is equal to 1 "with multiplicity at least $\mathfrak{m}(v)$" at all points v of S.
If for some *particular* $v \in S$, we have $\text{ord}_v(1 - g) \geq \mathfrak{m}(v)$, we say that $g \equiv 1(\text{mod } \mathfrak{m})$ at v.

Definition 3.2.2. *Let G be a commutative algebraic group defined over k. For any given $v \in \mathcal{P}$, a morphism Spec $K \xrightarrow{g} G$ is called* v-proper *if it admits a factorization Spec $K \longrightarrow$ Spec $O_v \xrightarrow{g_0} G$. In such a case, we use the notation $g(v)$ to denote the composition*

$$\textit{Spec } k(v) \longrightarrow \textit{Spec } O_v \xrightarrow{g_0} G$$

If we take C to be the smooth projective model for K over k, we may identify any such morphism Spec $K \xrightarrow{g} G$ with a rational map $C \longrightarrow G$. Furthermore, since every rational map from a curve to a variety is defined at all but finitely many points, we infer that g is v-proper for all but finitely many $v \in \mathcal{P}$.

Definition 3.2.3. *Let G be a commutative algebraic group defined over k, g : Spec $K \longrightarrow G$ a morphism and $h \in K^*$. Let $S(g) = \{v \in \mathcal{P} : g \text{ is not } v\text{-proper}\}$. By the remark above, $S(g)$ is finite. Set $R(g) = \mathcal{P} - S(g)$. Then for all $h \in K^*$ satisfying $\text{ord}_v(h - 1) \geq 1$ for* *all* $v \in S(g)$*, we define*

$$g((h)) = \sum_{v \in R(g)} ord_v(h) \cdot g(v)$$

(The notation is chosen to suggest that $S(g)$ *is in some sense the "singular set" of* g *and* $R(g)$ *the "regular set" of* g*)*

We will usually write the group law on G additively, except (for reasons of convention) when $G = \mathbf{G}_m$.

Definition 3.2.4. *Given a morphism,*

$$g : Spec\ K \longrightarrow G$$

we say that $\mathfrak{m}$ *is a modulus for* g *if* $\mathfrak{m}$ *is supported on* $S(g)$ *and* $g((h)) = 0$ *for all* $h \in K^*$ *such that* $h \equiv 1(mod\ \mathfrak{m})$*.*

Definition 3.2.5. *Let* g *be as above, and let* $\mathfrak{m}$ *be a modulus supported on* $S(g)$*. A* local symbol associated to the pair $(g, \mathfrak{m})$ *is an assignment, for each* $v \in \mathcal{P}$ *and each* $h \in K^*$*, of an element of* $G(k)$*, denoted* $(g, h)_v$*, satisfying the following conditions:*

1. $(g, hh')_v = (g, h)_v \cdot (g, h')_v$ *for any* $h, h' \in K^*$
2. $(g, h)_v = 0$ *if* $v \in S(g)$ *and* $h \equiv 1(mod\ \mathfrak{m})$ *at* v*.*
3. $(g, h)_v = ord_v(h) \cdot g(v)$ *if* $v \in R(g)$
4. $\sum_{v \in \mathcal{P}} (g, h)_v = 0$

The following criterion gives a necessary and sufficient condition for the existence of a local symbol.

Proposition 3.2.6. *Let* $g : Spec\ K \longrightarrow G$ *be a morphism and* $\mathfrak{m}$ *a modulus supported on* $S(g)$*. If there exists a local symbol associated to* $(g, \mathfrak{m})$*, then* $\mathfrak{m}$ *is a modulus for* g*. Conversely, if* $\mathfrak{m}$ *is a modulus for* g*, then there exists a unique local symbol associated to* $(g, \mathfrak{m})$*.*

Proof.

[Se 97], III.1, Proposition 1.

We will make use of several facts concerning local symbols.

Proposition 3.2.7. *Let k be an algebraically closed field and $K \in \mathcal{T}_1(k)$. The local symbol corresponding to the morphism Spec $K \xrightarrow{g} \mathbf{G}_m$ determined by $g \in K^*$ and the modulus $S(g)$ coincides with the "tame v-adic symbol" (cf. [Mil 71], p.98, also Section 2.2.3)*

$$(g, h)_v = (-1)^{ord_v(g) ord_v(h)} g^{ord_v(h)} h^{-ord_v(g)}$$

Proof.
[Se 97], III.1, Proposition 6.

In this case, the fourth property enjoyed by local symbols translates into the following formula:

Corollary 3.2.8. *For every $g, h \in K^*$,*

$$\prod_{v \in \mathcal{P}} (-1)^{ord_v(g) ord_v(h)} g^{ord_v(h)} h^{-ord_v(g)} = 1$$

Our last topic of discussion related to local symbols concerns the trace map. Let X, Y be smooth projective curves, $g : Y \longrightarrow G$ a rational map defined on $Y - S$, and $\pi : Y \longrightarrow X$ a finite morphism of varieties over k. Set $S' = \pi(S)$; for every $P \in X$, let $\pi^{-1}(P)$ denote the divisor $\sum_{\pi(Q)=P} e(Q/P) \cdot Q$, where $e(Q/P)$ represents the ramification index of Q with respect to P; that is, the length of the $\mathcal{O}_{Q,Y}$-module $\mathcal{O}_{Q,Y}/\pi^*(\mathcal{O}_{P,X})$. If $P \notin S'$, then the divisor $\pi^{-1}(P)$ has support disjoint from S; hence the expression $Tr_\pi g(P) = f(\pi^{-1}(P)) = \sum_{\pi(Q)=P} e(Q/P) \cdot g(Q)$ is well-defined. (The latter sum indicates summation on the group variety G) By [Se 97] III.2 Proposition 8, $Tr_\pi g(P)$ actually determines a rational map $X \longrightarrow G$ called the *trace* of g with respect to π; it is defined on $X - S'$.
The main result concerning the trace map which we will need is:

Proposition 3.2.9. *With notation as above, we have the following formula, for any $h \in k(X)^*$ and $v \in \mathcal{P}(k(X)/k)$:*

$$(Tr_\pi g, h)_v = \sum_{w \mapsto v} (g, \pi^*(h))_w$$

Proof.

[Se 97], III.1, Propoistion 3.

3.3 Definitions

3.3.1 Extension of valuations

Before giving the definition of Somekawa's K-groups, a few auxiliary definitions are in order. The first concerns the extension of a valuation on the multiplicative group of a field to a similar map on points of a semi-abelian variety.

Suppose k is a field and G is a semi-abelian variety defined over k: that is, there is an exact sequence of group schemes (viewed as sheaves in the flat topology, cf. [Shtz2 86], [Mi 71]) over k:

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

where T is a torus and A is an abelian variety. Fix $K \in \mathcal{T}_1(k)$ and $v \in \mathcal{P}(K/k)$. Let L/K_v be a finite unramified Galois extension such that T splits over the corresponding residue field F of L; that is, $T \times_k F \cong \mathbf{G}_m^n$ for some n; let w be the (unique) extension of v to L. Consider the following commutative diagram; the map ord_w is understood as the extension of the usual $\mathrm{ord}_w : L^* \longrightarrow \mathbf{Z}$ to a map $T(L) \cong (L^*)^n \longrightarrow \mathbf{Z}^n$, and the map $r_w = (r_w^1, \ldots, r_w^n)$ is described in the succeeding remarks:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T(O_w) & \longrightarrow & G(O_w) & \longrightarrow & A(O_w) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & T(L) & \longrightarrow & G(L) & \longrightarrow & A(L) & \longrightarrow & 0 \\
 & & \downarrow \mathrm{ord}_w & & \downarrow {\scriptstyle r_w=(r_w^1,\ldots,r_w^n)} & & & & \\
 & & \mathbf{Z}^n & \longrightarrow & \mathbf{Z}^n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The first row is exact: this follows from consideration of the long exact sequence of cohomology over the flat site (for a precise definition of the flat site, see [Mi 71], p.47, also Cor.

1.7, p. 52); the final map is surjective because $H^1_{fl}(O_w; \mathbf{G}_m^n) \cong (\text{Pic (Spec } O_w))^n = 0$. Exactness of the second row follows from Hilbert Theorem 90, and the last column is an isomorphism by the valuative criterion for properness. Finally, the cokernel of the first column is clearly isomorphic to $\mathbf{Z}^n$, so by the snake lemma (applied to the first two rows), the cokernel of the second column is also isomorphic to $\mathbf{Z}^n$ and enables us to define a map $r(w) = (r_w^1, \ldots, r_w^n)$.

Now fix $g \in G(K_v)$ and $h \in K_v^*$. We are going to construct a map $\partial_v : G(K_v) \otimes K_v^* \longrightarrow G(k(v))$ which simultaneously generalizes the local symbol (cf. Section 3.2) and the boundary homomorphism $K_2^M(K) \longrightarrow K_1^M(k(v))$ (cf. Section 2.2.3) of Milnor K-theory.

For each $i = 1, \ldots, n$,we define $h_i \in T(L)$ to be the n-tuple having h in the ith coordinate and 1 elsewhere. Then set

$$\varepsilon(g,h) = ((-1)^{\text{ord}_w(h) r_w^1(g)}, \ldots, (-1)^{\text{ord}_w(h) r_w^n(g)}) \in T(O_w) \subseteq G(O_w)$$

and

$$\tilde{\partial}_v(g,h) = \varepsilon(g,h) g^{\text{ord}_w(h)} \prod_{i=1}^{n} h_i^{-r_w^i(g)} \in G(O_w)$$

We define the "extended tame symbol" $\partial_v(g,h)$ to be the image of $\tilde{\partial}_v(g,h)$ under the canonical map $G(O_w) \longrightarrow G(F)$; we note that since $\tilde{\partial}_v(g,h)$ is invariant under the action of Gal (L/K_v), $\partial_v(g,h)$ must be invariant under the action of Gal $(F/k(v))$, so we henceforth interpret $\partial_v(g,h)$ as belonging to $G(k(v))$. We observe that since L/K_v is unramified, this definition of ∂_v is independent of the choice of L.

3.3.2 Definition of Somekawa's K-groups

In this section we define the Somekawa K-groups. We follow [Som 90], taking into account the correction noted in [RS 97], p.9.

Let k be a field, and $G_1, \ldots, G_r$ a finite (possibly empty) family of semi-abelian varieties defined over k. We define the Somekawa K-group $K(k; G_1, \ldots, G_r)$ as follows. If $r = 0$, we write $K(k;\emptyset)$ for our group and set $K(k;\emptyset) = \mathbf{Z}$. For $r \geq 1$, we define

$$K(k; G_1, \ldots, G_r) = F/R$$

where

$$F = \bigoplus_{E/k \text{ finite}} G_1(E) \otimes \ldots \otimes G_r(E)$$

and $R \subseteq F$ is the subgroup generated by the types of relations described below. The residue in F/R of an element $a_1 \otimes \ldots \otimes a_r$ in the Eth direct summand of F will henceforth be denoted $\{a_1, \ldots, a_r\}_{E/k}$.

- **R1**. For every diagram $k \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ of morphisms of finite extensions of k, all choices $i_0 \in \{1, \ldots, r\}$ and all choices $g_{i_0} \in G_{i_0}(E_2)$ and $g_i \in G_i(E_1)$ for $i \neq i_0$, the relation

$$(\phi^*(g_1) \otimes \ldots \otimes g_{i_0} \otimes \ldots \otimes \phi^*(g_r))_{E_2} - (g_1 \otimes \ldots \otimes N_{E_2/E_1}(g_{i_0}) \otimes \ldots \otimes g_r)_{E_1}$$

  (Here N_{E_2/E_1} denotes the norm map on the group scheme G_{i_0})

- **R2**. For every $K \in \mathcal{T}_1(k)$ and all choices $g_i \in G_i(K)$, $h \in K^*$ such that for each $v \in \mathcal{P}(K/k)$, there exists $i(v)$ such that $g_i \in G_i(O_v)$ for all $i \neq i(v)$, the relation

$$\sum_{v \in \mathcal{P}(K/k)} (g_1(v) \otimes \ldots \otimes \partial_v(g_{i(v)}, h) \otimes \ldots \otimes g_r(v))_{k(v)/k}$$

  Here $g_i(v) \in G_i(k(v))$ $(i \neq i(v))$ denotes the reduction of $g_i \in O_v$ modulo m_v and $\partial_v(g_{i(v)}, h)$ is the extended tame symbol as defined in Section 3.3.1.

Notation.

In the special case that $G_1 = \ldots = G_r = \mathbf{G}_m$, we write $K_r(k; \mathbf{G}_m)$ for the group $K(k; \underbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}_{r})$.

Remark.

Let $E \overset{j}{\longrightarrow} E'$ be an isomorphism of finite extensions of k. Then for any $g_i \in G_i(E)$, we have

$$\{j^*(g_1), \ldots, j^*(g_s)\}_{E'/k} = \{N_{E'/E}(j^*(g_1)), g_2, \ldots, g_s\}_{E/k}$$

by relation **R1**

$$= \{g_1, g_2, \ldots, g_r\}_{E/k}$$

since $[E' : E] = 1$. As Somekawa ([Som 90], p.107) observes, this computation shows that the symbols defining $K(k; G_1, \ldots, G_s)$ form a set.

3.4 Functoriality

In this section we give proofs of the various functorial properties of Somekawa K-groups first stated in [Som 90]. Although we will use these properties in later sections, they are not used in the main results of this work. We warn the reader that the proofs are rather technical and do not cast much light on the nature of the Somekawa groups.

3.4.1 Change of base field

For an arbitrary extension $\phi : k \hookrightarrow k'$ of fields, we wish to define a canonical homomorphism

$$K(k; G_1, \ldots, G_r) \xrightarrow{\phi_*} K(k'; G_1 \times_k k', \ldots, G_r \times_k k')$$

Given a finite extension $\psi : k \hookrightarrow E$ of fields, we have $E \otimes_k k' = \bigoplus_{i=1}^n A'_i$ where each A'_i is an Artin algebra of dimension e_i over the residue field E'_i. (cf. Section 2.3.2). Let $\psi_i : E \longrightarrow E'_i$ denote the canonical maps. Then we define

$$\phi_*(\{g_1, \ldots, g_r\}_{E/k}) = \sum_{i=1}^{n} e_i \{\psi_i^*(g_1), \ldots, \psi_i^*(g_r)\}_{E'_i/k'}$$

To show that this operation is well-defined, we need to demonstrate that it kills relations of type **R1** and **R2**.

From the following two facts, we may deduce that relations of type **R1** map to similar relations. Let $k \hookrightarrow E_1 \overset{\psi}{\hookrightarrow} E_2$ be finite extensions, and consider points $g_{i_0} \in G_{i_0}(E_2)$, $g_i \in G_i(E_1)$ for $i \neq i_0$.

First, consider the points g_i, $i \neq i_0$.

$$\text{Spec } E_2 \xrightarrow{\psi^*} \text{Spec } E_1 \xrightarrow{g_i} G$$

To investigate what happens to this point under the proposed map, first we apply the functor $- \times_k k'$:

$$\text{Spec } (E_2 \otimes_k k') \longrightarrow \text{Spec } (E_1 \otimes_k k') \longrightarrow G \times_k k'$$

Next, setting $F_i = \dfrac{E \otimes_k k'}{\mathfrak{N}}$ $(i = 1, 2)$, we obtain an induced map:

$$\text{Spec } F_2 \longrightarrow \text{Spec } F_1 \longrightarrow G \times_k k'$$

in the natural way.

This shows that pullbacks of points by field inclusions are mapped by ϕ_* to similar pullbacks of points.

Second, it follows from standard arguments (cf. Proposition 2.3.2) that the following diagram commutes for any finite extension E_2/E_1. Here we define fields $L_{1,j}$ by $\bigoplus_j L_{1,j} = \dfrac{E_1 \otimes_k k'}{\mathfrak{N}}$ and $L_{2,m}$ by $\bigoplus L_{2,m} = \dfrac{E_2 \otimes_k k'}{\mathfrak{N}}$:

$$\begin{array}{ccc} G_i(E_2) & \longrightarrow & \bigoplus_m (G_i \times_k k')(L_{2m}) \\ \downarrow N & & \downarrow N \\ G_i(E_1) & \longrightarrow & \bigoplus_j (G_i \times_k k')(L_{1j}) \end{array}$$

From the above two facts, it is clear that the relation $\{\psi^*(g_1), \ldots, g_{i_0}, \ldots, \psi^*(g_r)\}_{E_2/k} - \{g_1, \ldots, N_{E_2/E_1}(g_{i_0}), \ldots, g_r\}_{E_1/k}$ is mapped to a sum of similar relations in $K(k', G_1 \times_k k', \ldots, G_r \times_k k')$.

Now consider $K \in \mathcal{T}_1(k)$, $h \in K^*$ and points $g_i \in G_i(K)$, $(i = 1, \ldots, r)$ giving rise to a relation

$$\sum_{v \in \mathcal{P}(K/k)} \{g_1(v), \ldots, \partial_v(g_{i_0}, h), \ldots, g_r(v)\}_{k(v)/k}$$

of type **R2** in $K(k; G_1, \ldots, G_r)$. We will prove that this maps to a sum of relations in $K(k'; G_1 \times_k k', \ldots, G_r \times_k k')$ which are also of type **R2**.

The assertion will follow if we show that any point $g_i(v) \in G_i(k(v))$ maps to a sum of points $g_{ij}(w_j) \in (G_i \times_k k')(k'(w_j))$, $j = 1, \ldots, s$, where w_j is an element of $\mathcal{P}(K'_j/k')$

for some $K'_j \in \mathcal{T}_1(k')$, $g_{ij} \in (G_i \times_k k')(K'_j)$, and that the symbol $\partial_v(g_i, h)$ maps to an appropriate sum of symbols $\partial_{w_j}(g_{ij}, h)$.

We first examine points of the form $g_i(v)$. Since we are working with one G_i at a time, we drop the subscript i and write $G = G_i$, $g = g_i$, etc.

Let L be the algebraic closure of k in k'. Then we may choose a field $L \subseteq M \subseteq k'$ such that M/L is purely transcendental and k'/M is algebraic. Thus we have a tower of fields

$$\begin{array}{l} k' \\ \big| \text{ algebraic} \\ M \\ \big| \text{ purely transc.} \\ L \\ \big| \text{ algebraic} \\ k \end{array}$$

Since the proposed base change morphism is clearly functorial in the field extension $k \hookrightarrow k'$, it suffices (by the above) to prove the proposition in the following two cases:

- k'/k is algebraic
- k'/k is separably generated, with k algebraically closed in k'

Before proceeding further, we state a lemma

Lemma 3.4.1. *For any $v \in \mathcal{P}(K/k)$, the ring $O_v \otimes_k k'$ is Noetherian.*

Proof.

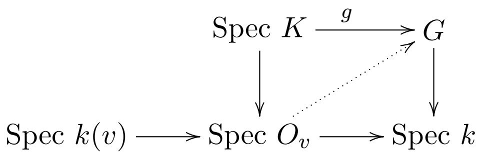
Let C be the smooth projective model for K over k and let $U = \text{Spec } A$ be an affine neighborhood of $v \in C$. Then O_v is a localization of A at some prime ideal $\mathfrak{p}$. Now A is a finitely generated k-algebra; therefore, $A \otimes_k k'$ is a finitely generated k'-algebra and is hence Noetherian. On the other hand, $O_v \otimes_k k'$ is equal to $T^{-1}(A \otimes_k k')$, where T is the image of $A - \mathfrak{p}$ under the natural map $A \longrightarrow A \otimes_k k'$. Since the localization of a Noetherian ring is Noetherian, this shows that $O_v \otimes_k k'$ is Noetherian.

Case 1a. k'/k is finite

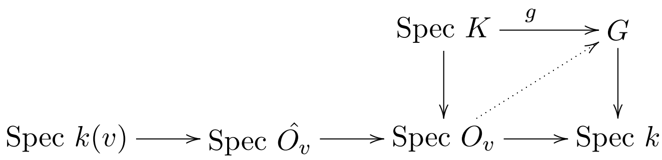
By functoriality of the base extension, we may assume further that k'/k is a simple extension and thus write:

$$k' = \frac{k[t]}{(f)}, \quad f(t) \in k[t] \text{ irreducible}$$

Fix $v \in \mathcal{P}(K/k)$ and $i \in \{1, \ldots, r\}$ such that $g(v) \in G(k(v))$. Cconsider the following diagram defining $g(v)$.

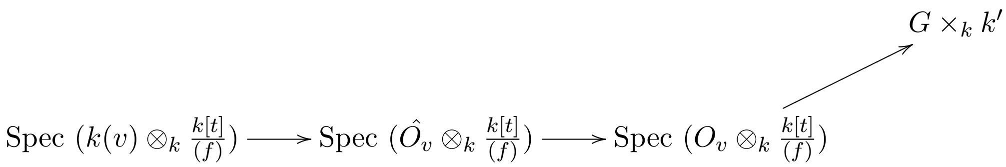


We may extend the diagram as follows, where ˆ denotes completion with respect to v.



By the Cohen Structure Theorem ([Ei 94], Proposition 10.16) we may write $\hat{O}_v = k(v)[[x]]$ for some indeterminate x.

Now taking the fiber product $- \times_k k'$ with a portion of the above diagram, we obtain a new diagram



To facilitate the proof in the case that k'/k is algebraic but not finite, we replace the ordinary tensor products $\hat{O}_v \otimes_k k'$ and $k(v) \otimes_k k'$ above with the completed tensor products (see [Br 66]) $\hat{O}_v \widehat{\otimes}_k k'$ and $k(v) \widehat{\otimes}_k k'$. We use completed tensor products because they commute with inverse limits (cf. [Br 66], Lemma A.4), unlike ordinary tensor products. By [Br 66], Lemma 2.1(i), finiteness of the extension $k(v)/k$ implies that

$$k(v) \widehat{\otimes}_k k' \cong k(v) \otimes_k k'$$

Now suppose $f(t) \in k[t]$ splits into irreducible elements as

$$f = f_1^{a_1} \cdot \ldots \cdot f_s^{a_s} \text{ in } O_v[t]$$

Note that this factorization is in fact unique (up to units), because O_v (and hence $O_v[t]$) is a unique factorization domain.

Then the total ring of quotients of $O_v \otimes_k k'$, denoted $TQ(O_v \otimes_k k')$ is isomorphic to the sum of quotient fields $\bigoplus_{i=1}^{s} QF(\frac{O_v[t]}{(f_i)}) \cong \bigoplus_{i=1}^{s} \frac{K[t]}{(f_i)}$ (cf. [Gre 78], Proposition 5.17(a)) Note that the latter expression is a direct sum of fields, since f_i, which is assumed to be irreducible in the unique factorization domain $O_v[t]$, remains irreducible in $K[t]$ by Gauss' Lemma. This shows in particular that $QF(\frac{O_v[t]}{(f_i)}) \cong \frac{K[t]}{(f_i)}$ is independent of v.

By Hensel's Lemma, the factorization of $f \in k(v)[[x]][t]$ into irreducible elements is

$$\phi_1^{e_1} \cdot \ldots \cdot \phi_s^{e_s}$$

such that $\phi_i^{b_i} = \bar{f}_i$, where the overhead bar denotes reduction modulo v and $e_i = a_i \cdot b_i$. Since $k(v) \subseteq \hat{O}_v \cong k(v)[[x]]$, the above factorization also holds in $\hat{O}_v[t]$. As above, the ring $TQ(\hat{O}_v \widehat{\otimes}_k k')$ is isomorphic to $\bigoplus_{i=1}^{s} QF(\frac{\hat{O}_v[t]}{(\phi_i)})$

Thus, if we divide the rings $k(v) \otimes_k \frac{k[t]}{(f)}$, $\hat{O}_v \widehat{\otimes}_k \frac{k[t]}{(f)}$, and $O_v \otimes_k \frac{k[t]}{(f)}$ by their nilradicals and use [Br 66], Lemma A.4, we obtain maps:

$$\begin{array}{ccccc}
\text{Spec } (\bigoplus_{i=1}^{s} \frac{k(v)[t]}{(\phi_i)}((x))) & \longrightarrow & \text{Spec } (\bigoplus_{i=1}^{s} QF(\frac{O_v[t]}{(f_i)})) & \longrightarrow & G \times_k k' \\
& \searrow & & \searrow & \uparrow \\
\text{Spec } (\bigoplus_{i=1}^{s} \frac{k(v)[t]}{(\phi_i)}) & \longrightarrow & \text{Spec } (\bigoplus_{i=1}^{s} \frac{k(v)[t]}{(\phi_i)}[[x]]) & \longrightarrow & \text{Spec } (O_v \otimes_k \frac{k[t]}{(f)})
\end{array}$$

Set $k'(w_i) = \frac{k(v)[t]}{(\phi_i)}$.

Now $k'(w_i)[[x]]$ is a discrete valuation ring with residue field $k'(w_i)$, and hence defines a valuation w_i on its (complete) quotient field $k'(w_i)((x))$. By means of the above diagram, this pulls back to a valuation on $K_i' = QF(\frac{O_v[t]}{(f_i)})$; this construction implies that $k'(w_i)((x))$ is isomorphic to the completion of K_i' at w_i.

Hence we have maps (for each i):

$$\begin{array}{ccccc}
& & \text{Spec } k'(w_i)((x)) & \longrightarrow \text{Spec } K_i' \xrightarrow{g_i'} & G \times_k k' \\
& & \downarrow & & \nearrow \\
\text{Spec } k'(w_i) & \longrightarrow & \text{Spec } \widehat{O_{w_i}} & &
\end{array}$$

Now, $g_i' \in G(K_i') \cap G(\widehat{O_{w_i}}) = G(O_{w_i})$, so we finally obtain a factorization

$$\begin{array}{ccccccc} & & \text{Spec } k'(w_i)((x)) & \longrightarrow & \text{Spec } K'_i & \xrightarrow{g'_i} & G \times_k k' \\ & & \downarrow & & \downarrow & \nearrow & \\ \text{Spec } k'(w_i) & \longrightarrow & \text{Spec } \widehat{O}_{w_i} & \longrightarrow & \text{Spec } O_{w_i} & & \end{array}$$

This shows that points $g(v) \in G(k(v))$ are mapped to points $g'_i(w_i) \in (G \times_k k')(k'(w_i))$, where w_i is a valuation extending v defined on some extension field K'_i of K.

Case 1b. k'/k is algebraic

In this case, the same argument may be used as in Case 1a; the only claim that remains to be verified in this case is that the map

$$\frac{O_v \otimes_k k'}{\mathfrak{N}} \xrightarrow{\beta_{k'}} \frac{\hat{O}_v \otimes_k k'}{\mathfrak{N}}$$

induces a map

$$TQ(\frac{O_v \otimes_k k'}{\mathfrak{N}}) \longrightarrow TQ(\frac{\hat{O}_v \otimes_k k'}{\mathfrak{N}})$$

We know from the argument of Case 1a that the corresponding map β_L has this property for L/k finite. To verify the above assertion, it suffices to check that $\beta_{k'}$ takes nonzerodivisors to nonzerodivisors. We do this via a limit argument.

Writing

$$k' = \overset{\lim}{\longrightarrow}_{L:k \subseteq L \subseteq k',\ L/k \text{ finite}} \ L$$

and noting that tensor products commute with direct limits, we have

$$O_v \otimes_k k' \cong \overset{\lim}{\longrightarrow} O_v \otimes_k L$$

It follows easily that

$$\frac{O_v \otimes_k k'}{\mathfrak{N}} \cong \overset{\lim}{\longrightarrow} \frac{O_v \otimes_k L}{\mathfrak{N}}$$

and similarly for $\hat{O}_v$.

Now if $\beta_{k'}(x)$ is a zerodivisor for some $x \in \dfrac{O_v \otimes_k k'}{\mathfrak{N}}$, it is a zerodivisor when considered as an element of $\dfrac{\hat{O}_v \otimes_k L}{\mathfrak{N}}$ for some finite extension L/k. Since we know that β_L takes nonzerodivisors to nonzerodivisors, x itself must have been a zerodivisor, which proves the assertion.

Case 2. k is separably generated, with k algebraically closed in k'

In this case, the proof is much easier. First fix $g \in G(K)$ and $v \in \mathcal{P}(K/k)$. Note that $B = O_v \otimes_k k'$ is a domain by [Ei 94], Ex. A1.2a, integrally closed by [Bou 85] V, 1.7, Proposition 19, and Noetherian by Lemma 3.4.1. Thus B is a Dedekind domain.

Consider the exact sequence

$$0 \longrightarrow m_v \longrightarrow O_v \longrightarrow k(v) \longrightarrow 0$$

of k-vector spaces

Since k' is flat over k, we obtain an exact sequence:

$$0 \longrightarrow m_v \otimes_k k' \longrightarrow B \longrightarrow k(v) \otimes_k k' \longrightarrow 0$$

Thus

$$k(v) \otimes_k k' \cong \frac{B}{m_v \otimes_k k'}$$

Since B is a Dedekind domain, $I = m_v \otimes_k k'$ factors uniquely as $\mathfrak{w}_1^{e_1} \cdot \ldots \cdot \mathfrak{w}_s^{e_s}$; hence

$$\frac{k(v) \otimes_k k'}{\mathfrak{N}} \cong \frac{B}{\mathfrak{w}_1} \oplus \cdot \oplus \frac{B}{\mathfrak{w}_s}$$

Hence the ideals $\mathfrak{w}_i$ correspond to valuations w_i on the fraction field of B, each of which extends the valuation v on K; from this it is easy to see that any point $g(v) \in G(k(v))$ is mapped to a sum of points $g_i(w_i) \in (G \times_k k')(k'(w_i))$ by the proposed morphism.

To complete the proof, we show that for any $g \in G(K)$ and $v \in \mathcal{P}(K/k)$, the local symbol $\partial_v(g, h)$ is mapped to a sum of local symbols $\partial_{w_i}(g_i, h)$, where $g_i \in (G \times_k k')(K'_i)$ and $w_i \in \mathcal{P}(K'_i/k')$.

Let notation be as above. By the discussion above, the k'-algebra $k(v) \otimes_k k'$ is a finite direct sum of Artin k'-algebras, each of dimension e_i over a field $k'(w_i)$, which coincides with the residue field of some discrete valuation w_i on an extension field K'_i/K. Choose any uniformizer π_v for v, and let $e(w_i/v) = w_i(\pi_v)$ be the ramification index.

Lemma 3.4.2. $e_i = e(w_i/v)$

Proof.

As before, we split the proof into cases.

In Case 2 above, the assertion is clear. We assume henceforth that we are in Case 1.

First suppose k'/k is finite and separable, hence monogenic. Then, as in Case 1a above, $k(v) \otimes_k k'$ splits as a direct sum of Artin rings, which in this case are fields, since k'/k is separable. That is, all of the e_i are equal to 1. Since an irreducible polynomial $f \in k[t]$ for k' is separable, it follows from our construction that $e(w_i/v) = 1$, too.

Next suppose k'/k is finite and purely inseparable; clearly, we may reduce to the case that $k' = \dfrac{k[t]}{(t^p - a)}$ for some $a \in k$. Let α be a root of $t^p - a$ in $\bar{k}$. Our constructions show that $e_i = p$ if and only if $\alpha \in k(v)$ if and only if $e(w_i/v) = p$. Since $e_i, e(w_i/v)$ can only assume values of 1, p, it is clear that $e_i = e(w_i/v)$

By multiplicativity of ramification indices in towers and functoriality of the various quantities, we obtain the desired result for all finite extensions k'/k.

To handle the case of an arbitrary algebraic extension, note that in $k(v) \otimes_k k' = \dfrac{k[t]}{(h)} \otimes_k k' \cong \dfrac{k'[t]}{(h)}$, h will split as $\phi_1^{e_1} \cdot \ldots \cdot \phi_s^{e_s}$ after some finite extension $k \subseteq L \subseteq k'$. Since we have the desired equality in the extension L/k and the e_i corresponding to any finite extension M/L are all equal to 1, it follows that all of the ramification indices associated to the base extension M/L must also be 1. We will use this fact to show that the ramification indices associated to the (algebraic but possibly not finite) base extension k'/L are all equal to 1.

Fix K_i' and a valuation w which extends a valuation u on any field K^L appearing as a summand of $\dfrac{K \otimes_k L}{\mathfrak{N}}$. Given any subfield $K^L \subseteq F \subseteq K_i'$ with F/K^L finite, let w_F denote the restriction of w to F. Then

$$e(w/u) = 1 \text{ iff } \frac{O_w}{(m_u O_w)} \text{ is reduced}$$

However,

$$O_w \cong \overset{\lim}{\longrightarrow}_{F/K_i' \text{ finite}} \; O_{w_F}$$

and direct limits commute with quotients; thus,

$$\frac{O_w}{(m_u O_w)} \cong \overset{\lim}{\longrightarrow} \frac{O_{w_F}}{m_u O_{w_F}}$$

Since $e(w_F/u) = 1$ for all such F, each ring in the limit on the right is reduced; therefore, the ring on the left is, too, This shows that $e(w/u) = 1$ and concludes the proof of Lemma 3.4.2.

Now we are well-equipped to show that the local symbols transform appropriately under the proposed base change morphism.

Given $K \in \mathcal{T}_1(k)$, $g \in G(K)$, $h \in K^*$, and $v \in \mathcal{P}(K/k)$, consider the symbol $\partial_v(g, h)$. It is easy to see from the definition that the symbol is multiplicative in either factor. We claim that $\partial_v(g, h)$ may be written as a product of symbols $\prod_{i=1}^n \partial_{w_i}((g_i)', h)$ where for each i, at least one of the statements $\mathrm{ord}_v(h) = 0$ or ($r_v^j(g_i) = 0$ for all $j = 1, \ldots, n$) holds.

Choose a finite unramified Galois extension L/K_v such that the torus T (in the exact sequence of group schemes defining G) splits over L. Then choose $\pi_v \in K^*$ such that $\mathrm{ord}_v(\pi_v) = 1$, and write $h = u \cdot \pi_v^b$, where $\mathrm{ord}_v(u) = 0$ and $b \in \mathbf{Z}$. Thus,

$$\partial_v(g, h) = \partial(g, u \cdot \pi_v^b) = \partial(g, u) \cdot (\partial_v(g, \pi_v))^b$$

Since the first term is already in the desired form, we concentrate on $\partial_v(g, \pi_v)$ Write $g = x \cdot (\pi_v^{c_1}, \ldots, \pi_v^{c_n})$, where the latter term is interpreted as an element of $T(L) \subseteq G(L)$ and $x \in G(L)$ is such that $r_v^j(x) = 0$ for all $j = 1, \ldots, n$. Then

$$\partial_v(g, \pi_v) = \partial_v(x, \pi_v) \cdot \partial_v((\pi_v^{c_1}, \ldots, \pi_v^{c_r}), \pi_v)$$

A routine verification (cf. Remark 2.1) shows that the last element is in fact equal (up to sign) to $\partial_v((\pi_v^{c_1}, \ldots, \pi_v^{c_r}), -1)$, which puts our element in the desired form.

Now in the case that $r_v^j(g) = 0$ for all $j = 1, \ldots, n$, we see that

$$\partial_v(g, h) = g^{\mathrm{ord}_v(h)}$$

transforms to a product of symbols

$$(g_i')^{e_i \cdot \mathrm{ord}_v(h)} = (g_i')^{e(w_i/v) \cdot \mathrm{ord}_v(h)} = (g_i')^{\mathrm{ord}_{w_i}(h)} = \partial_{w_i}(g_i', h)$$

In the case that $\mathrm{ord}_v(h) = 0$, we see that

$$\partial_v(g,h) = \prod_{j=1}^{n} h_j^{-r_v^j(g)}$$

transforms to a product of symbols

$$\prod_{j=1}^{n} h_j^{e_i\cdot(-r_v^j(g))}$$

which by Lemma 3.4.2

$$= \prod_{j=1}^{n} h_j^{-e(w_i/v)\cdot r_v^j(g)} = \prod_{j=1}^{n} h_j^{-r_{w_i}^j(g_i')} = \partial_{w_i}(g_i', h)$$

This concludes the verification that the base change homomorphism is well-defined.

3.4.2 Change of semi-abelian varieties

Suppose $f_i : G_i \longrightarrow G_i'$, $i = 1, \ldots, r$ are morphisms of semi-abelian varieties over k. Then for each k-algebra A/k and each $i = 1, \ldots r$, f_i induces a map $f_i^A : G_i(A) \longrightarrow G_i'(A)$, covariant functorial in A. These maps can be used to define a homomorphism

$$K(k; G_1, \ldots, G_r) \xrightarrow{f} K(k; G_1', \ldots, G_r')$$

by

$$\{a_1, \ldots, a_r\}_{E/k} \mapsto \{f_1^E(a_1), \ldots, f_r^E(a_r)\}_{E/k}$$

To verify that this map is well-defined, consider finite extensions $k \longrightarrow E_1 \xrightarrow{\phi} E_2$ and a relation

$$\{\phi^*(g_1) \ldots, g_{i_0}, \ldots, \phi^*(g_r)\}_{E_2/k} - \{g_1, \ldots, N_{E_2/E_1}(g_{i_0}), \ldots, g_r\}_{E_1/k}$$

of type **R1**. Since ϕ^* and $\phi_* = N_{E_2/E_1}$ commute with f, the above relation gets mapped by f to the relation

$$\{\phi^*(f_1^{E_1}(g_1)), \ldots, f_{i_0}^{E_2}(g_{i_0}), \ldots, \phi^*(f_r^{E_1}(g_r))\}_{E_2/k}$$

$$-\{f_1^{E_1}(g_1), \ldots, N_{E_2/E_1}(f_{i_0}^{E_2}(g_{i_0})), \ldots, f_r^{E_1}(g_r)\}_{E_1/k}$$

which is also of type **R1**.

Now fix choices of $K \in \mathcal{T}_1(k)$, $h \in K^*$ and $g_i \in G_i(K)$ and consider a relation

$$\sum_{v \in \mathcal{P}(K/k)} \{g_1(v), \ldots, \partial_v(g_{i(v)}, h), g_r(v)\}_{k(v)/k}$$

of type **R2**. Note that for any $v \in \mathcal{P}(K/k)$, we clearly have $f_i^{O_v}(G_i(O_v)) \subseteq G_i'(O_v)$; in particular, $f_i^{k(v)}(g(v)) = (f_i^K(g))(v)$. It remains to show

Proposition 3.4.3.

$$f_{i(v)}^{k(v)}(\partial_v(g_{i(v)}, h)) = \partial_v(f_{i(v)}^K(g), h)$$

The following two lemmas will be helpful in analyzing the maps f_i:

Lemma 3.4.4. *Let*

$$\mathbf{G}_m \xrightarrow{\psi} \mathbf{G}_m$$

be a morphism of group schemes over k. Then ψ is the map $x \mapsto x^n$ for some $n \in \mathbf{Z}$.

Proof.

Since $\mathbf{G}_m = \text{Spec } k[T, T^{-1}]$, a k-morphism $\mathbf{G}_m \longrightarrow \mathbf{G}_m$ of group schemes corresponds to a homomorphism

$$k[T, T^{-1}] \xrightarrow{\psi^*} k[T, T^{-1}]$$

of k-algebras.

Since T is a unit in $k[T, T^{-1}]$, $\psi^*(T)$ is also a unit of $k[T, T^{-1}]$. If $\psi^*(T) \in k$, then also $\psi^*(T^{-1}) \in k$ and ψ is a constant map (the case $n = 0$ above). If $\psi^*(T) \notin k$, then $\psi^*(T) = cT^n$ for some $n \neq 0$, since the only units in $k[T, T^{-1}]$ are elements of the form cT^n. However, since ψ is a map of group schemes, the identity section must be preserved, which forces $c = 1$.

Lemma 3.4.5. *Let $G \xrightarrow{\alpha} G'$ be a morphism of semi-abelian varieties defined over k. Let*

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow T' \longrightarrow G' \longrightarrow A' \longrightarrow 0$$

be the exact sequences (of group schemes over k) defining G and G' respectively. Let a and a' be determined by

$$T \times_k \bar{k} \cong \mathbf{G}_m^a \text{ and } T' \times_k \bar{k} \cong \mathbf{G}_m^{a'}$$

Fix choices of $K \in \mathcal{T}_1(k)$ and $v \in \mathcal{P}(K/k)$. Then there exist nonnegative integers s_{ij}, $i = 1, \ldots, a$, $j = 1, \ldots, a'$, such that for any $b \in G(L)$,

$$r_v^j(\alpha(b)) = \sum_{i=1}^{a} s_{ij} \cdot r_v^i(b)$$

Proof.

Since α may be interpreted as a morphism of the exact sequences defining G and G', choose a field L as in Section 3.3.1; then we have a commutative square

$$\begin{array}{ccc} T(L) \cong (\mathbf{G}_m(L))^a & \longrightarrow & G(L) \\ \downarrow \alpha & & \downarrow \alpha \\ T'(L) \cong (\mathbf{G}_m(L))^{a'} & \longrightarrow & G'(L) \end{array}$$

By Lemma 3.4.4, the left vertical arrow sends an element $l = (l_1, \ldots, l_a) \in (\mathbf{G}_m(L))^a$ to $(\prod_{i=1}^a l_i^{s_{i1}}, \ldots, \prod_{i=1}^a l_i^{s_{ia'}})$ for some integers s_{ij}, $i = 1, \ldots, a$, $j = 1, \ldots, a'$. Thus,

$$\mathrm{ord}_v^j(\alpha(l)) = \sum_{i=1}^{a} s_{ij} \cdot \mathrm{ord}_v^i(l)$$

(We use the notation ord_v^i to refer to the map ord_v on the ith summand of $(\mathbf{G}_m)^a$; similarly for ord_v^j).

Now let $b \in G(L)$ be arbitrary. Choose $l = (l_1, \ldots, l_a) \in T(L)$ such that $\mathrm{ord}_v^i(l) = -r_v^i(b)$. Then $bl \in G(O_v)$; thus,

$$r_v^j(\alpha(b)) = r_v^j(\alpha(bl \cdot l^{-1})) = r_v^j(\alpha(bl)) + r_v^j(\alpha(l^{-1}))$$

$$= 0 + \mathrm{ord}_v^j(\alpha(l^{-1})) = \sum_{i=1}^{r} s_{ij} \cdot (-\mathrm{ord}_v^i(l)) = \sum_{i=1}^{r} s_{ij} \cdot r_v^i(b)$$

which proves Lemma 3.4.5.

We now give a proof of Proposition 3.4.3 by computation:

$$f_{i(v)}^{k(v)}(\partial_v(g,h)) = f_{i(v)}^{k(v)}(((-1)^{\mathrm{ord}_v(h) r_v^1(g)} h^{-r_v^1(g)}, \ldots, (-1)^{\mathrm{ord}_v(h) r_v^a(g)} h^{-r_v^a(g)}) g^{\mathrm{ord}_v(h)})$$

By Lemma 3.4.5 this expression equals:

$$= (\prod_{i=1}^{a} (-1)^{\mathrm{ord}_v(h) r_v^i(g) s_{i1}} h^{-s_{i1} r_v^i(g)}, \ldots, \prod_{i=1}^{a} (-1)^{\mathrm{ord}_v(h) r_v^i(g) s_{ia'}} h^{-s_{ia'} r_v^i(g)}) (f^K(g))^{\mathrm{ord}_v(h)}$$

$$= (-1)^{\mathrm{ord}_v(h) r_v^1(f^K(g))} h^{-r_v^1(f^K(g))}, \ldots, (-1)^{\mathrm{ord}_v(h) r_v^{a'}(f^K(g))} h^{-r_v^{a'}(f^K(g))}) (f^K(g))^{\mathrm{ord}_v(h)}$$

$$= \partial_v(f_{i(v)}^K(g), h)$$

Galois actions.

Suppose k'/k is a finite extension of fields and $G_1, \ldots, G_r$ semi-abelian varieties defined over k. Then $K(k'; G_1 \times_k k', \ldots, G_r \times_k k')$ is a Gal (k'/k)-module in the following sense: Any $\sigma \in$ Gal (k'/k) is an automorphism $k' \xrightarrow{\sigma} k'$ This induces an invertible morphism Spec $k' \xrightarrow{\sigma^*}$ Spec k' and by the mapping property of fiber products, invertible morphisms

$$G_i \times_k k' \xrightarrow{\sigma_i^*} G_i \times_k k'$$

for each $i = 1, \ldots, r$. By the theory of 3.4.2, these in turn induce an automorphism

$$K(k'; G_1 \times_k k', \ldots, G_r \times_k k') \xrightarrow{\sigma^*} K(k'; G_1 \times_k k', \ldots, G_r \times_k k')$$

3.4.3 Contravariant Functoriality

Suppose k'/k is a finite extension of fields. Then one may define a map

$$N_{k'/k} : K(k'; G_1 \times_k k', \ldots, G_r \times_k k') \longrightarrow K(k; G_1, \ldots, G_r)$$

by

$$\{a_1, \ldots, a_r\}_{E/k'} \mapsto \{a_1, \ldots, a_r\}_{E/k}$$

It is obvious that relations of type **R1** map to similar relations. We assert that a relation

$$\sum_{v \in \mathcal{P}(K/k)} \{g_1(v), \ldots, \partial_v(g_{i(v)}, h), g_r(v)\}_{k(v)/k}$$

of type **R2** maps to another relation of type **R2** arising from the same $K \in \mathcal{T}_1(k') \subseteq \mathcal{T}_1(k)$, $h \in K^*$ and $g_i^k \in G_i(K)$, where each g_i^k is the composition Spec $K \xrightarrow{g_i} G \times_k k' \longrightarrow G$. The only assertion that needs to be checked is that every $v \in \mathcal{P}(K/k)$ arises from some $v \in \mathcal{P}(K/k')$; that is, any place v of K such that $v(k) = 0$ also satisfies $v(k') = 0$.

To this end, consider $v \in \mathcal{P}(K/k)$, and fix $y \in k' - k$; we will show that $v(y) = 0$. Suppose to the contrary that $v(y) \neq 0$; we may assume without loss of generality that $v(y) > 0$. Then since k'/k is finite, any $y \in k' - k$ satisfies

$$y^r + \ldots + a_1 y + a_0 = 0$$

for some $r \geq 1$ and $a_0, \ldots, a_{r-1} \in k$, $a_0 \neq 0$. Thus $r \cdot v(y) = v(y^r) = v(-a_{r-1}y^{r-1} - \ldots - a_0) = v(-a_{r-1}y^{r-1}) = v(-a_{r-1}) + v(y^{r-1}) = (r-1)v(y)$. This forces $v(y) = 0$, which yields a contradiction.

3.5 Elementary properties

Here we make some general remarks on Somekawa K-groups.

Proposition 3.5.1. *Let k be a field, and $G_1, \ldots, G_r$ semi-abelian varieties defined over* *k such that* $G_i = 0$ *for some* i*. Then*

$$K(k; G_1, \ldots, G_r) = 0$$

Proof.

If $G_i = 0$, then $G_i(E) = 0$ for all E, so $K(k; G_1, \ldots, G_r)$ has no generators.

The next result follows immediately from the definitions.

Proposition 3.5.2. *If* k *is a field and* $G_1, \ldots, G_r$ *are semi-abelian varieties defined over* k*, then for any permutation* σ *of the set* $\{1, \ldots, r\}$ *there is a canonical isomorphism*

$$K(k; G_1, \ldots, G_r) \cong K(k; G_{\sigma_1}, \ldots, G_{\sigma_r})$$

Proposition 3.5.3. *Let* k *be any field and* $G_1, G_2, \ldots, G_r, G'$ *semi-abelian varieties defined over* k*. Suppose* $G_i \xrightarrow{f} G'$ *is a surjective map for some* i*. Then the induced map*

$$K(k; G_1, \ldots, G_i, \ldots, G_r) \otimes \mathbf{Q} \longrightarrow K(k; G_1, \ldots, G', \ldots, G_r) \otimes \mathbf{Q}$$

is surjective.

Proof.

Given a symbol $\{g_1, \ldots, g', \ldots, g_r\}_{E/k} \in K(k; G_1, \ldots, G', \ldots, G_r)$, choose $g_i \in G_i$ such that $f(g_i) = g'$ (Here we identify g' with a closed point of G'). Let F/E be an extension containing the residue field $k(g_i)$; let $i : k(g_i) \hookrightarrow F$ and $i' : E \hookrightarrow k(g_i) \hookrightarrow F$ denote the obvious inclusions.

Then the induced map takes

$$\{\phi^*(g_1), \ldots, i^*(g_i), \ldots, \phi^*(g_r)\}_{F/k} \otimes \frac{1}{[F:E]} \mapsto$$

$$\{\phi^*(g_1), \ldots, (i')^*(g'), \ldots, \phi^*(g_r)\}_{F/k} \otimes \frac{1}{[F:E]} \mapsto$$

$$\{g_1, \ldots, g', \ldots, g_r\}_{E/k} \otimes 1$$

If we assume that the field k is algebraically closed, we do not need to tensor with $\mathbf{Q}$.

Proposition 3.5.4. *Given the notation and hypotheses of Proposition 3.5.3, if we assume further that k is algebraically closed, then the induced map:*

$$K(k;G_1,\ldots,G_r) \longrightarrow K(k;G_1,\ldots,G',\ldots,G_r)$$

is surjective.

Proposition 3.5.5. *Let k be a field, $G_1,\ldots,G_r$ semi-abelian varieties defined over k and $k \overset{\phi}{\hookrightarrow} k'$ a finite extension of fields. Then the kernel of the base change homomorphism*

$$K(k;G_1,\ldots,G_r) \xrightarrow{\phi_*} K(k';G_1 \times_k k',\ldots,G_r \times_k k')$$

(cf. 3.4.1) is $[k':k]$-torsion.

Proof.
If $\phi_*(x) = 0$, then $[k':k]x = N_{k'/k}(\phi_*(x)) = 0$, as desired.

Proposition 3.5.6. *Let k be a field and $H_1, G_1,\ldots,G_r$ semi-abelian varieties defined over k. Then there are canonical maps (in both directions) establishing an isomorphism*

$$K(k;H_1 \times_k G_1, G_2,\ldots,G_r) \cong K(k;H_1,G_2,\ldots,G_r) \oplus K(k;G_1,G_2,\ldots,G_r)$$

Proof.
There are natural projection maps

$$\pi_H : H_1 \times_k G_1 \longrightarrow H_1$$

$$\pi_G : H_1 \times_k G_1 \longrightarrow G_1$$

and inclusion maps

$$i_H : H_1 \hookrightarrow H_1 \times_k G_1$$

$$h \mapsto (h,0)$$

and

$$i_G : G_1 \hookrightarrow H_1 \times_k G_1$$

$$g \mapsto (0, g)$$

all of which are morphisms of semi-abelian varieties.

Then setting $\alpha = (\pi_H)_* \oplus (\pi_G)_*$ and $\beta = ((i_H)_*, 0) + (0, (i_G)_*)$, we have induced maps (cf. 3.4.2):

$$K(k; H_1 \times_k G_1, G_2, \ldots, G_r) \xrightarrow{\alpha} K(k; H_1, G_2, \ldots, G_r) \oplus K(k; G_1, G_2, \ldots, G_r)$$

$$K(k; H_1 \times_k G_1, G_2, \ldots, G_r) \xleftarrow{\beta} K(k; H_1, G_2, \ldots, G_r) \oplus K(k; G_1, G_2, \ldots, G_r)$$

Calculating with symbols,

$$\alpha \circ \beta(\{h_1, \ldots, h_r\}_{E/k}, \{g_1, \ldots, g_r\}_{F/k})$$

$$= \alpha(\{(h_1, 0), \ldots, h_r\}_{E/k} + \{(0, g_1), \ldots, g_r\}_{F/k})$$

$$= (\{h_1, \ldots, h_r\}_{E/k}, \{g_1, \ldots, g_r\}_{F/k})$$

Similarly,

$$\beta \circ \alpha(\{(h_1, g_1), g_2, \ldots, g_r\}_{E/k})$$

$$= \beta(\{h_1, g_2, \ldots, g_r\}_{E/k}, \{g_1, g_2, \ldots, g_r\}_{E/k})$$

$$= \{(h_1, 0) + (0, g_1), g_2, \ldots, g_r\}_{E/k}$$

$$= \{(h_1, g_1), g_2, \ldots, g_r\}_{E/k}$$

Therefore β and α are inverses, as desired.

3.6 Relation to Milnor K-theory

The following theorem due to Somekawa (Theorem 1.4 of [So]) shows that the Somekawa K-groups indeed generalize the Milnor K-groups and makes precise the remarks of Section 3.1.

Theorem 3.6.1. *Let k be a field and $r \geq 1$. Then there exists a canonical isomorphism*

$$K_r^M(k) \xrightarrow{\cong} K_r(k; \mathbf{G}_m)$$

Proof.

The strategy of the proof is to construct homomorphisms

$$\alpha : K_r^M(k) \longrightarrow K_r(k; \mathbf{G}_m)$$

and

$$\beta : K_r(k; \mathbf{G}_m) \longrightarrow K_r^M(k)$$

and show that these maps are inverse to each other.

Let

$$\alpha(\{a_1, \dots, a_r\}) = \{a_1, \dots, a_r\}_{k/k}$$

To show that this rule is well-defined, we need to show that $\alpha(\{a_1, \dots, a_r\}_{k/k}) = 0$ if $a_i + a_j = 1$ for some $i < j$. The desired result follows from the following slightly stronger lemma:

Lemma 3.6.2. *Set $G_1 = \dots = G_r = \mathbf{G}_m$ and suppose $G_{r+1}, \dots, G_{r+s}$ are any semi-abelian varieties defined over k. In the group $K(k; \underbrace{\mathbf{G}_m, \dots, \mathbf{G}_m}_{r}, G_{r+1}, \dots, G_{r+s})$, every relation of the form $\{a_1, \dots, a_r, b_1, \dots, b_s\}_{E/k}$, where $a_i + a_j = 1$ for some $i < j$ and E/k is any finite extension, is equivalent to zero.*

To this end, consider the relation of type **R2** corresponding to the following data:

- $K = E(T)$
- $h = T^{-1} \in K^*$
- $g_i = 1 - a_i T^{-1} = T^{-1}(T - a_i)$
- $g_j = 1 - T$
- $g_l = a_l$ for $l \neq i, j$, $1 \leq l \leq r$

- $g_m = b_{m-r}$ for $r+1 \leq m \leq r+s$

The only places $v \in \mathcal{P}(E(T)/k)$ such that the local symbol $\partial_v(g_{i(v)}, h)$ and hence the expression $\{g_1(v), \ldots, \partial_v(g_{i(v)}, h), \ldots, g_{r+s}(v)\}_{k(v)/k}$ may be nonzero correspond either to the valuation "at infinity" (that is, $v_\infty(g(T)) = -\deg g$), or to the various valuations defined by the polynomials $(T-1)$, T, $(T-a_i)$.

Calculating $\{g_1(v), \ldots, \partial_v(g_{i(v)}, h), \ldots, g_{r+s}(v)\}_{k(v)/k}$ explicitly in each of these cases:

$$\{g_1(v), \ldots, \partial_v(g_{i(v)}, h), \ldots, g_{r+s}(v)\}_{k(v)/k}$$

$$= \begin{cases} \{a_1, \ldots, a_r, b_1, \ldots, b_s\}_{E/k} \text{ if } v = (T-a_i) \\ 0 \text{ otherwise} \end{cases}$$

Thus

$$\{a_1, \ldots, a_r\}_{E/k} = \sum_{v \in \mathcal{P}(K/k)} \{g_1(v), \ldots, \partial_v(g_{i(v)}, h), \ldots, g_{r+s}(v)\}_{k(v)/k}$$

and so α is well-defined.

We now define

$$\beta(\{a_1, \ldots, a_r\}_{E/k}) = N^M_{E/k}(\{a_1, \ldots, a_r\})$$

where $N^M_{E/k} : K^M_r(E) \longrightarrow K^M_r(k)$ is the norm map on Milnor K-theory.

Towards a verification that β is well-defined, first consider relations of type **R1**. Given a finite extension $E_1 \overset{\phi}{\hookrightarrow} E_2$,

$$\beta(\{\phi^*(a_1), \ldots, a_i, \ldots, \phi^*(a_r)\}_{E_2/k}) = N^M_{E_2/k}(\{\phi^*(a_1), \ldots, a_i, \ldots, \phi^*(a_r)\})$$

$$= N^M_{E_1/k} \circ N^M_{E_2/E_1}(\{\phi^*(a_1), \ldots, a_i, \ldots, \phi^*(a_r)\})$$

By the projection formula 2.2.1 for Milnor K-theory, this is equal to

$$N^M_{E_1/k}(\{a_1, \ldots, N_{E_2/E_1}(a_i), \ldots, a_r\})$$

which in turn equals

$$\beta(\{a_1, \ldots, N_{E_2/E_1}(a_i), \ldots, a_r\}_{E_1/k})$$

Next, given $K \in \mathcal{T}_1$, $h \in K^*$, and $g_1, \dots, g_r \in \mathbf{G}_m(K) = K^*$ with the property that for all $v \in \mathcal{P}(K/k)$ there exists $i(v)$ such that $g_i \in \mathbf{G}_m(O_v) = O_v^*$ for all $i \neq i(v)$, and the associated relation of type **R2**, we calculate (using additive notation):

$$\beta\Big(\sum_{v \in \mathcal{P}(K/k)} \{g_1(v), \dots, \partial_v(g_{i(v)}, h), \dots, g_r(v)\}_{k(v)/k}\Big)$$

$$= \sum_{v \in \mathcal{P}(K/k)} N^M_{k(v)/k}(\{g_1(v), \dots, \partial_v(g_{i(v)}, h), \dots, g_r(v)\})$$

By anticommutativity of the Milnor ring (see second remark following Definition 2.1.2),

$$= \sum_{v \in \mathcal{P}(K/k)} (-1)^{i(v)-1} N^M_{k(v)/k}(\{\partial_v(g_{i(v)}, h), g_1(v), \dots, \widehat{g_i(v)}, \dots, g_r(v)\})$$

$$= \sum_{v \in \mathcal{P}(K/k)} (-1)^{i(v)-1} N^M_{k(v)/k}(\{\partial_v(g_{i(v)}, h, g_1, \dots, \widehat{g_{i(v)}}, \dots, g_r)\})$$

Using anticommutativity again,

$$= \sum_{v \in \mathcal{P}(K/k)} -N^M_{k(v)/k}(\partial_v\{h, g_1, \dots, g_{i(v)}, \dots, g_r\})$$

$$= 0$$

by the reciprocity law for Milnor K-groups (2.2.6).
Thus β is well-defined.

It is easy to see that $\beta \circ \alpha = id$; therefore to show that $\alpha \circ \beta = id$, it suffices to show that α is surjective or that β is injective.

Case 1: k has property E_p (cf. 2.2.2).
Given an extension E/k and an element $x_s = \{x_1, \dots, x_r\}_{E/k}$, our aim is to prove, by induction on $\log_p[E : k]$, that α maps onto x_s.
If $[E : k] = 1$ the assertion is obvious. If $[E : k] = p^r$, choose (by Proposition 2.2.5) an intermediate field F such that $[E : F] = p$. Proposition 2.2.3 then shows that the element $x = \{x_1, \dots, x_r\} \in K^M_r(E)$ may be written as a product of elements of the form

$y_i = \{y_{i_1}, \phi_*(z_{i_2}), \ldots, \phi_*(z_{i_r})\}$, where $y_{i_1} \in K_r^M(E)$, $z_{i_j} \in K_r^M(F)$, and $\phi : F \hookrightarrow E$ is the inclusion map. Since $x_s \in K_r(k; \mathbf{G}_m)$ is the image of $x = \{x_1, \ldots, x_r\}$ under the composed homomorphism

$$K_r^M(E) \xrightarrow{\alpha_E} K_r(E; \mathbf{G}_m) \xrightarrow{N_{E/k}} K_r(k; \mathbf{G}_m)$$

we conclude that x_s may be written as a product of elements of the form $(y_i)_s = \{y_{i_1}, \phi^*(z_{i_2}), \ldots, \phi^*(z_{i_r})\}_{E/k}$. However, the relations of type **R1** in $K_r(k; \mathbf{G}_m)$ imply that

$$\{y_{i_1}, \phi^*(z_{i_2}), \ldots, \phi^*(z_{i_r})\}_{E/k} = \{N_{E/F}(y_{i_1}), z_{i_2}, \ldots, z_{i_r}\}_{F/k}$$

By induction, the element on the right hand side is in the image of α; since x_s is a product of such elements, it follows that α maps onto x_s. Thus α is surjective.

Case 2: k is arbitrary.

This time our strategy is to prove that β is injective. Fix a prime p, and choose (using Proposition 2.2.4) an algebraic extension $i : k \hookrightarrow k(p)$ such that $k(p)$ has property E_p.

Proposition 3.6.3. *The following diagram commutes:*

$$\begin{array}{ccc}
K_r(k(p); \mathbf{G}_m) & \xrightarrow{\beta_{k(p)}} & K_r^M(k(p)) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\
K_r(k; \mathbf{G}_m) & \xrightarrow{\beta} & K_r^M(k)
\end{array}$$

Proof.

Write $E \otimes_k k(p) = \bigoplus_{j=1}^n A_j$, where A_j is an Artin $k(p)$-algebra of length e_j with residue field E_j. Let $i_j : E \hookrightarrow E_j$ denote the inclusion maps. Then

$$\beta_{k(p)} \circ i_*(\{a_1, \ldots, a_r\}_{E/k})$$

$$= \sum_{j=1}^n N^M_{E_j/k(p)}(e_j\{(i_j)_*(a_1), \ldots, (i_j)_*(a_r)\})$$

By Proposition 2.3.2, this is the same as

$$= i_*(N^M_{E/k}\{a_1, \ldots, a_r\})$$

$$= i_* \circ \beta(\{a_1, \ldots, a_r\}_{E/k})$$

Now suppose x is an element such that $\beta(x) = 0$. By commutativity of the diagram above, $\beta_{k(p)}(i_*(x)) = 0$. However, $\beta_{k(p)}$ is an isomorphism by Case 1; therefore, $i_*(x) = 0$.

Lemma 3.6.4. *Let notation be as above. There exists a finite extension $i_F : k \hookrightarrow F$ contained in $k(p)$ such that $(i_F)_*(x) = 0$.*

Proof.

Since $(i_F)_*(x) = 0 \in K_r(k(p); \mathbf{G}_m)$, it must be a sum of relations of type **R1** and **R2**. We show that any such relation comes from a relation at a "finite stage" of the direct limit; that is, it is the image of a relation in $K_r(F; \mathbf{G}_m)$ for some finite F/k under the canonical map.

Consider finite extensions $k(p) \hookrightarrow E_1(p) \overset{\phi}{\hookrightarrow} E_2(p)$, elements $g_1, \ldots, \hat{g}_i, \ldots, g_r \in E_1(p)^*$, $g_i \in E_2(p)^*$, and a relation

$$\{\phi^*(g_1), \ldots, g_i \ldots, \phi^*(g_r)\}_{E_2(p)/k(p)} - \{g_1, \ldots, N_{E_2(p)/E_1(p)} g_i, \ldots, g_r\}_{E_1(p)/k(p)}$$

of type **R1**. Since the field $k(p)$ has property E_p, we assume by functoriality and Proposition 2.2.5 that $[E_2(p) : E_1(p)] = p$.

The field $E_1(p)$ may be obtained from $k(p)$ by a sequence of s monogenic extensions $k(p) = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_s = E_1(p)$; let $f_j \in D_{j-1}[t]$, $j = 1, \ldots, s$, be an irreducible polynomial such that $D_j \cong \dfrac{D_{j-1}[t]}{(f_j)}$. Let $h \in E_1(p)[t]$ be an irreducible polynomial for g_i if $g_i \in E_2(p) - E_1(p)$, or any polynomial which generates $E_2(p)$ over $E_1(p)$ if $g_i \in E_1(p)$. In either case, $E_2(p) \cong \dfrac{E_1(p)[t]}{(h)}$.

Now choose a finite extension F of k such that $g_m \in F^*$ for all $m = 1, \ldots, \hat{i}, \ldots, r$, $g_i \in F^*$ if $g_i \in E_1(p)^*$, all the coefficients of the polynomials f_j lie in F, and all the coefficients of h lie in F. Define a sequence of fields $F = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_s$ by $C_j = \dfrac{C_{j-1}[t]}{(f_j)}$; set $F_1 = C_s$. Write $F_2 = \dfrac{F_1[t]}{(h)}$. (Note that the polynomials f_j are still irreducible) From now on, we view $g_1, \ldots, \widehat{g_i}, \ldots, g_r$ as elements of F^*.

Letting $\phi_F : F_1 \hookrightarrow F_2$ denote the inclusion map, the relation of $K_r(F; \mathbf{G}_m)$ given by

$$\{\phi_F^*(g_1), \ldots, g_i \ldots, \phi_F^*(g_r)\}_{F_2/F} - \{g_1, \ldots, N_{F_2/F_1} g_i \ldots, g_r\}_{F_1/F}$$

maps to the original relation by commutativity of the following diagram (cf. 3.4.1), where the horizontal maps are induced by the base extension $F \hookrightarrow k(p)$:

$$\begin{array}{ccc} \mathbf{G}_m(F_2) & \longrightarrow & \mathbf{G}_m(E_2(p)) \\ \downarrow N & & \downarrow N \\ \mathbf{G}_m(F_1) & \longrightarrow & \mathbf{G}_m(E_1(p)) \end{array}$$

Definition 3.6.5. *Suppose C, L, M, N are fields such that $C \subseteq L$ and $C \subseteq M \subseteq N$, with L/C and M/C finite and N/C algebraic. Write $\dfrac{L \otimes_C M}{\mathfrak{N}}$ as a direct sum $\bigoplus_{i=1}^{s} F_i$ of fields; likewise, write $\bigoplus_{i=1}^{s} F_i \otimes_M N$ as a direct sum $\bigoplus_{j=1}^{u} A_j$ of Artin local rings. The field L is called N-decomposed in M if $s = u$, the A_j are all fields, and there exists some permutation σ of $\{1, \ldots, s\}$ such that $F_i \otimes_M N \cong A_{\sigma(i)}$.*

Remark.

This definition is much easier to understand if we assume that L/C is monogenic. Write $L = \dfrac{C[t]}{(h)}$ for some irreducible polynomial $h \in C[t]$. Viewing h as a polynomial in $M[t]$, let $h = h_1^{e_1} \cdot \ldots \cdot h_s^{e_s}$ be the decomposition of h into distinct irreducible elements in $M[t]$, and let $h = \tilde{h}_1^{f_1} \cdot \ldots \cdot \tilde{h}_u^{f_u}$ be the decomposition of h in $N[t]$. Then L being N-decomposed in M simply means that all of the splitting and ramification which occurs when we view h (the generating polynomial for L over C) as a polynomial in $N[t]$ already occurs when we view it as a polynomial in $M[t]$.

Remark. For every C, L, N as above, it is clear that there exists a finite extension M/C such that L is N-decomposed in M.

Remark.

If L is N-decomposed in C then $L \otimes_C N \cong LN$.

Now consider the following relation in $K_r(k(p); \mathbf{G}_m)$ of type **R2**

$$\sum_{v \in \mathcal{P}(K(p)/k(p))} \{g_1(v), \ldots, \partial_v(g_{i(v)}, h), g_r(v)\}_{(k(p))(v)/k(p)}$$

corresponding to the data $K(p) \in \mathcal{T}_1(k(p))$, $g_1, \ldots, g_r \in K(p)^*$ and $h \in K(p)^*$. We may write $K(p) = k(p)(t)(\alpha_1, \ldots, \alpha_m)$ where t is transcendental over k and $\alpha_1, \ldots, \alpha_m$ are algebraic over $k(p)(t)$.

Choose a finite extension F/k contained in $k(p)$, large enough so that the following hold:

- $\alpha_1, \ldots \alpha_r$ are algebraic over $F(t)$
- Write $x = \sum_{i=1}^{s} \{b_1, \ldots, b_r\}_{E_i/k} \in K_r(k; \mathbf{G}_m)$. (Here x is the element referred to in the statement of Lemma 3.6.4) Then each E_i is $k(p)$-decomposed in F.
- $g_1, \ldots, g_r, h \in F(t)(\alpha_1, \ldots, \alpha_m)$.

Now for each $v \in \mathcal{P}(K(p)/k(p))$, let v_F denote the restriction of v to $F(t)(\alpha_1, \ldots, \alpha_m)$. We denote by $F(v)$ the residue field with respect to this valuation. The theory of Section 3.4.1 tells us that the valuation v_F has only finitely many extensions to a place of $K(p)$. By taking F to be a larger extension, we may assume without loss of generality that for all those (finitely many) v yielding a nontrivial term in the above relation, v is the unique extension of v_F to a place of $K(p)$. In other words, $F(v)/F$ is $k(p)$-decomposed in F.

Fix a choice of $v \in \mathcal{P}(K(p)/k(p))$, and choose a basis $b_1, \ldots, b_n$ for $F(v)/F$. Clearly the above basis also serves as a basis for the M-vector space $MF(v) = F(v) \otimes_F M$ where M is any algebraic extension M/F contained in $k(p)$.

Since x maps to 0 under the map $(i_{k(p)})_*$ of K-groups induced by the inclusion $k \hookrightarrow k(p)$, some symbol $\{a_1, \ldots, a_r\}_{L/F} \in K_r(F; \mathbf{G}_m)$ in the expression for $(i_F)_*(x)$ maps to $\{g_1(v), \ldots, g_r(v)\}_{(k(p))(v)/k(p)}$. By our choice of F, we assumed that for all (finitely many) such symbols, L is $k(p)$-decomposed in F. Then $L \otimes_F k(p) \cong Lk(p) \cong k(p)(v)$. Since $F(v) \subseteq k(p)(v)$, , $Lk(p)$ contains the aforementioned basis $b_1, \ldots, b_n$, which means that LM contains the basis for some finite extension M of F. Replacing F by M and L by LM, we may assume that L contains a basis for $F(v)/F$, i.e. $L \supseteq F(v)$. Note also that $[L : F] = [k(p)(v) : k(p)]$ have the same dimension. Likewise, since $F(v)$ is $k(p)$-decomposed in F, $[k(p)(v) : k(p)] = [F(v) : F]$, so L and $F(v)$ must have the same dimension over F; that is, $L = F(v)$. Thus the relation

$$\sum_{v_F \in \mathcal{P}(K/F)} \{g_1(v_F), \ldots, \partial_{v_F}(g_{i(v_F)}, h), g_r(v_F)\}_{F(v)/F}$$

of $K_r(F; \mathbf{G}_m)$ of type **R2** maps to the original relation under the canonical map $K_r(F; \mathbf{G}_m) \longrightarrow K_r(k(p); \mathbf{G}_m)$.

Thus $(i_F)_*(x) = 0$. This concludes the proof of Lemma 3.6.4.

We now finish the proof of Theorem 3.6.1 Since $N_{F/k} \circ i_F =$ (multiplication by $[F : k]$), it follows that

$$[F : k] \cdot x = N_{F/k}((i_F)_*(x)) = 0$$

However, $k \subseteq F \subseteq k(p)$, so $p \nmid [F : k]$ and hence x is prime-to-p torsion. Since this argument is valid for all p, we conclude that $x = 0$ and hence that β is injective.

Remark.

In the case that $G_1 = \ldots = G_r = \mathbf{G}_m$, the maps $N_{k'/k}$ on Somekawa K-groups and the norm maps $N^M_{k'/k}$ on Milnor K-theory are compatible in the sense that the following diagram commutes for any finite extension k'/k, where the horizontal maps are as defined in Theorem 3.6.1. The proof of this fact is routine and follows immediately from the definitions.

$$\begin{array}{ccc}
K_r(k'; \mathbf{G}_m) & \xrightarrow[\alpha_{k'}]{\cong} & K^M_r(k') \\
\Big\downarrow{\scriptstyle N_{k'/k}} & & \Big\downarrow{\scriptstyle N^M_{k'/k}} \\
K_r(k; \mathbf{G}_m) & \xrightarrow[\alpha_k]{\cong} & K^M_r(k)
\end{array}$$

Chapter 4

One-variety Families

In this section, we calculate the group $K(k;S)$ where S is a (single) semi-abelian variety defined over k. We accomplish this by defining a map $K(k;S) \longrightarrow S(k)$ and showing that it is an isomorphism.

We begin by analyzing the two simplest cases, S being a torus or an abelian variety, and defer the general case till later. When dealing with abelian varieties, we usually favor additive notation, and when dealing with tori or semi-abelian varieties, multiplicative notation. We open with an elementary remark.

Proposition 4.0.6. *Let k be a field, and S a semi-abelian variety defined over k. Then there exists a natural surjective map $S(k) \xrightarrow{c} K(k;S)$.*

Proof.

Set $c(s) = \{s\}_{k/k}$. Every element of the group $K(k;S)$ is represented by a symbol $\{x\}_{E/k}$ for some finite extension E of k. By using relations of type **R1**, $\{x\}_{E/k} = \{N_{E/k}(x)\}_{k/k} = c(N_{E/k}(x))$ and so c is surjective.

The following lemma is extremely helpful in reducing our computations to the case in which k is algebraically closed. There are many advantages to making such a reduction, beyond the obvious elimination of type **R1** relations: not only are we justified in using the classical language of varieties; we may also assume that all tori over k are split.

Lemma 4.0.7. *Let k be an arbitrary field, S a semi-abelian variety over k, and $\bar{k}$ an algebraic closure of k. If*

$$S(\bar{k}) \xrightarrow{\bar{c}} K(\bar{k}; S \times_k \bar{k})$$

is an isomorphism, then

$$S(k) \xrightarrow{c} K(k;S)$$

is also an isomorphism.

Proof.

By Proposition 4.0.6 it suffices to prove that c is injective. The following diagram is clearly commutative; the second vertical arrow is the base change homomorphism of Somekawa K-groups defined in Section 3.4.1.

$$\begin{array}{ccc} S(k) & \xrightarrow{\quad c \quad} & K(k;S) \\ \big\downarrow & & \big\downarrow \\ S(\bar{k}) & \xrightarrow{\quad \bar{c} \quad} & K(\bar{k}, S\times_k\bar{k}) \end{array}$$

If $\bar{c}$ is an isomorphism, then since the first vertical arrow is injective, c must also be injective by commutativity of the diagram.

4.1 Tori

Theorem 4.1.1. *Let T be a torus over a field k. Then the canonical map*

$$T(k) \xrightarrow{c} K(k;T)$$

is an isomorphism.

Proof.

By Lemma 4.0.7 we may assume that k is algebraically closed. Thus $K(k;T)$ may be interpreted as a quotient of $T(k) \cong (\mathbf{G}_m(k))^r$ by relations of type **R2**. It suffices to show that all relations of type **R2** are degenerate in $T(k)$; that is, to show that for all $(t_1, \ldots, t_r) \in (\mathbf{G}_m(K))^r \cong T(K)$ and $h \in K^*$, we have

$$\prod_{v \in \mathcal{P}(K/k)} (\partial_v(t_1, h), \ldots, \partial_v(t_r, h)) = 1$$

Note that in this case, ∂_v coincides with the boundary map

$$\partial_v : K_2^M(K) \longrightarrow K_1^M(k(v))$$

of Milnor K-theory.

However, each of the coordinates in the above product is the product (over all v) of the local symbols $\partial_v(t_i, h)$ on $\mathbf{G}_m$, which is equal to 1 by Corollary 3.2.8.

4.2 Abelian Varieties

Theorem 4.2.1. *Let k be a field, and A an abelian variety defined over k. Then the canonical map $A(k) \xrightarrow{c} K(k; A)$ is an isomorphism.*

Before proving Proposition 4.2.1, we make some preliminary remarks.

Remark.

The relations of type **R2** in $K(k; A)$ take on a form simpler than that of the general case. In particular, the map r_v defined in 3.3.1 is identically 0. Relations of type **R2** thus correspond to some data $K \in \mathcal{T}_1(k)$, $v \in \mathcal{P}(K/k)$, $a \in A(K)$, and $h \in K^*$, and take the form

$$\sum_{v \in \mathcal{P}(K/k)} \mathrm{ord}_v h \{a(v)\}_{k(v)/k}$$

where $a(v) \in A(k(v))$ denotes the specialization of $a \in A(K)$ with respect to the valuation v. (These observations will be made precise below)

Proposition 4.2.2. *Let k be a field, and V a proper variety defined over k. Let $K \in \mathcal{T}_1(K/k)$ and $v \in \mathcal{P}(K/k)$. Then for every $g \in V(K)$, there exists a unique morphism $g_0(v) \in V(O_v)$ such that the following diagram commutes:*

$$\begin{array}{ccccc} & & \mathit{Spec}\ K & \xrightarrow{g} & V \\ & & \downarrow & \overset{g_0(v)}{\nearrow} & \downarrow \\ \mathit{Spec}\ k(v) & \xrightarrow{\pi_v} & \mathit{Spec}\ O_v & \longrightarrow & \mathit{Spec}\ k \end{array}$$

Proof.

All assertions all follow from the valuative criterion for properness.

Remark.

In the situation of Proposition 4.2.2, the notation $g(v)$ is used to denote the element of $V(k(v))$ defined by the composition $g_0(v) \circ \pi_v$. We call $g(v)$ the *specialization* of g at v.

Proof of Theorem 4.2.1

By Lemma 4.0.7 we may assume that k is algebraically closed. Thus, it suffices to prove the following:

Proposition 4.2.3. *Let k be algebraically closed. With the above notation, all relations of the form*

$$\sum_{v\in\mathcal{P}(K/k)} \{\partial_v(g,h)\}_{k(v)/k} = \sum_{v\in\mathcal{P}(K/k)} ord_v(h)\ \{g(v)\}_{k/k}$$

are zero in $K(k;A)$.

Proof.

Fix $K \in \mathcal{T}_1(k)$, $g \in A(K)$ and $h \in K^*$. The morphism Spec $K \xrightarrow{g} A$ corresponds to a nonconstant rational map $X \xrightarrow{\gamma} A$ from the smooth projective model X of K to A, which is a morphism because A is projective and X is smooth of dimension 1. The image $\gamma(X) \subseteq A$ has dimension 0 or 1. If $\gamma(X)$ has dimension 0, then g is a closed point of A, and so $g(v) = c$ is independent of v; hence

$$\sum_{v\in\mathcal{P}(K/k)} \mathrm{ord}_v(h)g(v) = c \sum_{v\in\mathcal{P}(K/k)} \mathrm{ord}_v(h) = 0$$

If $\gamma(X)$ has dimension 1, then $Y = \gamma(X)$ is a (projective) curve contained in A.
In this case, $\gamma(X)$ contains some closed point $d : \mathrm{Spec}\ k \longrightarrow A$ of A. Replacing g by $g - d$ (subtraction on A) we may assume that $0 \in A$ is in the image of γ.

Identifying points of X with valuations $v \in \mathcal{P}(K/k)$, we see that $\sum_{v\in\mathcal{P}(K/k)} \mathrm{ord}_v(h)\cdot g(v)$ is the image of the divisor div $h \in \mathrm{Div}\ (X)$ under the composition $f : \mathrm{Div}\ (X) \cong Z_0^0(X) \xrightarrow{\gamma_*} Z_0^0(A) \xrightarrow{\Sigma} A$. The notation Z_0^0 refers to the group of zero-cycles of degree 0, (cf. Section 5.1 for precise definitions); the first map is induced by γ and the second involves summation of the points of a zero-cycle using the group law on A.

We know from Section 8.2 that $Alb(A) \cong A$ and the map $Z_0^0(A) \xrightarrow{\Sigma} A$ factors through rational equivalence to yield the Albanese map $A_0(A) \xrightarrow{\mathrm{alb}} A$. Thus, the map f factors as:

$$Z_0^0(X) \xrightarrow{\gamma_*} Z_0^0(A) \xrightarrow{\pi} A_0(A) \xrightarrow{\mathrm{alb}} A$$

(Here $\pi : Z_0^0(A) \longrightarrow A_0(A)$ simply represents factoring out by rational equivalence)

Now, the divisor div $h \in Div(X)$ is a cycle of $Z_0^0(X)$ rationally equivalent to zero; by

Proposition 5.1.2, $\gamma_*(\text{div } h) \in Z_0^0(A)$ is a cycle rationally equivalent to zero. However, π kills cycles rationally equivalent to zero, so we conclude that $f(\text{div } h) = 0$; that is,

$$\sum_{v \in \mathcal{P}(K/k)} \text{ord}_v(h) \cdot g(v) = 0$$

This concludes the proof of Proposition 4.2.3.

Corollary 4.2.4. *Let k be a field, and A an abelian variety defined over k. Let T denote $\mathbf{G}_m^r$; let S denote $\mathbf{G}_m^r \times_k A$, and consider the following exact sequence of group schemes over k (the maps are the obvious ones):*

$$0 \longrightarrow \mathbf{G}_m^r \longrightarrow \mathbf{G}_m^r \times_k A \longrightarrow A \longrightarrow 0$$

Then there are canonical isomorphisms

$$K(k; S) \cong S(k) \cong T(k) \times A(k)$$

Proof. Using Lemma 4.0.7 we may assume that k is algebraically closed. Thus $K(k; S)$ is isomorphic to $S(k) \cong T(k) \times A(k)$ modulo relations of type **R2**, that is, given $K \in \mathcal{T}_1(k)$, $h \in K^*$ and $g = (t, a) \in S(K) \cong T(K) \times A(K)$, we consider the relation

$$\sum_{v \in \mathcal{P}(K/k)} \{\partial_v((t, a), h)\}_{k/k}$$

To prove the claim, it suffices (as above) to show that the sum within the expression

$$\{ \sum_{v \in \mathcal{P}(K/k)} \{\partial_v((t, a), h))\}_{k/k}$$

is zero.

Hence

$$\partial_v((t, a), h) = (((-1)^{\text{ord}_v(t_1)\ \text{ord}_v(h)} t_1^{\text{ord}_v(h)} h^{-\text{ord}_v(t_1)}, \ldots,$$

$$(-1)^{\text{ord}_v(t_r)\ \text{ord}_v(h)} t_r^{\text{ord}_v(h)} h^{-\text{ord}_v(t_r)}), \text{ord}_v(h)\ a(v))$$

When we substitute the above expression for $\partial_v((t, a), h)$ into the above sum, the first ith coordinate ($1 \leq i \leq r$) of the considered expression is equal to the sum over all v of the local symbols $\partial_v(t_i, h)$ on $\mathbf{G}_m$, which is 0 by the reciprocity law (Proposition 2.2.6); the last entry evaluates to 0 by Theorem 4.2.1.

4.3 Semi-abelian varieties

In this section, we calculate the group $K(k; S)$, where S is a semi-abelian variety defined over k. As in the previous two cases, we may assume by Lemma 4.0.7 that k is an algebraically closed field, and hence that our semi-abelian variety is defined by some exact sequence:

$$0 \longrightarrow (\mathbf{G}_m)^r \xrightarrow{i} S \xrightarrow{\pi} A \longrightarrow 0$$

The case $r = 0$ is treated in Section 4.2; thus, we may assume $r \geq 1$. To maintain the analogy with the case $S = \mathbf{G}_m$, we use multiplicative notation for the group law on S.

Theorem 4.3.1. *Let k be a field and S a semi-abelian variety defined over k. The canonical map $S(k) \xrightarrow{c} K(k; S)$ is isomorphism.*

Before we begin the proof of Theorem 4.3.1, we need two basic lemmas. The first proceeds directly from the definition of the local symbol.

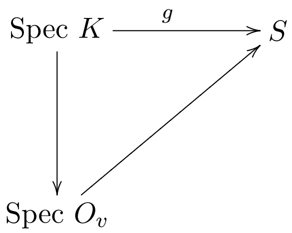
Lemma 4.3.2. *The symbol $\partial_v(g, h)$ is multiplicative in g; that is, for all g, $g' \in S(K)$ and $h \in K^*$, $\partial_v(gg', h) = \partial_v(g, h)\partial_v(g', h)$*

Lemma 4.3.3. *For any $g \in S(K)$, $r_v^1(g) = \ldots = r_v^r(g) = 0$ for all but finitely many $v \in \mathcal{P}(K/k)$.*

Proof. If g is a closed point, then $g \in S(k)$, and since $k \subseteq O_v$ for all v, we have $g \in S(O_v)$ for all v and hence $r_v^j(g) = 0$ for all v and all $j = 1, \ldots, r$. If g is not a closed point, the morphism Spec $K \xrightarrow{g} S$ corresponds to a nonconstant rational map $X \xrightarrow{\gamma} S$, where X is the smooth projective model for K over k. This map is defined for $v \in X - T$, where $T \subseteq X$ is some finite subset. For such v there is a local homomorphism

$$O_{S,\gamma(v)} \xrightarrow{\gamma^\#} \mathcal{O}_{X,v} = \mathcal{O}_v$$

and hence a morphism Spec $O_v \longrightarrow S$ such that the diagram



commutes. Hence $g \in S(O_v)$ and so $r_v^1(g) = \ldots = r_v^r(g) = 0$ for all but finitely many v.

Proof of Theorem 4.3.1. Note that the group $K(k; S)$ may be interpreted as the free abelian group on symbols $\{s\}_{k/k}$ for $s \in S(k)$ modulo the subgroup generated by relations of the form

$$\prod_{v \in \mathcal{P}(K/k)} \partial_v(g, h) = \{ \prod_{v \in \mathcal{P}(K/k)} (\varepsilon_v(s, h) s^{\mathrm{ord}_v(h)} \prod_{j=1}^{r} h_j^{-r_v^j(s)})_v \}_{k/k}$$

for all $h \in K^*$ and $s \in S(K)$. (The subscript v serves as a reminder that the parenthesized expression to which it is attached is to be interpreted as an element of $S(k(v))$.

Thus it suffices to prove that the expression within the braces is equal to the identity element 1 of $S(k)$. To this end, fix $s \in S(K)$ and $h \in K^*$.

Case 1: $s \in S(K)$ is a closed point.
Under this assumption, $s_v = c$ is constant, and $s \in S(k)$. Since $k \subseteq \bigcap_{v \in \mathcal{P}} O_v$, we have $s \in S(O_v)$ for all $v \in \mathcal{P}$, so $r_v(s) = 0$ and $\varepsilon_v(s, h) = 1$ for all such v, and the assertion reduces to checking that $\prod_{v \in \mathcal{P}} c^{\mathrm{ord}_v(h)} = 1$, which follows from the formula $\sum_{v \in \mathcal{P}} \mathrm{ord}_v(h) = 0$.

Case 2: $\pi_*(s) = 0$
By Hilbert Theorem 90, we have an exact sequence

$$0 \longrightarrow \mathbf{G}_m^r(K) \xrightarrow{i_*} S(K) \xrightarrow{\pi_*} A(K) \longrightarrow 0$$

Since $\pi_*(s) = 0$, we must have $s = i_*(t_s)$, where $t_s \in \mathbf{G}_m^r(K)$, and hence $r_v^j(s) = \mathrm{ord}_v^j(i_*(t_s))$.
Therefore we have

$$\prod_{v\in\mathcal{P}}(\varepsilon_v(s,h)s^{\mathrm{ord}_v(h)}\prod_{j=1}^{r}h^{-r_v^j(s)})_v$$

$$=\prod_{v\in\mathcal{P}}(\varepsilon_v(i_*(t_s),h)i_*(t_s)^{\mathrm{ord}_v(h)}\prod_{j=1}^{r}h^{-\mathrm{ord}_v^j(i_*(t_s))})_v$$

which is trivial by Corollary 3.2.8. (The notation ord_v^j refers to the usual map ord_v interpreted as a map on the jth summand of $T(K)\cong(K^*)^r$)

Case 3: s is not closed and is of the form Spec $k(t)\longrightarrow S$

We may interpret s as a nonconstant rational map $\mathbf{P}^1\longrightarrow S$. Composing with the given morphism $S\xrightarrow{\pi}A$, we get a rational map from $\mathbf{P}^1$ to the abelian variety A; but we know ([Mi1 86], Cor. 3.8) that all such maps must be constant. Thus the image of s lies in $\pi^{-1}(a)$ where a is some closed point of a. Choose any closed point $c\in\pi^{-1}(a)$; by Case 1, the assertion holds for c. Since $\pi_*(c^{-1}s)=-a+a=0$, the assertion holds for $c^{-1}s$ by Case 2. Finally, the assertion for $s=c(c^{-1}s)$ follows from the above two cases in conjunction with Lemma 4.3.2.

Case 4: $s:\mathrm{Spec}\ K\longrightarrow S$ is arbitrary

Let $v_1,\ldots,v_n$ be the places such that $r_{v_i}^j(s)=a_i^j\neq 0$ for some j. For each $j=1,\ldots,r$, use the weak approximation theorem for valuations (cf. [Bou 85], VI. 7.2, Corollary 1) to choose elements $k_j\in K^*$ such that $\mathrm{ord}_{v_i}^j(k_j)=a_i^j$ for all $i=1,\ldots n$. Letting $\mathbf{k}=(k_1,k_2,\ldots,k_r)\in T(K)\subseteq S(K)$, and writing $s=(s\mathbf{k}^{-1})\mathbf{k}$, we have $\partial_v(s,h)=\partial_v(s\mathbf{k}^{-1},h)\partial_v(\mathbf{k},h)$. Since $\prod_{v\in\mathcal{P}}\partial_v(\mathbf{k},h)=1$ by Case 2, it suffices to check the assertion for s,h under the assumption that $r_v^1(s)=\ldots=r_v^r(s)=0$ for all v such that $\mathrm{ord}_v(h)\neq 0$. Because K is finitely generated of transcendence degree 1 over the algebraically closed field k, the element $h\in K^*$ is either in k or transcendental over k. If $h\in k$, let t be any element transcendental over k; otherwise let $t=h$. Then $k(t)$ is a subfield of K of finite index in K; thus the inclusion map $\pi^*:k(t)\hookrightarrow K$ is in fact the map on function fields associated to a finite morphism

$$\pi:X\longrightarrow\mathbf{P}^1$$

where X is the smooth projective model for K. Note that in either case, h is contained in $\pi^*(k(\mathbf{P}^1))$; we define h' by $h=\pi^*(h')$. We may interpret s as a rational map $X\longrightarrow S$, and use the notation $Tr_\pi s$ to denote the trace of s with respect to the map π (see Section 3.2); this is a rational map $Tr_\pi s : \mathbf{P}^1 \longrightarrow S$. Finally, we denote by Z the subset of X on which s is not defined; it follows (cf. Section 3.2) that $Tr_\pi s$ is defined away from $Z' = \pi(Z) \subseteq \mathbf{P}^1$.

Proposition 4.3.4. *For all $w \in \mathcal{P}(k(t)/k)$, we have*

$$\partial_w(Tr_\pi s, h') = \prod_{v \in \mathcal{P}(K/k):\ v|_{\mathbf{P}^1} = w} \partial_v(s, h)$$

Proof.

We first note that if $s \in \mathbf{G}_m(K) = K^*$, this is exactly Proposition 3.2.9.

If $w \notin Z'$ (equivalently, $r_w(s) = 0$), the proposition follows from a direct calculation. Let $e_\pi(v|w)$ represent the ramification index with respect to the map π; thus, if $v \in \mathcal{P}(K/k)$ restricts to $w \in \mathcal{P}(k(t)/k)$, then $\mathrm{ord}_v(h) = e_\pi(v|w)\mathrm{ord}_w(h')$.

Hence,

$$\begin{aligned} &\partial_w(Tr_\pi s, h') \\ &= (Tr_\pi s(w))^{\mathrm{ord}_w(h')} \\ &= \Big(\prod_{v|_{\mathbf{P}^1} = w} s(v)^{e_\pi(v|w)}\Big)^{\mathrm{ord}_w(h')} \\ &= \prod_{v|_{\mathbf{P}^1} = w} s(v)^{\mathrm{ord}_v(h)} \\ &= \prod_{v|_{\mathbf{P}^1} = w} \partial_v(s, h) \end{aligned}$$

If $w \in Z'$, use the weak approximation theorem for valuations to select $k_1, \ldots, k_r \in K^*$ such that $\mathrm{ord}_v(k_j) = r_v^j(s)$ for all $j = 1, \ldots, r$ and all v which restrict to w, and $\mathrm{ord}_v(k_j) = 0$ for all $j = 1, \ldots, r$ and all v such that $\mathrm{ord}_v(h) \neq 0$ (by hypothesis, these conditions are independent). Writing $\mathbf{k} = (k_1, \ldots, k_r)$, we have $r_v^j(s\mathbf{k}^{-1}) = 0$ for all v which restrict to w; thus $Tr_\pi(s\mathbf{k}^{-1})$ is defined at w and so by the argument above, we have

$$\partial_w(Tr_\pi(s\mathbf{k}^{-1}), h') = \prod_{v \in \mathcal{P}(K/k):\ v|_{\mathbf{P}^1} = w} \partial_v(s\mathbf{k}^{-1}, h)$$

Since the proposition for $\mathbf{k}$ follows from Proposition 3.2.9, Lemma 4.3.2 together with the last calculation and the fact that $Tr_\pi ab = Tr_\pi a \cdot Tr_\pi b$ imply the proposition for s.

To conclude the proof of Theorem 4.3.1, we observe that

$$\prod_{v \in \mathcal{P}} \partial_v(s, h)$$

$$= \prod_{w \in \mathcal{P}'} \prod_{v|_{\mathbf{P1}} = w} \partial_v(s, h)$$

$$= \prod_{w \in \mathcal{P}'} \partial_w(Tr_\pi s, h') \text{by Proposition 4.3.4}$$

$$= 1 \text{ by Case 3}$$

Chapter 5

Mixed K-groups

In this section we investigate the relationship between certain Somekawa-type K-groups and the Zariski cohomology of Milnor K-sheaves on smooth varieties defined over a field. In preparation for our results, we need to define "mixed K-groups". The definition is useful in that it generalizes both the Somekawa K-groups defined in [Som 90] (also Section 3) and the Somekawa-type K-groups defined in [RS], (2.1.1). We begin with a brief overview of the intersection theory necessary for our definition.

5.1 Intersection Theory

Order functions

Let k be a field and X a variety over k. Let Y be a subvariety of X of codimension 1. Then the local ring $A = \mathcal{O}_{Y,X}$ is a one-dimensional local domain. For $r \in k(X)^*$, we define

$$\mathrm{ord}_Y(r) = l_A(A/(r))$$

where l_A denotes the length function on A-modules (cf. [AM 69], p.77). This definition is the most general possible, and does not require Y to be smooth. If Y is smooth, there is a more intuitive definition of ord_Y; in this case, note that $\mathcal{O}_{Y,X}$ is a discrete valuation ring, the valuation v_Y of which extends to a valuation on the (nonzero elements of the) fraction field of $\mathcal{O}_{Y,X}$; i.e., on $k(X)^*$. Then we have

$$\text{ord}_Y(r) = v_Y(r)$$

Cycles and Rational Equivalence

Let X be an algebraic scheme over a field k. An *m-cycle* on X is a finite formal sum $\sum n_i[V_i]$, where n_i are integers and V_i are m-dimensional subvarieties of X. The *group of m-cycles on X*, denoted $Z_m(X)$, is by definition the free abelian group on the m-dimensional subvarieties of X: for each such subvariety V, there is one generator $[V]$.

Now let W be an $(m+1)$-dimensional subvariety of X. Given $r \in k(W)^*$, define an m-cycle on X, denoted [div r] by

$$[\text{div } r] = \sum \text{ord}_V(r)[V]$$

the sum being taken over all codimension 1 subvarieties V of W. (Here ord_V is the order function defined with respect to the local ring $\mathcal{O}_{V,W}$).

An element $\alpha \in Z_m(X)$ is said to be *rationally equivalent to zero*, written $\alpha \sim 0$, if there are a finite number of $(m+1)$-dimensional subvarieties $W_1, \ldots, W_n$ of X and elements $r_i \in k(W_i)^*$, $i = 1, \ldots, n$ such that

$$\alpha = \sum_{i=1}^{n} [\text{div } r_i]$$

Since $[\text{div } \frac{1}{r}] = -[\text{div } r]$, the cycles of $Z_m(X)$ rationally equivalent to zero form a subgroup $Rat_m(X)$ of X; we define the *Chow group of m-cycles* on X, denoted $CH_m(X)$ by

$$CH_m(X) = Z_m(X)/Rat_m(X)$$

Define $Z_*(X)$ (respectively $CH_*(X)$) to be the direct sum of the groups $Z_m(X)$ (respectively $CH_m(X)$) for $m = 1, \ldots, \dim(X)$. Thus $Z_*(X)$ and $CH_*(X)$ are graded abelian groups in a natural way. A *cycle* on X is an element of $Z_*(X)$. We abuse terminology and often refer to elements of $CH_*(X)$ as cycles.

Cycle associated to a subscheme

If Y is any closed subscheme of X and $Y_1, \ldots, Y_t$ are the irreducible components of Y, we use the notation $[Y]$ as shorthand for the element of $CH_*(Y)$ defined by

$$\sum_{i=1}^{t} l_{\mathcal{O}_{Y_i,Y}}(\mathcal{O}_{Y_i,Y}) \cdot [Y_i]$$

The coefficients in the above sum represent, in some sense, the "multiplicity" with which Y_i appears as an irreducible component of Y. The cycle $[Y]$ is called the *cycle associated to* Y.

There is another useful characterization of rational equivalence, described by the following proposition.

Proposition 5.1.1. *([Fu 84], Proposition 1.6) A cycle α in $Z_m(X)$ is rationally equivalent to zero if and only if there are $(m+1)$-dimensional subvarieties $V_1, \ldots, V_t$ of $X \times_k \mathbf{P}^1_k$ such that the projections $\pi_i : V_i \longrightarrow \mathbf{P}^1_k$ are dominant, and such that*

$$\alpha = \sum_{i=1}^{t} [\pi_i^{-1}(\{0\})] - [\pi_i^{-1}(\{\infty\})]$$

in $Z_m(X)$.

Proper Push-Forward

Let k be a field and $f : X \longrightarrow Y$ a proper morphism of algebraic schemes over k. For any subvariety V of X, then the image $W = f(V)$ is then closed in Y, and therefore inherits the structure of a subvariety of Y. Thus the generic point of V maps to the generic point of W and hence we have an inclusion $k(W) \hookrightarrow k(V)$. If W has the same dimension as V, this field extension is finite. Define

$$\deg\,(V/W) = \begin{cases} [k(V) : k(W)] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V \end{cases}$$

Set $f_*[V] = \deg\,(V/W) \cdot [W]$. This rule extends by linearity to a *push-forward* homomorphism:

$$f_* : Z_m(X) \longrightarrow Z_m(Y)$$

These homomorphisms have desirable functorial properties. If $i : X \longrightarrow X$ is the identity map, then i_* is also the identity, and if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are proper morphisms, then $(gf)_* = g_* f_*$.

Much less obvious is the following:

Proposition 5.1.2. *([Fu 84], Theorem 1.4) Let $f : X \longrightarrow Y$ be a proper morphism. If $\alpha \in Z_m(X)$ and $\alpha \sim 0$ on X, then $f_*(\alpha) \sim 0$ on Y.*

Thus f_* induces a map

$$f_* : CH_m(X) \longrightarrow CH_m(Y)$$

Evidently the map f_* on Chow groups inherits the functoriality properties enjoyed by the map f_* on cycle groups. In the case that $X = Z \times_k E_2$ and $Y = Z \times_k E_1$ for some finite extension E_2/E_1 of fields, we sometimes refer to f_* as a "norm map" and use the notation N_{E_2/E_1} in place of f_*.

In the case $m = 0$, we may define a homomorphism:

$$\deg : Z_0(X) \longrightarrow \mathbf{Z}$$

called the *degree homomorphism* or *degree map*, by the rule

$$\deg\,(\sum_{i=1}^{t} n_i[P_i]) = \sum_{i=1}^{t} n_i \cdot [k(P_i) : k]$$

We denote by $Z_0^0(X)$ the kernel of deg. It is not hard to check that deg factors through an exact sequence

$$0 \longrightarrow A_0(X) \longrightarrow CH_0(X) \longrightarrow \mathbf{Z}$$

which is split exact if and only if X admits a zero-cycle of degree 1, in particular if X has a k-rational point.

It is easy to verify that the map deg as defined above coincides with the push-forward map $p_* : CH_0(X) \longrightarrow CH_0(\text{Spec } k)$ induced by the structure morphism $p : X \longrightarrow \text{Spec } k$.

Flat Pull-back

Let X and Y be algebraic schemes over k, and suppose $f : X \longrightarrow Y$ is a flat morphism of relative dimension n (cf. [Ha 83], III.10). Then, given a subvariety V of Y, set

$$f^*[V] = [f^{-1}(V)]$$

where $f^{-1}(V) = V \times_Y X$ is the scheme-theoretic inverse image. This extends by linearity to a *pull-back homomorphism*:

$$f^* : Z_m(Y) \longrightarrow Z_{m+n}(X)$$

With some effort, one can show the following:

Proposition 5.1.3. *([Fu 84], Theorem 1.7) Let $f : X \longrightarrow Y$ be a flat morphism of relative dimension n, and $\alpha \in Z_m(Y)$. If $\alpha \sim 0$ on Y, then $f^*(\alpha) \sim 0$ on X.*

Thus f^* induces a map

$$f^* : CH_m(Y) \longrightarrow CH_{m+n}(X)$$

External Product

Let X, Y be algebraic schemes over a field k. We define the *exterior product* of cycles:

$$Z_l(X) \otimes Z_m(Y) \xrightarrow{\times} Z_{l+m}(X \times_k Y)$$

on generators by the formula

$$[V] \otimes [W] \mapsto [V \times_k W]$$

and extending by bilinearity to cycles. Note that $V \times_k W$ may not be irreducible; hence we must interpret $[V \times_k W]$ as the cycle associated to a subscheme, as defined in Section 5.1.

Proposition 5.1.4. *([Fu 84], Proposition 1.10)*

1. *If $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$.*

2. *Let* $f : X' \longrightarrow X$, $g : Y' \longrightarrow Y$ *be morphisms, and* $f \times g : X' \times_k X \longrightarrow Y' \times_k Y$ *the induced morphism. Then*

   - *If* f *and* g *are proper, so is* $f \times g$, *and*

     $$(f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times g_*(\beta)$$

     for all $\alpha \in Z_*(X')$, $\beta \in Z_*(Y')$.

   - *If* f *and* g *are flat of (respective) relative dimensions* m *and* n, *then* $f \times g$ *is flat of relative dimension* $m + n$, *and*

     $$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta)$$

     for all $\alpha \in Z_*(X)$, $\beta \in Z_*(Y)$.

3. *There is a homomorphism*

   $$CH_l(X) \otimes CH_m(Y) \stackrel{\times}{\longrightarrow} CH_{l+m}(X \times_k Y)$$

   (also called the exterior product) satisfying the formulas of statement 2 *above.*

Intersection Product

Let X be an equidimensional algebraic scheme over k of dimension n. In this context, we may index the cycle groups (and Chow groups) by codimension, and write

$$Z^p(X) := Z_{n-p}(X), \quad CH^p(X) := CH_{n-p}(X)$$

If furthermore X is smooth, one may compose the exterior product $CH_l(X) \otimes CH_m(X) \longrightarrow CH_{l+m}(X \times_k X)$ with a special pullback operation $CH_{l+m}(X \times_k X) \longrightarrow CH_{l+m-n}(X)$ known as the *Gysin homomorphism* (see [Fu 84], 8.1 for details). Thus we obtain the *intersection product* on X:

$$CH^p(X) \otimes CH^q(X) \longrightarrow CH^{p+q}(X)$$

The intersection product makes $CH^*(Y) = \bigoplus_{i=0}^{n} CH^i(Y)$ into a (commutative) ring called the *Chow ring*, with unit element given by the class $[Y]$. The construction is functorial in the sense that $Y \longrightarrow CH^*(Y)$ is a functor from the category of smooth varieties over k to the category of graded rings.

Specialization

In this section we describe a specialization operation on Chow groups; for details, we refer the reader to [Fu 84], 20.3. Let R be a discrete valuation ring with valuation v, quotient field K and residue field $k(v)$, and suppose X is a scheme over R. Then there is a homomorphism known as the *specialization map*

$$s_v : CH_m(X \times_R K) \longrightarrow CH_m(X \times_R k(v))$$

with the following property. Given a subvariety V of $X \times_R K$, let V^c denote the closure of the image of V under the naturally induced map $X \times_R K \longrightarrow X$ and set $V' = V^c \times_R k(v)$. Then we have

$$s_v([V]) = [V']$$

Specialization is functorial in X in the sense that it commutes with flat pull-back and proper push-forward homomorphisms.

5.2 Definition of Mixed K-groups

Let k be a field, and let X a smooth projective variety defined over k. Let $\mathcal{CH}_0(X)$ denote the contravariant functor which associates to each (not necessarily finite) field extension E/k the group $CH_0(X_E)$.

Let $r \geq 0$, $s \geq 0$ be integers. Suppose $X_1, \ldots, X_r$ are smooth projective varieties defined over k and $G_1, \ldots, G_s$ are semi-abelian varieties defined over k. The *mixed K-group* $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$ is defined to be $\mathbf{Z}$ if $r = s = 0$. Otherwise, define $K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_r),G_1,\ldots,G_s)$ to be the quotient F/R, where

$$F = \bigoplus_{E/k \text{ finite}} CH_0((X_1)_E)\otimes\ldots\otimes CH_0((X_r)_E)\otimes G_1(E)\otimes\ldots\otimes G_s(E)$$

and $R \subseteq F$ is the subgroup generated by the following elements.

- **M1.** For convenience of notation, set $H_i(E) = CH_0((X_i)_E)$ for $i = 1,\ldots,r$ and $H_j(E) = G_{j-r}(E)$ for $j = r+1,\ldots,r+s$,

  For every diagram $k \hookrightarrow E_1 \stackrel{\phi}{\hookrightarrow} E_2$ of finite extensions of k, all choices $i_0 \in \{1,\ldots,r+s\}$ and all choices $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ (for $i \neq i_0$), the relation

  $$(\phi^*(h_1)\otimes\ldots\otimes h_{i_0}\otimes\ldots\otimes\phi^*(h_s))_{E_2} - (h_1\otimes\ldots\otimes N_{E_2/E_1}(h_{i_0})\otimes\ldots\otimes h_r)_{E_1}$$

- **M2.** For every $K \in \mathcal{T}_1(k)$, all choices of $h \in K^*$, $f_i \in CH_0((X_i)_K)$, $i = 1,\ldots,r$ and $g_j \in G_j(K)$, $j = 1,\ldots,s$ such that for every $v \in \mathcal{P}(K)$, there exists $j_0(v)$ such that $g_j \in G_j(O_v)$ for all $j \neq j_0(v)$, the relation:

  $$\sum_{v\in\mathcal{P}(K/k)} (s_v(f_1)\otimes\ldots\otimes s_v(f_r)\otimes g_1(v)\otimes\ldots\otimes\partial_v(g_{j_0},h)\otimes\ldots\otimes g_s(v))_{k(v)}$$

  if $s > 0$. Here $s_v : CH_0((X_i)_K) \longrightarrow CH_0((X_i)_{k(v)})$ is the specialization map (cf. Section 5.1) and ∂_v is the extended tame symbol as defined in Section 3.3.1.

  If $s = 0$, the relation:

  $$\sum_{v\in\mathcal{P}(K/k)} \mathrm{ord}_v(h)\cdot(s_v(f_1)\otimes\ldots\otimes s_v(f_r))_{k(v)}$$

As before, we denote the class of $(a_1\otimes\ldots\otimes a_r)_E$ in F/R by $\{a_1,\ldots,a_r\}_{E/k}$.

Remark.

The analogue of Remark 3.3.2 holds for mixed K-groups:

Let $E \stackrel{j}{\longrightarrow} E'$ be a k-isomorphism. Then for any $x_i \in CH_0((X_i)_E)$ and any $g_j \in G_j(E)$, we have

$$\{j^*(x_1), \ldots, j^*(x_r), j^*(g_1), \ldots, j^*(g_s)\}_{E'/k} = \{j_*(j^*(x_1)), x_2, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

by relation **M1**

$$= [E' : E]\{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k} = \{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

Remark.

We may replace CH_0 by A_0 everywhere in the above definition to define a group $K(k; \mathcal{A}_0(X_1), \ldots, \mathcal{A}_0(X_r), G_1, \ldots, G_s)$. Taking this one step further, one may also "mix" CH_0-type and A_0-type groups and likewise define $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_t), \mathcal{A}_0(X_{t+1}), \ldots, \mathcal{A}_0(X_r), G_1, \ldots, G_s)$.

Similarly, one may replace CH_0 by CH_n or CH^n to define analogous mixed K-groups, although we will not make use of these.

Remark.

The following lemma is the analogue of Lemma 3.6.2 for mixed K-groups. The proof is virtually identical, so we omit it.

Lemma 5.2.1. *Let k be a field, and $X_1, \ldots, X_r$ smooth projective varieties defined over k. Set $G_1 = \ldots = G_t = \mathbf{G}_m$ and suppose $G_{t+1}, \ldots, G_{t+s}$ are any semi-abelian varieties defined over k. In the group $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \underbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}_{t}, G_{t+1}, \ldots, G_{t+s})$, every relation of the form $\{x_1, \ldots, x_r, a_1, \ldots, a_r, b_1, \ldots, b_s\}_{E/k}$, where $a_i + a_j = 1$ for some $i < j$ and E/k is any finite extension, is equivalent to zero.*

Remark.

In the case $G_1 = \ldots = G_r = \mathbf{G}_m$, it seems more natural to work with a smaller quotient of F, most easily defined as a quotient of the group $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r))$, which we define to be shorthand noation for the group $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \underbrace{\mathbf{G}_m, \ldots, \mathbf{G}_m}_{s})$.

The relations we are about to describe are in some sense a generalization of relations of type **M2**.

Observe that given a finite extension E/k and $y_1, \ldots, y_r \in CH_0((X_i)_E)$, Lemma 5.2.1 assures the existence of a (well-defined) map

$$S_{E,y_1,\ldots,y_r} : K_s^M(E) \longrightarrow K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r))$$

defined by

$$\{a_1, \ldots, a_s\} \mapsto \{y_1, \ldots, y_r, a_1, \ldots, a_s\}_{E/k}$$

Now fix a choice of $K \in \mathcal{T}_1(k)$, $h \in K^*$, $f_i \in CH_0((X_i)_K)$ for $i = 1, \ldots, r$ and $g_j \in G_j(K) \cong K^*$ for $j = 1, \ldots, s$. We now define our relations as follows:

S2.

$$\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v(f_1), \ldots, s_v(f_r)}(\partial_v(g_1, \ldots, g_s, h))$$

where ∂_v is the boundary map on Milnor K-theory (cf. Section 2.2.3).

We denote by $\tilde{K}_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \mathbf{G}_m)$ the group obtained from $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \mathbf{G}_m)$ by factoring out relations of type **S2**.

For $s = 0$ and $s = 1$, the relations of type **S2** in $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r))$ correspond naturally to relations of type **M2** in F; hence the natural homomorphism

$$K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \mathbf{G}_m) \xrightarrow{\sigma} \tilde{K}_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), \mathbf{G}_m)$$

is in fact an isomorphism.

However, for $s > 1$, it is not obvious that σ is an isomorphism. We will prove this fact when $r = 1$ in Section 7.

Remark.

Sometimes it is of interest to consider the group obtained from F by factoring out (only) relations of type **M1**. We will call this group $M(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$ because of its connection with Mackey functors (cf.[Ka 92]). We use the notation $[x_1, \ldots, x_r, g_1, \ldots, g_s]_{E/k}$ for the residue of $(x_1 \otimes \ldots \otimes x_r \otimes g_1 \otimes \ldots \otimes g_s)_E$ in $M(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$.

5.3 Functoriality

All of the functoriality properties for Somekawa K-groups proved in 3.4 have their analogues in the context of mixed K-groups; the latter have some additional properties coming from intersection theory. The proofs in the new setting are very similar to those of Section 3.4 or to the corresponding properties of intersection theory, so we state the results here without proof. The corresponding results also hold for the $\tilde{K}$ groups. In each of the propositions below, k is a field, $X_1, \ldots, X_r$ are smooth projective varieties defined over k, and $G_1, \ldots, G_s$ are semi-abelian varieties defined over k.

Proposition 5.3.1. *(Covariant functoriality in field)*
Suppose $k \hookrightarrow k'$ is any extension. Then there exists a natural homomorphism

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$$

$$\longrightarrow K(k'; \mathcal{CH}_0((X_1)_{k'}), \ldots, \mathcal{CH}_0((X_r)_{k'}), (G_1)_{k'}, \ldots, (G_s)_{k'})$$

Proposition 5.3.2. *(Change of semi-abelian varieties)*
Let G'_j, $j = 1, \ldots, s$ be semi-abelian varieties defined over k, and let $G_j \xrightarrow{\psi_j} G'_j$, $j = 1, \ldots, s$ be morphisms of semi-abelian varieties over k. There exists a naturally induced homomorphism

$$(\phi, \psi) : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$$

$$\longrightarrow K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G'_1, \ldots, G'_s)$$

Proposition 5.3.3. *(Contravariant functoriality in field)*
Suppose $k \hookrightarrow k'$ is a finite extension. There exists a natural homomorphism

$$K(k'; \mathcal{CH}_0((X_1)_{k'}), \ldots, \mathcal{CH}_0((X_r)_{k'}), (G_1)_{k'}, \ldots, (G_s)_{k'})$$

$$\longrightarrow K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$$

Proposition 5.3.4. *(Covariant functoriality in varieties)*
Suppose $Y_1, \ldots, Y_r$ are smooth projective varieties defined over k and $X_i \xrightarrow{\phi_i} Y_i$, $i =$

$1, \ldots, r$ *are proper morphisms. Then there exists a naturally induced homomorphism*

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r) \longrightarrow K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_r), G_1, \ldots, G_s)$$

Proposition 5.3.5. *(Contravariant functoriality in varieties)*
Suppose $Y_1, \ldots, Y_r$ *are smooth projective varieties defined over* k *and* $X_i \xrightarrow{\phi_i} Y_i$ *(*$i = 1, \ldots, r$*) are flat morphisms of relative dimension* 0*. Then there exists a naturally induced homomorphism:*

$$K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_r), G_1, \ldots, G_r) \longrightarrow K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s)$$

5.4 The degree homomorphism

The goal of this section is to prove Corollary 5.4.4, which relates mixed K-groups with CH_0-type arguments (cf. Section 5.2) to mixed K-groups with A_0-type arguments.

To ease notation when dealing with fields, we write, for a field F, $CH_0(F)$ in place of $CH_0(\text{Spec } F)$. As before, the results of this section hold true when one replaces K-groups with $\tilde{K}$-groups. We begin with an elementary but useful lemma which says essentially that the argument $\mathcal{CH}_0(k)$ in a mixed K-group acts somewhat like an "identity" in that adding or it removing it yields a K-group isomorphic to the original.

Lemma 5.4.1. *Let* k *be a field and* $X_1, \ldots, X_r$ *smooth projective varieties defined over* k*. Suppose* $G_1, \ldots, G_s$ *are any semi-abelian varieties defined over* k*. Then there is a natural isomorphism*

$$K(k; \mathcal{CH}_0(k), \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

$$\xrightarrow{\cong} K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

Proof.

For each finite extension E/k, consider the point P_E (unique up to k-isomorphism) of $(\text{Spec } k)(E)$. By the remark following the definition in Section 5.2, the symbol

$\{P_E, x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$ is independent of the choice of P_E.

For a scheme X over k and a field extension E/k, we use the notation $\deg_E$ to denote the degree homomorphism $CH_0(X_E) \xrightarrow{(p_E)_*} CH_0(\text{Spec } E) \cong \mathbf{Z}$, where $p_E : X_E \longrightarrow \text{Spec } E$ is the structure map.

Define

$$\alpha : K(k; \mathcal{CH}_0(k), \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

$$\longrightarrow K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

on symbols by

$$\{a, x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k} \mapsto \deg_E(a) \cdot \{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

It is clear that α kills relations of type **M2**. Now consider a diagram $k \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ of finite extensions of j and a relation of type **M1** as below, where $a \in (\text{Spec } k)(E_2)$, $x_i \in CH_0((X_i)_{E_1})$, $g_j \in G_j(E_1)$. We calculate:

$$\alpha(\{a, \phi^*(x_1), \ldots, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k})$$

$$= \deg_{E_2}(a) \cdot \{\phi^*(x_1), \ldots, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k})$$

$$= \deg_{E_2}(a)([E_2 : E_1])\{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E_1/k}$$

$$= \deg_{E_1}(\phi_*(a))\{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E_1/k}$$

$$= \alpha(\{\phi_*(a), x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E_1/k})$$

Similarly, if $a \in (\text{Spec } k)(E_1)$, $x_i \in CH_0((X_i)_{E_1})$ for $i \neq i_0$, $x_{i_0} \in CH_0((X_{i_0})_{E_2})$, and $g_j \in G_j(E_1)$, we have:

$$\alpha(\{\phi^*(a), \phi^*(x_1), \ldots, x_{i_0}, \ldots, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k})$$

$$= \deg_{E_2}(\phi^*(a))\{\phi^*(x_1), \ldots, x_{i_0}, \ldots, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k})$$

$$= \deg_{E_1}(a)\{x_1, \ldots, \phi_*(x_{i_0}), \ldots, x_r, g_1, \ldots, g_s\}_{E_1/k})$$

$$= \alpha(\{a, x_1, \ldots, \phi_*(x_{i_0}), \ldots, x_r, g_1, \ldots, g_s\}_{E_1/k})$$

The case that $a \in (\text{Spec } k)(E_1)$, $x_i \in CH_0((X_i)_{E_1})$ and $g_j \in G_j(E_1)$ for $j \neq j_0$ is similar to the last calculation above.

Now we construct a homomorphism

$$\beta : K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

$$\longrightarrow K(k; \mathcal{CH}_0(k), \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_r)$$

by sending

$$\{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k} \mapsto \{[P_E], x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

Once again, it is clear that relations of type **M2** are killed by β. For relations of type **M1**, we have, for $x_i \in CH_0((X_i)_{E_1})$ for $i \neq i_0$, $x_{i_0} \in CH_0((X_{i_0})_{E_2})$, and $g_j \in G_j(E_1)$:

$$\beta(\{\phi^*(x_1), \ldots, x_{i_0}, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k})$$

$$= \{[P_{E_2}], \phi^*(x_1), \ldots, x_{i_0}, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k}$$

$$= \{[\phi^*(P_{E_1})], \phi^*(x_1), \ldots, x_{i_0}, \phi^*(x_r), \phi^*(g_1), \ldots, \phi^*(g_s)\}_{E_2/k}$$

$$= \{[P_{E_1}], x_1, \ldots, \phi_*(x_{i_0}), x_r, g_1, \ldots, g_s\}_{E_1/k}$$

$$= \beta(\{x_1, \ldots, \phi_*(x_{i_0}), x_r, g_1, \ldots, g_s\}_{E_1/k}$$

The other case is similar, and thus β is well-defined.

A routine computation shows that $\beta \circ \alpha = id$ and $\alpha \circ \beta = id$, and hence that α and β are mutually inverse isomorphisms.

Our next step is to construct homomorphisms connecting CH_0- and A_0-type groups.

Lemma 5.4.2. *Let k be a field, $X_1, \ldots, X_r$ smooth projective varieties defined over k, and $G_1, \ldots, G_s$ semi-abelian varieties defined over k; suppose further that X_{i_0} admits a zero-cycle w of degree 1. Let $\mathcal{B}_0(X_i)$ be either $\mathcal{CH}_0(X_i)$ or $\mathcal{A}_0(X_i)$ for $i \neq i_0$. Then there are natural homomorphisms*

$$K(k; \mathcal{B}_0(X_1), \ldots, \mathcal{CH}_0(X_{i_0}), \ldots, \mathcal{B}_0(X_r), G_1, \ldots, G_s) \xrightarrow{z}$$

$$K(k; \mathcal{B}_0(X_1), \ldots, \mathcal{A}_0(X_{i_0}), \ldots, \mathcal{B}_0(X_r), G_1, \ldots, G_s)$$

and

$$K(k; \mathcal{B}_0(X_1), \ldots, \mathcal{A}_0(X_{i_0}), \ldots, \mathcal{B}_0(X_r), G_1, \ldots, G_s) \xrightarrow{c}$$

$$K(k; \mathcal{B}_0(X_1), \ldots, \mathcal{CH}_0(X_{i_0}), \ldots, \mathcal{B}_0(X_r), G_1, \ldots, G_s)$$

Proof.

Let $\tau_E : \text{Spec } E \longrightarrow \text{Spec } k$ be the canonical morphism. We define the first map by:

$$\{x_1, \ldots, x_{i_0}, \ldots, x_r, g_1, \ldots, g_s\}_{E/k} \mapsto$$

$$\{x_1, \ldots, x_{i_0} - \deg_E(x_{i_0}) \cdot \tau_E^*(w), \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

It is easy to check that relations of type **M1** and **M2** are killed. We define the second map by

$$\{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k} \mapsto \{x_1, \ldots, x_r, g_1, \ldots, g_s\}_{E/k}$$

The verification that this rule is well-defined is equally obvious.

We have now developed enough machinery to state and prove the crucial result of this section:

Proposition 5.4.3. *Let k be a field. Suppose $Y_1, \ldots, Y_l$, V and $Z_1, \ldots, Z_m$ are smooth projective varieties defined over k. Assume further that V admits a zero-cycle w of degree 1. Let $G_1, \ldots, G_s$ be semi-abelian varieties defined over k. For convenience of notation, set*

$$F = K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_l), \mathcal{CH}_0(V), \mathcal{A}_0(Z_1), \ldots, \mathcal{A}_0(Z_m), G_1, \ldots, G_s)$$

$$F_0 = K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_l), \mathcal{A}_0(Z_1), \ldots, \mathcal{A}_0(Z_m), G_1, \ldots, G_s)$$

$$F_1 = K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_l), \mathcal{A}_0(V), \mathcal{A}_0(Z_1), \ldots, \mathcal{A}_0(Z_m), G_1, \ldots, G_s)$$

Then there exists a canonical isomorphism

$$F \xrightarrow{\cong} F_0 \bigoplus F_1$$

Proof.

By Lemma 5.4.1, we may replace F_0 by

$$K(k; \mathcal{CH}_0(Y_1), \ldots, \mathcal{CH}_0(Y_l), \mathcal{CH}_0(k), \mathcal{A}_0(Z_1), \ldots, \mathcal{A}_0(Z_m), G_1, \ldots, G_s)$$

Now apply the functoriality assertion of Proposition 5.3.2 to the structure morphism $V \xrightarrow{\sigma} \text{Spec } k$ to obtain a map $F \xrightarrow{\sigma_*} F_0$. The first part of Lemma 5.4.2 yields a map $F \xrightarrow{y} F_1$. Hence we have a map

$$\gamma : F \xrightarrow{\sigma_* \oplus y} F_0 \bigoplus F_1$$

We define a map $F_0 \xrightarrow{v} F$ by sending

$$\{x_1, \ldots, x_r, z, g_1, \ldots, g_s\}_{E/k} \mapsto \{x_1, \ldots, x_r, \deg{}_E(z) \cdot \tau_E^*(w), g_1, \ldots, g_s\}_{E/k}$$

It is easy to verify that this map is well-defined.

Finally, the second part of Lemma 5.4.2 yields a map $F_1 \xrightarrow{c} F$. Hence we have a map

$$\delta : F_0 \bigoplus F_1 \xrightarrow{v+c} F$$

It is easy to check that $\gamma \circ \delta = id$ and $\delta \circ \gamma = id$; hence, the proposition is proven.

The next statement follows by induction on Proposition 5.4.3

Corollary 5.4.4. *Let k be a field and $X_1, \ldots, X_r$ smooth projective varieties defined over k, each admitting a zero-cycle of degree 1. Suppose $G_1, \ldots, G_s$ are any semi-abelian varieties defined over k.*

Then there is a canonical map yielding an isomorphism

$$K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r), G_1, \ldots, G_s) \xrightarrow{\cong}$$

$$K(k; G_1, \ldots, G_s) \oplus \bigoplus_{1 \leq \nu \leq r} \bigoplus_{1 \leq i_1 < \ldots < i_\nu \leq r} K(\mathcal{A}_0(X_{i_1}), \ldots, \mathcal{A}_0(X_{i_\nu}), G_1, \ldots, G_s)$$

Finally, if C is a smooth projective curve with $C(E) \neq \emptyset$, we may identify $A_0(C_E)$ functorially with the E-valued points of the Jacobian J of C (cf. [RS 97], p.9). This gives us the following corollary, in which the Somekawa K-groups appear.

Corollary 5.4.5. *Suppose k is a field and $C_1, \ldots, C_r$ are smooth projective curves defined over k with $C_i(k) \neq \emptyset$ for all $i = 1, \ldots, r$. Let J_i denote the Jacobian of C_i for each $i = 1, \ldots, r$, and let $G_1, \ldots, G_s$ be semi-abelian varieties defined over k. Then there is a canonical map establishing an isomorphism*

$$K(k; \mathcal{CH}_0(C_1), \ldots, \mathcal{CH}_0(C_r), G_1, \ldots, G_s) \cong$$

$$K(k; G_1, \ldots, G_s) \oplus \bigoplus_{\nu=1}^{r} \bigoplus_{1 \leq i_1 < \ldots < i_\nu \leq r} K(k; J_{i_1}, \ldots, J_{i_\nu}, G_1, \ldots, G_s)$$

In the case $s = 0$, this result comes out of the proof of [RS 97], Corollary 2.4.1; however, that proof involves Chow motives whereas ours is more direct.

Another curious consequence proceeds from Corollary 5.4.4 and Theorem 3.6.1:

Corollary 5.4.6. *Suppose k is a field, and $X_1, \ldots, X_r$ smooth projective varieties defined*

over k, each admitting a zero-cycle of degree 1. *Let* $s \geq 0$ *be an integer. Then the group* $K_s^M(k)$ *is a direct summand of the group*
$K_s(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_r),\mathbf{G}_m)$

Chapter 6

Mixed K-groups and Cohomology of Milnor K-sheaves.

6.1 Calculating Zariski cohomology of Milnor K-sheaves

Our task in this section is to define Milnor K-sheaves on the Zariski site of a scheme and to present some results concerning the highest nontrivial cohomology group of such sheaves. In preparation for this construction, we define the Milnor K-theory of a ring in imitation of Definition 2.1.1.

Definition 6.1.1. *Let R be a ring, and s an integer. The Milnor K-groups $K_s^M(R)$ are defined as follows:*

$$K_s^M(R) = 0 \textit{ for } s < 0,\ K_0^M(R) = \mathbf{Z},\ K_1^M(R) = R^*$$

For $s \geq 2$,

$$K_s^M(R) = \frac{\bigotimes_{i=1}^{s} R^*}{I_s}$$

where $I_s \subseteq \bigotimes_{i=1}^{s} R^*$ is the subgroup generated by elements of the form $a_1 \otimes \ldots \otimes a_s$ with $a_i + a_j = 1$ for some $i < j$. The class of $a_1 \otimes \ldots \otimes a_s$ in $K_s^M(R)$ is denoted $\{a_1, \ldots, a_s\}$.

The following properties are immediate from the above definition.

Proposition 6.1.2. *1. This definition is covariant functorial: if $\phi : R \longrightarrow R'$ is a* *homomorphism of rings, then there is a canonically induced map*

$$\phi_* : K_s^M(R) \longrightarrow K_s^M(R')$$

defined by

$$\phi_*(\{a_1, \ldots, a_r\} = \{\phi(a_1), \ldots, \phi(a_r)\}$$

2. *If* $R = \overset{\lim}{\rightarrow} R_i$*, then* $K_s^M(R) \cong \overset{\lim}{\rightarrow} K_s^M(R_i)$.

For the remainder of this section, "sheaf" means "Zariski sheaf".

Definition 6.1.3. *Let X be a scheme and $s \geq 0$ an integer. The sheaf $\mathcal{K}_s^M$ on X is defined to be the sheaf associated to the presheaf $U \mapsto K_s^M(\mathcal{O}_X(U))$.*

We now restrict our attention to quasi-projective varieties X defined over a field k, and focus on a long-standing problem: does there exist an explicit flasque resolution of the sheaf $\mathcal{K}_s^M$? If the answer to this question is affirmative, we obtain an explicit formula for all the cohomology groups $H^*_{Zar}(X, \mathcal{K}_s^M)$. One of the reasons for studying such a question stems from its counterpart in Quillen (algebraic) K-theory:

Theorem 6.1.4. *(Quillen, [Qu 73], Theorem 5.11)*
Let k be a field and X a smooth quasiprojective variety of dimension d defined over k. Let $\mathcal{K}_s$ denote the sheaf on X associated to the presheaf $U \mapsto K_s(U)$, where $K_s(U)$ is the sth Quillen K-group of the scheme U. For any point $x \in X$, let i_x : Spec $k(x) \longrightarrow X$ denote the canonical inclusion, and for any abelian group A, let $(i_x)_*A$ denote the direct image (with respect to i_x) of the constant sheaf A on Spec $k(x)$. Then there is an exact sequence furnishing a flasque resolution, often termed the Gersten resolution *of $\mathcal{K}_s$, as follows. (The maps ∂ come from the coniveau spectral sequence for Quillen K-theory)*

$$0 \longrightarrow \mathcal{K}_s \longrightarrow \bigoplus_{x \in X^0} (i_x)_* K_s \xrightarrow{\partial} \bigoplus_{x \in X^1} (i_x)_* K_{s-1} \xrightarrow{\partial}$$
$$\ldots \xrightarrow{\partial} \bigoplus_{x \in X^{d-1}} (i_x)_* K_{s-d+1} \xrightarrow{\partial} \bigoplus_{x \in X^d} (i_x)_* K_{s-d} \longrightarrow 0$$

Unfortunately, the analogous sequence obtained by substituting Milnor K-theory for Quillen K-theory above is not known to be exact at the second term $\mathcal{K}_s^M$, although it is conjectured to be so. Since we will refer to this conjecture later, we state it formally here:

Conjecture 6.1.5. *(Gersten Conjecture for Milnor K-theory) With the notation of Theorem 6.1.4, the following sequence of sheaves is exact:*

$$0 \longrightarrow \mathcal{K}_s^M \longrightarrow \bigoplus_{x\in X^0} (i_x)_* K_s^M \xrightarrow{\partial} \bigoplus_{x\in X^1} (i_x)_* K_{s-1}^M \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{x\in X^d} (i_x)_* K_{s-d}^M \longrightarrow 0$$

The following proposition summarizes the extent of our knowledge about the problem, and gives, in spite of the absence of an explicit resolution for $\mathcal{K}_s^M$, an expression for its highest cohomology group as a sheaf on X.

Theorem 6.1.6. *Let notation be as in Proposition 6.1.4.*

1. *(Rost [Rost 96], Cor. 6.5; Müller-Stach / Elbaz-Vincent, [MSEV 98], Prop. 4.1)*

   For any field k, the following sequence of sheaves on X is exact:

   $$\bigoplus_{x\in X^0} (i_x)_* K_s^M \xrightarrow{\partial} \bigoplus_{x\in X^1} (i_x)_* K_{s-1}^M \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{x\in X^d} (i_x)_* K_{s-d}^M \longrightarrow 0$$

   where the maps ∂ are induced by the appropriate boundary maps of Milnor K-theory (cf. Section 2.2.3).

2. *(Gabber, [Ga 98])*

   If k is infinite, then the augmented complex

   $$\mathcal{K}_s^M \longrightarrow \bigoplus_{x\in X^0} (i_x)_* K_s^M \xrightarrow{\partial} \bigoplus_{x\in X^1} (i_x)_* K_{s-1}^M \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{x\in X^d} (i_x)_* K_{s-d}^M \longrightarrow 0$$

   is exact.

3. *(Soulé, [Sou 85], Théorème 5)*

Assume that k is infinite. Then the sequence of Conjecture 6.1.5 is exact up to localization by $(s-1)!$; that is, after tensoring with $\mathbf{Z}[\frac{1}{(s-1)!}]$ (The precise statement of Soulé's theorem is slightly stronger)

4. *(Kato, [Kato 86], Theorem 3)*

   Let k be any field. Set $d = \dim X$ and define

$$\mathcal{N}_{X,s} = Ker\ (\bigoplus_{x\in X^0}(i_x)_*K_s^M \longrightarrow \bigoplus_{x\in X^1}(i_x)_*K_{s-1}^M)$$

   Then for any $s \geq \dim X$, there are isomorphisms

$$H^d_{Zar}(X,\mathcal{K}_s^M) \cong H^d_{Zar}(X,\mathcal{N}) \cong \frac{\bigoplus_{x\in X^d} K^M_{s-d}(k(x))}{\partial(\bigoplus_{x\in X^{d-1}} K^M_{s-d+1}(k(x)))}$$

Remark.

Comparatively little is known about the Zariski cohomology groups $H^p_{Zar}(X,\mathcal{K}_q^M)$ when X is not smooth. However, arguments of Barbieri Viale ([Bar 90], Theorem 2.5) show that at least under the assumption of Conjecture 6.1.5, $H^p_{Zar}(X,\mathcal{K}_q^M) = 0$ for $p \geq q + d_s + 2$, where d_s is the dimension of the singular locus of X. (In [Bar], this is proven for Quillen K-theory, but all of the arguments given work, *mutatis mutandis*, for Milnor K-theory)

6.2 Mixed K-groups and the Zariski cohomology of Milnor K-sheaves

In this section, we prove the following theorem relating mixed K-groups to the Zariski cohomology of Milnor K-sheaves.

Theorem 6.2.1. *Let k be a field and X a smooth projective variety of dimension d. Then for any integer $s \geq 0$ there exists a natural isomorphism:*

$$H^d_{Zar}(X,\mathcal{K}^M_{d+s}) \xrightarrow{\cong} \tilde{K}_s(k;\mathcal{CH}_0(X),\mathbf{G}_m)$$

In the hope of making the proof more readable, we introduce some auxiliary notation. Let $\mathcal{I}$ be an indexing set and $i \mapsto A_i$ an assignment of an abelian group A_i to each

$i \in \mathcal{I}$. Fix an index $j \in \mathcal{I}$ and suppose $x \in A_j$. The image of x under the canonical map $A_j \longrightarrow \bigoplus_{i\in\mathcal{I}} A_i$ will be denoted $(x)_j$, and the image of A_j itself will be denoted $(A_j)_j$.

Proof.

Our strategy in the proof is to construct maps in both directions and show that either composition is the identity.

The first step in the proof is the construction of a map

$$\gamma : H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \longrightarrow \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$$

By Theorem 6.1.6, this amounts to constructing a map

$$\gamma : \frac{\bigoplus_{x\in X^d} K^M_s(k(x))}{\partial(\bigoplus_{y\in X^{d-1}} K^M_{s+1}(k(y)))} \longrightarrow \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$$

For every $x \in X^d$, we send:

$$(\{a_1, \dots, a_s\})_x \mapsto \{i^*([x]), a_1, \dots, a_s\}_{k(x)/k}$$

where $i : k \hookrightarrow k(x)$ is the natural inclusion. Note that if $a_i + a_j = 1$ for some $i < j$, then the expression on the right is equivalent to zero by Lemma 5.2.1.

It remains to check that elements of $\partial(\bigoplus_{y\in X^{d-1}} K^M_{s+1}(k(y)))$ are killed by γ. Fix $y \in X^{d-1}$ and calculate

$$\gamma(\partial((\{b_1, \dots, b_{s+1}\})_y)) = \gamma(\sum_{z\in\mathcal{P}(k(y)/k)} \sum_{l=1}^{n_v} \partial_z(\{b_1, \dots, b_{s+1}\})_z)$$

$$= \sum_{z\in\mathcal{P}(k(y)/k)} S_{k(z),s_z(y)}(\partial_z\{b_1, \dots, b_{s+1}\})$$

where $s_z(y)$ denotes the specialization map and $S_{k(z),s_z(y)}$ is the map defined in Section 5.2. The expression above is a relation of type **S2** and hence is zero in the group $\tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$. Therefore γ is well-defined.

It is helpful to give an alternate characterization of the map γ by identifying each summand $K^M_s(k(x))$ with $K_s(k(x); \mathbf{G}_m)$ via the isomorphism $K^M_s(k(x)) \xrightarrow{\alpha} K_s(k(x); \mathbf{G}_m)$ of

Theorem 3.6.1. We remark that the construction can be done directly, without resorting to Theorem 3.6.1; if we chose that route, we would recover Theorem 3.6.1 as a consequence of our more general result. It is only in the interest of simplifying our arguments that we choose the former option.

Consider the rule

$$\tilde{\gamma} : (\{b_1, \dots, b_s\}_{E/k(x)})_x \mapsto \{i_E^*[x], b_1, \dots, b_s\}_{E/k}$$

where $i_E : k \hookrightarrow E$ is the natural inclusion map.

Assuming for a moment that this is well-defined, it clearly agrees with the γ on generators $\{a_1, \dots, a_s\}_{k(x)/k(x)}$ of $K_s(k(x); \mathbf{G}_m)$, hence it defines the same map.

We now check that this rule is well-defined. To treat the case of relations of type **M1**, suppose we have finite extensions $k(x) \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ and points $g_i \in E_1^*$ for $i \neq i_0$, $g_{i_0} \in E_2^*$. Then

$$\tilde{\gamma}((\{\phi^*(g_1), \dots, g_{i_0}, \dots, \phi^*(g_s)\}_{E_2/k(x)})_x)$$

$$= \{i_{E_2}^*[x], \phi^*(g_1), \dots, g_{i_0}, \dots, \phi^*(g_s)\}_{E_2/k}$$

$$= \{i_{E_1}^*[x], g_1, \dots, N_{E_2/E_1}(g_{i_0}), \dots, g_s\}_{E_1/k}$$

$$= \tilde{\gamma}((\{g_1, \dots, N_{E_2/E_1}(g_{i_0}), \dots, g_s\}_{E_1/k(x)})_x)$$

Now consider a relation of type **R2** corresponding to the data $K \in \mathcal{T}_1(k(x))$, $h \in K^*$, and $g_1, \dots, g_s \in K^*$ such that for each $v \in \mathcal{P}(K/k(x))$, $g_i \in O_v^*$ for all $i \neq i(v)$.

Then

$$\tilde{\gamma}((\sum_{v \in \mathcal{P}(K/k(x))} \{g_1(v), \dots, \partial_v(g_{i(v)}, h), \dots, g_s(v)\}_{k(v)/k(x)})_x)$$

$$= \sum_{v \in \mathcal{P}(K/k(x))} \{i_{k(v)}^*[x], g_1(v), \dots, \partial_v(g_{i(v)}, h), \dots, g_s(v)\}_{k(v)/k}$$

which is a relation of type **M2** in the group $\tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$ corresponding to the data

$K \in \mathcal{T}_1(k)$, $h \in K^*$, $g_1, \dots, g_s \in K^*$ and $i_K^*[x] \in CH_0(X_K)$.

Next, we turn our attention to the construction of a map

$$\eta : \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow \frac{\bigoplus_{x \in X^d} K_s^M(k(x))}{\partial(\bigoplus_{y \in X^{d-1}} K_{s+1}^M(k(y)))}$$

We do this in the spirit of [Bl 81].

Let $\{x, a_1, \dots, a_s\}_{E/k}$ be an element of $\tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$.
Consider the following diagram:

$$\begin{array}{ccc}
\bigoplus_{y \in X_E^{d-1}} k(y)^* \otimes (E^*)^{\otimes s} & \xrightarrow{\partial \otimes id^{\otimes s}} & \bigoplus_{x \in X_E^d} (E^*)^{\otimes s} \\
\downarrow \cup & & \downarrow \cup \\
\bigoplus_{y \in X_E^{d-1}} K_{s+1}^M(k(y)) & \xrightarrow{\partial} & \bigoplus_{x \in X_E^d} K_s^M(k(x)) \\
\downarrow N & & \downarrow N \\
\bigoplus_{w \in X^{d-1}} K_{s+1}^M(k(w)) & \xrightarrow{\partial} & \bigoplus_{z \in X^d} K_s^M(k(z))
\end{array}$$

The symbol ∂ over the top horizontal arrow stands for the (various) boundary homomorphisms $k(y)^* \cong K_1^M(k(y)) \longrightarrow K_0^M(k(x)) \cong \mathbf{Z}$. The symbol ∂ over the middle and bottom arrow stands for the (various) boundary homomorphisms $K_{s+1}^M(\cdot) \longrightarrow K_s^M(\cdot)$. The top left vertical map $\cup$ is defined as follows: fix $y \in X_E^{d-1}$ and let $E^* \overset{i}{\hookrightarrow} k(y)^*$ be the natural inclusion. We send

$$(a_1 \otimes e_1 \otimes \ldots \otimes e_s)_y \mapsto (\{a_1, i(e_1), \dots, i(e_s)\})_y$$

The top right vertical map is defined similarly.
The bottom left vertical maps are defined thus: fix $y \in X_E^{d-1}$ and let $\pi_E : X_E \longrightarrow X$ be the natural map. Then we send

$$(\{b_1, \dots, b_{s+1}\})_y \mapsto (N_{k(y)/k(\pi_E(y))}(\{b_1, \dots, b_{s+1}\}))_{\pi_E(y)}$$

if $[k(y) : k(\pi_E(y))]$ is finite, and to zero if $[k(y) : k(\pi(y))]$ is infinite. In the expression immediately above, the symbol N stands for the norm map on Milnor K-theory.
The bottom right vertical map is defined similarly.

The top square is easily seen to be commutative by explicit computation. Commutativity of the bottom square is nontrivial and follows from Proposition 2.3.3.

As the aforementioned diagram is commutative, it induces (vertical) maps between the various cokernels of the horizontal maps. The cokernel of the top map is (by definition) $CH_0(X_E) \otimes (E^*)^{\otimes s}$. The cokernel of the middle map is $H^d_{Zar}(X_E, \mathcal{K}^M_{d+s})$ by Theorem 6.1.6, and that of the bottom map is $H^d_{Zar}(X, \mathcal{K}^M_{d+s})$ by the same theorem. Summarizing all of this, we have a commutative diagram:

$$\begin{array}{ccccccc}
\bigoplus_{y \in X_E^{d-1}} k(y)^* \otimes (E^*)^{\otimes s} & \xrightarrow{\partial \otimes id^{\otimes s}} & \bigoplus_{x \in X_E^d} (E^*)^{\otimes s} & \longrightarrow & CH_0(X_E) \otimes (E^*)^{\otimes s} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle \eta^{\cup}} & & \\
\bigoplus_{y \in X_E^{d-1}} K^M_{s+1}(k(y)) & \xrightarrow{\partial} & \bigoplus_{x \in X_E^d} K^M_s(k(x)) & \longrightarrow & H^d_{Zar}(X_E, \mathcal{K}^M_{d+s}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \eta^{N}} & & \\
\bigoplus_{w \in X^{d-1}} K^M_{s+1}(k(w)) & \xrightarrow{\partial} & \bigoplus_{v \in X^d} K^M_s(k(v)) & \longrightarrow & H^d_{Zar}(X, \mathcal{K}^M_{d+s}) & \longrightarrow & 0
\end{array}$$

Composition of maps in the rightmost column gives a map

$$\tilde{\eta}_E : CH_0(X_E) \otimes (E^*)^{\otimes s} \longrightarrow H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

and hence a map

$$\tilde{\eta} : \bigoplus_{E/k \text{ finite}} CH_0(X_E) \otimes (E^*)^{\otimes s} \longrightarrow H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

We now check that $\tilde{\eta}$ factors through relations **M1**. Suppose first that we have finite extensions $k \longrightarrow E_1 \xrightarrow{\phi} E_2$ and points $x \in CH_0(X_{E_2})$ and $a_1, \ldots, a_s \in E_1^*$. We may assume without loss of generality that $x = [P]$, where $P \in X_{E_2}$ is some point. For any finite extension L/K, let $i_{L/K} : K^* \longrightarrow L^*$ denote the inclusion and $\pi_{L/K} : X_L \longrightarrow X_K$ denote the canonical map. Let $F_0 = k(\pi_{E_2/k}(P))$, $F_1 = k(\pi_{E_2/E_1}(P))$, $F_2 = k(P)$. Then

$$\tilde{\eta}((x \otimes \phi(a_1) \otimes \ldots \otimes \phi(a_s))_{E_2})$$

$$= (N_{F_2/F_0}(\{i_{F_2/E_1}(a_1), \ldots, i_{F_2/E_1}(a_s)\}))_{\pi_{E_2/k}(P)}$$

$$= (N_{F_1/F_0} \circ N_{F_2/F_1}(\{i_{F_2/E_1}(a_1), \ldots, i_{F_2/E_1}(a_s)\}))_{\pi_{E_1/k}(\pi_{E_2/E_1}(P))}$$

$$= ([F_2 : F_1] N_{F_1/F_0}(\{i_{F_1/E_1}(a_1), \ldots, i_{F_1/E_1}(a_s)\}))_{\pi_{E_1/k}(\pi_{E_2/E_1}(P))}$$

$$= \tilde{\eta}((((i_{E_2/E_1})_*(x) \otimes a_1 \otimes \ldots \otimes a_s)_{E_1})$$

Now suppose $x = [P] \in CH_0(X_{E_1})$, $a_i \in E_1^*$ for $i \neq i_0$, and $a_{i_0} \in E_2^*$. Let $\sum_{y \in X^d_{E_2}} n_y \cdot y$, be a cycle representing $\phi^*(x)$. Set $F_0 = k(\pi_{E_1/k}(P))$, $F_1 = k(P)$, and $F_2^y = k(y)$ for each $y \in X^d_{E_2}$.

$$\tilde{\eta}((\phi^*(x) \otimes \phi(a_1) \otimes \ldots \otimes a_{i_0} \otimes \ldots \otimes \phi(a_s))_{E_2})$$

$$\tilde{\eta}(\sum_y n_y(y \otimes \phi(a_1) \otimes \ldots \otimes a_{i_0} \otimes \ldots \otimes \phi(a_s))_{E_2})$$

$$= \sum_y n_y \cdot (N_{F_2^y/F_0}(\{i_{F_2^y/E_1}(a_1), \ldots, i_{F_2^y/E_2}(a_{i_0}), \ldots, i_{F_2^y/E_1}(a_s)\}))_{\pi_{E_2/k(y)}}$$

$$= \sum_y n_y \cdot (N_{F_1/F_0} \circ N_{F_2^y/F_1}(\{i_{F_2^y/E_1}(a_1), \ldots, i_{F_2^y/E_2}(a_{i_0}), \ldots, i_{F_2^y/E_1}(a_s)\}))_{\pi_{E_1/k(P)}}$$

$$= \sum_y n_y \cdot (N_{F_1/F_0}\{i_{F_1/E_1}(a_1), \ldots, N_{F_2^y/F_1} \circ i_{F_2^y/E_2}(a_{i_0}), \ldots, i_{F_1/E_1}(a_s)\}))_{\pi_{E_1/k(P)}}$$

$$= (N_{F_1/F_0}\{i_{F_1/E_1}(a_1), \ldots, i_{F_1/E_1}(N_{E_2/E_1}(a_{i_0})), \ldots, i_{F_1/E_1}(a_s)\}))_{\pi_{E_1/k(P)}}$$

$$= \tilde{\eta}((x \otimes a_1 \otimes \ldots \otimes N_{E_2/E_1}(a_{i_0}) \otimes \ldots \otimes a_s)_{E_1})$$

This completes the verification that relations of type **M1** are killed. Next we need to check that relations of type **M2** are killed, and finally that relations of type **S2** (in the group $\tilde{K}_s(k; \mathcal{CH}_0(X), \mathbf{G}_m))$ are killed. The proof that relations of type **M2** are killed is virtually identical (in fact, slightly simpler) than that for relations of type **S2**, so we assume the former and prove the latter. Our assumption implies that $\tilde{\eta}$ factors through a map $\eta_0 : K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) \longrightarrow H^d_{Zar}(X, \mathcal{K}^M_{d+s})$. Consider $K \in \mathcal{T}_1(k)$, $h \in K^*$, $x \in CH_0(X_K)$ and $g_1, \ldots, g_s \in K^*$. As before, we may assume without loss of generality that $x = [P]$ for some $P \in X_K$. A closed point of X_K is determined by a morphism Spec $L \longrightarrow X \times_k K$, where L is some finite extension of K.

Case 1 $L = K$; that is, x is the cycle of a K-rational point of X_K

In this case, x may be identified with a morphism Spec $K \xrightarrow{c} X$. By Proposition 4.2.2, this morphism may be specialized at v in a natural way to obtain a morphism Spec $k(v) \xrightarrow{c(v)} X$, and by [RS 97] p.7, the specialization $s_v(x)$ may be identified with $[c(v)]$, viewed as a cycle on $X \times_k k(v)$.

If the image y of c is a closed point, we have:

$$\eta_0(\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v([P])}(g_1, \ldots, g_s, h))$$

$$= (\sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k(y)} \partial_v(\{g_1, \ldots, g_s, h\}))_{\pi_{k(v)/k(y)}}$$

$$= (\sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k(y)} \partial_v(\{g_1, \ldots, g_s, h\}))_y$$

By the reciprocity law for Milnor K-groups (Proposition 2.2.6), viewing $k(y)$ as the base field and $K \in \mathcal{T}_1(k(y))$, the above sum is equal to 0 in $K^M_s(k(y))$

If the image y of c is not closed, then c maps onto the generic point γ of some dimension 1 subvariety of X. Furthermore, we have:

$$\eta_0(\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v([P])}(g_1, \ldots, g_s, h))$$

$$= \sum_{v \in \mathcal{P}(K/k)} (N_{k(v)/k(v)}(\partial_v(g_1, \ldots, g_s, h)))_{c(v)}$$

By Proposition 2.3.3, this is the same as

$$= \sum_{z \in \mathcal{P}(k(\gamma)/k)} (\partial_z(N_{K/k(\gamma)}(\{g_1, \ldots, g_s, h\})))_{\gamma(z)}$$

$$= \partial(N_{K/k(\gamma)}(\{g_1, \ldots, g_s, h\}))$$

where ∂ now denotes the left horizontal arrow on the bottom row of the commutative diagram.

The diagram shows that the class of this element is zero in $H^d_{Zar}(X, \mathcal{K}^M_{d+s})$,

Case 2. L/K is arbitrary.

We reduce to Case 1 by the following argument. It follows from standard properties of intersection theory (cf. [Fu 84], 20.3 and Theorem 6.2) that the homomorphisms which define the specialization σ are functorial in the sense that the following diagram commutes:

$$\begin{array}{ccc} CH_0(X_L) & \xrightarrow{e(w/v)\cdot(s_w)} & \bigoplus_{w \mapsto v} CH_0(k(w)) \\ \Big\downarrow{\scriptstyle N_{L/K}} & & \Big\downarrow{\scriptstyle N_{k(w)/k(v)}} \\ CH_0(X_K) & \xrightarrow{s_v} & CH_0(k(v)) \end{array}$$

Since x is clearly in the image of the left vertical arrow of its extension x_L to an L-rational cycle, we see from the diagram that $s_v(x) = \sum_{w \mapsto v} e(w/v) \cdot N_{k(w)/k(v)}(s_w(x_L))$

Thus

$$\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v([x])}(g_1, \ldots, g_s, h)$$

$$\sum_{v \in \mathcal{P}(K/k)} \sum_{w \mapsto v} S_{k(v), e(w/v) \cdot N_{k(w)/k(v)}(s_w([x_L]))}(g_1, \ldots, g_s, h)$$

By a relation of type **M1** and Proposition 2.3.1, we see that this is equal to

$$\sum_{w \in \mathcal{P}(L/k)} S_{k(w), s_w([x_L])}(g_1, \ldots, g_s, h)$$

where now $g_1, \ldots, g_s, h$ are considered elements of L. The latter expression is evidently a relation of type **S2** corresponding to the data $L \in \mathcal{T}_1(k)$, $h \in L^*$, $x_L \in CH_0(X_L)$ and $g_1, \ldots, g_s \in L$. Thus the problem is reduced to that of Case 1.

This completes our verification that $\tilde{\eta}$ kills relations of type **S2**. Hence we have a map

$$\eta : \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

It remains to check that $\eta \circ \gamma = id$ and $\gamma \circ \eta = id$. The first follows immediately from direct computation. To verify the second assertion, it suffices, in light of the first, to show that γ is surjective. We use the alternate description $\tilde{\gamma}$ of the map γ, described at the beginning of the proof.

Given a symbol $\{x, a_1, \ldots, a_s\}_{E/k} \in \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$, we may assume as usual that $x = [P]$ for some $P \in X_E$. Let $Q \in X$ denote the (closed point) image of the composed morphism Spec $E \longrightarrow X_E \longrightarrow X$, and $k(Q)$ its residue field. Then

$$\{x, a_1, \ldots, a_s\}_{E/k}$$

$$= \{[i^*_E(Q)], a_1, \ldots, a_s\}_{E/k}$$

$$= \tilde{\gamma}((\{a_1, \ldots, a_s\}_{E/k(Q)})_Q)$$

Thus γ is surjective and the proof of Theorem 6.2.1 is complete.

Chapter 7

Mixed K-groups and Higher Chow groups

In the last section, we proved Theorem 6.2.1, which related mixed K-groups to the Zariski cohomology of Milnor K-sheaves. In this section we establish an isomorphism between certain mixed K-groups and various higher Chow groups of zero-cycles. We begin with a description of the higher Chow groups.

7.1 Higher Chow groups: Definition

The original definition of the so-called higher Chow groups for algebraic schemes over a field k goes back to Bloch ([Bl 86]). Bloch defined the higher Chow groups as the homotopy groups of a certain simplicial abelian group, which by the Dold-Kan correspondence (cf. [Wei 95], p.270), coincide with the homology groups of a related chain complex. Not long afterward, Totaro ([To 92]) defined an analogous complex using "cubical" instead of "simplicial" constructions, and showed that it is in fact quasi-isomorphic to Bloch's original complex. As noted by Totaro, the cubical definition is favorable in applications dealing with the product structure on higher Chow groups, as this structure may then be defined very explicitly. Since our methods are related to those of [To 92], we use the cubical complex throughout; in particular, the definition of higher Chow groups given below is the definition in terms of the cubical complex. The definition given below for schemes of finite type over a regular Noetherian scheme of dimension 1 is taken from

Levine [Le 99].

Let k be a field and Y an equidimensional algebraic scheme over k. We use the notation $z^*(Y)$ to denote the free abelian group, graded by codimension, whose generators are the closed integral subschemes of Y.

Definition 7.1.1. *Let* $Y \longrightarrow$ *Spec* k *be an equidimensional algebraic scheme and* $W \overset{i}{\hookrightarrow} Y$ *a closed subscheme which is a local complete intersection. A subvariety* Z *of* Y *is said to* intersect W properly *if*

$$codim_Z(Z \cap W) \geq codim_Y(W)$$

In such a situation, one may define (cf. [Fu 84], 6.6) a pullback map $i^* : z^*(Y)' \longrightarrow z^*(W)$, where $z^*(Y)' \subseteq z^*(Y)$ is the subgroup generated by irreducible subvarieties of Y which intersect W properly.

Definition 7.1.2. *Let* $n \geq 0$ *be an integer. The* n-cube $\square_k^n$ *is defined to be* $(\mathbf{P}_k^1 - \{1\})^n$.

Remark.

The above definition looks a bit bizarre; however, as explained in [To 92], it helps simplify the proof of the main theorem of this section. The key point worth noting is that the scheme $\square_k^n$ may be identified with the more intuitive "cube" $\mathbf{A}_k^n$ via the isomorphism $\mathbf{A}_k^n \cong {\square_k}^n$ given by

$$(x_1, \ldots, x_n) \mapsto (1 - \frac{1}{x_1}, \ldots, 1 - \frac{1}{x_n})$$

We typically use coordinates $t_1, \ldots, t_n$ to describe (closed) points on ${\square_k}^n$. The subscheme defined by the equations $t_{i_1} = \ldots = t_{i_r} = 0, t_{j_1} = \ldots = t_{j_s} = \infty$ is called a (codimension $r+s$) *face* of ${\square_k}^n$.

Given a strictly increasing map $\rho : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$, and $\varepsilon_i \in \{0, \infty\}$ for all $i \notin \text{Im }(\rho)$, we define the *face map* $\tilde{\rho}^\varepsilon : \square_k^m \longrightarrow \square_k^n$ by

$$(\tilde{\rho}^\varepsilon)^* t_i = \begin{cases} t_j & \text{if } i = \rho(j) \text{ for some } j \\ \varepsilon_i & \text{if } i \notin \rho(\{1 \ldots, m\}) \end{cases}$$

Now define $z^*(Y, n) \subseteq z^*(Y \times_k \square_k^n)$ as the subgroup generated by the irreducible closed subschemes which intersect all faces of the cube properly. As implied by the statement following Definition 7.1.1, imposition of this condition allows us to define (intersection-theoretic) pullback homomorphisms with respect to the face maps. One checks that the various pullback homomorphisms induced by the face maps define a complex $z^*(Y,\cdot)$:

$$\ldots \longrightarrow z^*(Y,2) \xrightarrow{d_2} z^*(Y,1) \xrightarrow{d_1} z^*(Y,0)$$

where the maps d_n are defined by

$$d_n = \sum_{i=1}^{n} (-1)^i (\partial_i^\infty - \partial_i^0)$$

Here ∂_i^0 denotes pullback via the face map associated to $\rho : \{1,\ldots,\hat{i},\ldots,n\} \to \{1,\ldots,n\}$
$\rho(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$
and $\varepsilon_i = 0$. Speaking informally, $t_i = 0$ is a face of the n-cube, and ∂_i^0 means "intersect with the hyperplane $t_i = 0$". The map ∂_i^∞ is defined the same way, but replacing 0 with ∞.

Now we define the *higher Chow groups* $CH^*(Y,\cdot)$, to be the homology groups of the complex $z^*(Y,\cdot)$. Precisely, we write $CH^i(Y,n)$ for the codimension i graded piece of the nth homology group of the complex $z^*(Y,\cdot)$.

Remark.

To ease notation in the case that $Y = \text{Spec } A$ is an affine scheme, we often write $z^*(A,\cdot)$ (respectively $CH^*(A,\cdot)$) in place of $z^*(\text{Spec } A,\cdot)$ (respectively $CH^*(\text{Spec } A,\cdot)$).

Remark.

In the notation for the higher Chow groups, no mention is made of the base field. For example, if L is a finite extension of k, one may interpret Spec L as a scheme of finite type over Spec k or over Spec L. Are the Chow groups associated to these two interpretations isomorphic? We sketch a proof below that this is in fact the case.

Let k be a field, k' a finite extension and Y an equidimensional scheme of finite type over k. We consider the groups $z_k^*(Y \times_k k',\cdot)$, $z_{k'}^*(Y \times_k k',\cdot)$, where the subscript indicates the base field under consideration.

It is easy to see that there is a natural isomorphism of complexes:

$$z_k^*(Y \times_k k') \xrightarrow{\iota} z_{k'}^*(Y \times_k k')$$

induced by the isomorphism

$$Y \times_k k' \times_k \square_k^s \cong (Y \times_k k') \times_{k'} \square_{k'}^s$$

The dimension of the intersection of an element Z of $z^*(Y \times_k k', s)$ (considered as a k-scheme) with a face $F \xrightarrow{\sigma} \square_k^s$ of the s-cube over k is given by

$$\dim(F \times_{\square_k{}^s} Z)$$

where the fibered product is taken with respect to σ and the natural projection map $Z \longrightarrow \square_k^s$. On the other hand, if $F' \xrightarrow{\sigma'} \square_{k'}^s$ is the face of the s-cube over k' obtained by base extension, we have, considering Z as a k'-scheme,

$$\dim(F' \times_{\square_{k'}{}^s} Z)$$

$$= \dim((F \times_{\square_k{}^s} Z) \times_k k')$$

which is the same (by [Ha 83], II, ex. 3.20(f)) as the first dimension calculated above. Hence ι induces an isomorphism:

$$z_k^*(Y \times_k k', s) \longrightarrow z_{k'}^*(Y \times_k k', s)$$

Using subscripts as before to indicate the base field, we conclude that there is a natural identification

$$CH_k^*(Y \times_k k', \cdot) \cong CH_{k'}^*(Y \times_k k', \cdot)$$

Remark.

The above discussion actually shows that if $k \hookrightarrow k'$ is *any* extension of fields, then there is a natural homomorphism of complexes

$$z_k^*(Y, \cdot) \longrightarrow z_{k'}^*(Y \times_k k', \cdot)$$

induced by fiber product $- \times_k k'$.

Remark.

Let B be a regular equidimensional noetherian scheme of dimension one. Levine [Le 99] has defined higher Chow groups for schemes $Y \longrightarrow B$ of finite type over B. The definition is the same as above (replacing the base k by B), but with the words dimension and codimension taken to mean the following: given an irreducible scheme X and a morphism $X \longrightarrow B$ of finite type, let η denote image of the generic point of X. If η is a closed point of B, the *dimension* $\dim X$ of X is defined to be Krull dim $(X \times_B k(\eta))$, and if η is not a closed point of B, we define the dimension of X to be Krull dim $(X \times_B k(\eta)) + 1$. If $X' \hookrightarrow X$ is a closed immersion of equidimensional B-schemes, then the *codimension* of X' in X is defined to be $\dim X - \dim X'$.

7.2 Higher Chow groups: Properties

Here we summarize some of the properties enjoyed by the higher Chow groups; we state results on the level of complexes whenever possible. In the following, k is a field, X, Y are (at least) algebraic schemes over k, and all morphisms mentioned are morphisms of k-schemes. All schemes are assumed to be equidimensional. For proofs of these facts, we refer the reader to [Bl 86] and [Bl 94].

1. *Covariant Functoriality*

   Let $X \xrightarrow{f} Y$ be a proper morphism of algebraic schemes. Then pushforward of cycles (cf. [Fu 84], 1.4) induces a homomorphism

   $$z^*(X, n) \xrightarrow{f_*} z^*(Y, n)$$

   of complexes. (with the appropriate shift in codimension) If L/K is a finite extension of fields then the map on cycles induced by the canonical map $X = \text{Spec } L \longrightarrow$

Spec $K = Y$ is often denoted $N_{L/K}$.

2. *Contravariant Functoriality*

Let $X \xrightarrow{f} Y$ be an flat morphism of algebraic schemes. Then pullback of cycles (cf. [Fu 84], 1.7) induces a homomorphism

$$z^*(Y, n) \xrightarrow{f^*} z^*(X, n)$$

and hence a homomorphism

$$CH^*(Y, n) \xrightarrow{f^*} CH^*(X, n)$$

In the case that Y is smooth over k and and $X \longrightarrow Y$ is an arbitrary morphism of quasiprojective varieties, pullback induces a map on the level of Chow groups. The proof of this is somewhat more subtle: because pullback of cycles may not be well-defined on the level of complexes, one needs to work with subcomplexes of $z^*(X, \cdot)$ and $z^*(Y, \cdot)$; see Bloch [Bl 86], Theorem 4.1.

3. *Change of Base Field*

Let $i : k \hookrightarrow k'$ be an extension of fields. Then pullback by i induces a homomorphism

$$z^*(X, \cdot) \longrightarrow z^*(X \times_k k', \cdot)$$

of complexes.

4. *Homotopy*

Let X be an algebraic scheme. Then pullback of cycles induces an natural isomorphism

$$z^*(X, n) \xrightarrow{\cong} z^*(X \times_k \mathbf{A}^1_k, n)$$

5. *Localization*

Let X be quasiprojective, and $Y \subset X$ closed of (pure) codimension d. Then the natural map of complexes

$$\frac{z^*(X,\cdot)}{z^{*-d}(Y,\cdot)} \hookrightarrow z^*(X-Y,\cdot)$$

is a quasi-isomorphism. In particular, there is a long exact sequence

$$\ldots \longrightarrow CH^*(X-Y,n+1) \longrightarrow CH^{*-d}(Y,n) \longrightarrow CH^*(X,n) \longrightarrow$$

$$\ldots \longrightarrow CH^*(X-Y,n) \longrightarrow CH^{*-d}(Y,n-1) \longrightarrow \ldots \longrightarrow$$

$$\longrightarrow \ldots \longrightarrow CH^{*-d}(Y,0) \longrightarrow CH^*(X,0) \longrightarrow CH^*(X-Y,0) \longrightarrow 0$$

6. *Products*

For any quasiprojective schemes X,Y and integers $p,q,r,s \geq 0$, there exists an external product:

$$CH^p(X,q) \otimes CH^r(Y,s) \stackrel{\cdot}{\longrightarrow} CH^{p+r}(X \times_k Y, q+s)$$

$$x \otimes y \mapsto x \cdot y$$

If one uses the cubical complex $z^*(-,\cdot)$ defined above, the external product has an easy description in terms of cycles. Let U be a subvariety of $X \times_k \square_k^q$ and V a subvariety of $Y \times_k \square_k^s$ Using the obvious identification $(X \times_k \square_k^q) \times_k (Y \times_k \square_k^s) \cong (X \times_k Y) \times_k \square_k^{q+s}$ we map $[U] \otimes [V] \mapsto [U \times_k V]$. One then extends this rule to a map $z^*(X,q) \otimes z^*(Y,s) \longrightarrow z^*(X \times_k Y, q+s)$ by bilinearity and checks that it factors through a map on higher Chow groups. This simple description of the product structure is one of the advantages of using the cubical (rather than the simplicial) complex to calculate the higher Chow groups in the present context.

If X is smooth over k, then pulling back along the diagonal $\Delta : X \longrightarrow X \times_k X$ defines an internal product:

$$CH^p(X,q) \otimes CH^r(X,s) \longrightarrow CH^{p+r}(X,q+s)$$

7. *Projection Formula*

Let Y be smooth over k and $f : X \longrightarrow Y$ proper. Then for all $x \in CH^*(X, \cdot)$, $y \in CH^*(Y, \cdot)$,

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

8. *Specific Formulas*

- There is a natural isomorphism

$$CH^*(X) \xrightarrow{\cong} CH^*(X, 0)$$

  of rings, where the graded ring on the left is the (ordinary) Chow ring. (This follows essentially from Proposition 5.1.1)

- If X is smooth over k, $CH^1(X,q) = \begin{cases} \text{Pic }(X) & \text{if } q = 0 \\ \Gamma(X, \mathcal{O}_X^*) & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases}$

7.3 Connection with Mixed K-groups

As a prelude to the statement of our main result, we quote the following important precursor.

Theorem 7.3.1. *(Totaro, [To 92], Theorem 1) Let k be a field and $s \geq 0$ an integer. There is a canonical map inducing an isomorphism*

$$T : K_s^M(k) \xrightarrow{T} CH^s(k, s)$$

such that $T(\{a_1, \ldots, a_s\} = 0)$ if some $a_i = 1$ and $T(\{a_1, \ldots, a_s\}) = [a_1] \cdot \ldots \cdot [a_s]$ otherwise. Here $\cdot$ represents the product structure on the higher Chow groups and the class $[a_i]$ is understood as being defined with respect to the cubical complex.

Remark.

The isomorphism of Theorem 7.3.1 was also proved by Nesterenko and Suslin ([NeSu 89], Theorem 4.9). However, their proof is not as explicit as Totaro's proof and relies on various other nontrivial results from their paper.

We henceforth refer to the map T as the *Totaro isomorphism.* If more than one field enters the discussion, we will use a subscript (e.g. T_k) to specify the Totaro isomorphism under consideration.

By identifying $K_s^M(k)$ with $K_s(k;\mathbf{G}_m)$ via Theorem 3.6.1, we see that Theorem 7.3.1 is a special case of the following more general result which we prove:

Theorem 7.3.2. *Let k be a field and X a smooth projective variety of dimensions d defined over k. Let $s \geq 0$ be an integer. Then there are natural maps*

$$CH^{d+s}(X,s) \xrightarrow{\alpha} K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

and

$$CH^{d+s}(X,s) \xrightarrow{\tilde{\alpha}} \tilde{K}_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

which are in fact isomorphisms.

Remark.

Even though the Totaro isomorphism maps *into* a higher Chow group and the isomorphism α maps *from* a higher Chow group, the proof of either theorem involves the construction of an inverse. We choose the maps T and α for the statements of the theorems (instead of their respective inverses) because they each happen to be easier to define than their inverses.

Proof.

The first step in the proof is to construct a map:

$$\alpha : CH^{d+s}(X,s) \longrightarrow K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

and therefore, by composition with the natural map

$$K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m) \longrightarrow \tilde{K}_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

a map:

$$\tilde{\alpha} : CH^{d+s}(X,s) \longrightarrow \tilde{K}_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

We do this in the spirit of [To 92]. The group $CH^{d+s}(X,s)$ is by definition the homology of the (cubical) complex below at $z^{d+s}(X,s)$

$$\dots \xrightarrow{d_{s+2}} z^{d+s}(X,s+1) \xrightarrow{d_{s+1}} z^{d+s}(X,s) \xrightarrow{d_s} z^{d+s}(X,s-1) = 0$$

Thus every element of $CH^{d+s}(X,s)$ is represented by a (finite) sum of closed points of $X \times_k \square^s_k$, hence is generated by classes $[P]$ where P is a closed point $P : \text{Spec } k(P) \longrightarrow X \times_k \square^s_k$. The point P is determined by k-morphisms $\text{Spec } k(P) \longrightarrow X$ and $\text{Spec } k(P) \longrightarrow \square^s_k \cong \text{Spec } k[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_s}]$; the latter morphism is determined by the images $a_1, \dots, a_s \in k(P)$ of the coordinate functions $t_1, \dots, t_s$. Thus P is determined by the data $x : \text{Spec } k(P) \longrightarrow X$ and elements $a_1, \dots, a_s \in k(P)^*$. We define $x' = x \times id : \text{Spec } k(P) \longrightarrow X \times_k k(P)$ to be the morphism naturally obtained from x.

Define

$$\alpha : CH^{d+s}(X,s) \xrightarrow{\alpha} K_s(k; \mathcal{CH}_0(X_1), \mathcal{CH}_0(X_2); \mathbf{G}_m)$$

by

$$[P] \mapsto \{[x'], a_1, \dots, a_s\}_{k(P)/k}$$

To check that this rule is well-defined, we must show that elements of $d_{s+1}(z^{d+s}(X,s+1))$ are killed by α.

Consider a generator C of $d_{s+1}(z^{d+s}(X,s+1))$. C is the generator corresponds to a closed subvariety of $X \times_k \square^s_k$ of dimension 1 which meets codimension 1 faces of the $(s+1)$-cube in dimension 0 (i.e. in points) and does not meet faces of codimension ≥ 2. Let $\tilde{C} \xrightarrow{\nu} C$ be the normalization of C. The map $\tilde{C} \xrightarrow{\nu} C$ is proper because it is finite (cf. Hartshorne II, Ex. 3.8, Ex. 4.1), and the inclusion map $C \overset{i}{\hookrightarrow} X \times_k \square^{s+1}_k$ is proper because it is a closed immersion. (Hartshorne, Cor. 4.8 (a)). By Hartshorne, Cor. 4.8(b), the composition

$$\phi : \tilde{C} \longrightarrow C \hookrightarrow X \times_k \square^{s+1}_k$$

is proper.

We may interpret i as defined by a collection of $s+2$ morphisms:

$$C \xrightarrow{f} X$$

$$C \xrightarrow{g_1} \mathbf{P}^1_k$$

$$.$$

$$.$$

$$.$$

$$C \xrightarrow{g_{s+1}} \mathbf{P}^1_k$$

Likewise, we may interpret ϕ as given by a collection of $s+2$ morphisms:

$$\tilde{C} \xrightarrow{\tilde{f}} X$$

$$\tilde{C} \xrightarrow{\widetilde{g_1}} \mathbf{P}^1_k$$

$$.$$

$$.$$

$$.$$

$$\tilde{C} \xrightarrow{\widetilde{g_{s+1}}} \mathbf{P}^1_k$$

where $\tilde{f}_i = \nu^*(f_i)$, $\widetilde{g_i} = \nu^*(g_i)$

For convenience, we make the following definition:

Definition 7.3.3. *Let Z be a curve and $h : Z \longrightarrow \mathbf{P}^1_k$ a morphism. Let $w \in Z$ be a closed point. If $h(w) = 0$, we say that h* vanishes *at w. If $h(w) = \infty$, we say that h* explodes *at w. If h either vanishes or explodes at w, we say that h is* critical *at w.*

The assumption that C meets the codimension 1 faces of the cube in points translates into the fact that none of the g_i are identically 0 or ∞. The condition that C does not meet the codimension ≥ 2 faces of the cube at all implies that given $x \in C^1$, at most one of the g_i is critical at x. In particular, this means that for every $v \in \tilde{C}^1$, at most one of the $\widetilde{g_i}$ is critical at v.

For $x \in C^1$, define $j(x)$ to be the index j such that g_j is critical at x if such an index exists, and 1 if no such index exists. Likewise, given $v \in \tilde{C}^1$, define $\tilde{j}(v)$ to be the index $\tilde{j}$ such that $\widetilde{g_{\tilde{j}}}$ is critical at v if such an index exists, and 1 if no such index exists.

Then by definition of the boundary map d_{s+1}, we have:

$$d_{s+1}(C) = \sum_{x \in C^1} (-1)^{j(x)} \mathrm{ord}_x(g_{j(x)})(f(x), g_1(x), \ldots, \widehat{g_{j(x)}(x)}, \ldots, g_{s+1}(x))$$

By [Fu], Ex. 1.2.3, p. 9, this is equal to:

$$\sum_{v \in \tilde{C}^1} (-1)^{\tilde{j}(v)} \mathrm{ord}_v(g_{\tilde{j}(v)})(\tilde{f}(v), \widetilde{g_1}(v), \ldots, \widehat{\widetilde{g_{\tilde{j}(v)}}(x)}, \ldots, \widetilde{g_{s+1}}(v))$$

(the sums above are to be interpreted as formal sums of points)

We now wish to consider *all* valuations of $K = k(C) = k(\tilde{C})$.

Let $\tilde{C} \longrightarrow \bar{\tilde{C}}$ be the projective closure of $\tilde{C}$ and $D = \tilde{\bar{\tilde{C}}} \longrightarrow \bar{\tilde{C}}$ the normalization map. Since $\tilde{C}$ is normal, we have a map $\tilde{C} \longrightarrow D$ by the universal property of normalization (cf. Hartshorne, II, Ex. 3.8), thus enabling us to identify $\tilde{C}$ with a subset of D. We assert the following:

Lemma 7.3.4. *Suppose $v \in D - \tilde{C}$. Then there exists $i(v) \in \{1, \ldots, s+1\}$ such that $\widetilde{g_{i(v)}}(v) = 1$.*

Proof.

Let K denote the function field $k(C) = k(\tilde{C}) = k(D)$. If $\widetilde{g_i}(v) \neq 1$ for all $i = 1, \ldots, s+1$, then all the $\widetilde{g_i}$ are regular at $v \in D$, and therefore we have a morphism Spec $O_v \xrightarrow{\phi_v} X \times_k \square_k^{s+1}$. Putting all of this data into one diagram, we have:

$$\begin{array}{ccc} \mathrm{Spec}\, K & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow \phi \\ \mathrm{Spec}\, O_v & \xrightarrow{\phi_v} & X \times_k \square_k^{s+1} \end{array}$$

Now the right vertical map is proper, so the valuative criterion for properness gives a (unique) map Spec $O_v \longrightarrow \tilde{C}$ making the resulting diagram commutative. This is a contradiction, because v was chosen to be in $D - \tilde{C}$.

Returning to the expression for $d_{s+1}(C)$, we have

$$\alpha(d_{s+1}(C))$$

$$= \alpha(\sum_{v \in \tilde{C}^1} (-1)^{\tilde{j}(v)} \mathrm{ord}_v(g_{\tilde{j}(v)})(\tilde{f}(v), \widetilde{g_1}(v), \ldots, \widehat{\widetilde{g_{\tilde{j}(v)}}(v)}, \ldots, \widetilde{g_{s+1}}(v)))$$

$$= \sum_{v \in \tilde{C}^1} (-1)^{\tilde{j}(v)} \mathrm{ord}_v(g_{\tilde{j}(v)})\{[\tilde{f}(v)'], \widetilde{g_1}(v), \ldots, \widehat{\widetilde{g_{\tilde{j}(v)}}(v)}, \ldots, \widetilde{g_{s+1}}(v))\}_{k(v)/k}$$

By Lemma 7.3.4, this is the same as the sum:

$$= \sum_{v \in \mathcal{P}(K/k)} (-1)^{\tilde{j}(v)} \mathrm{ord}_v(g_{\tilde{j}(v)})\{[\tilde{f}(v)'], \widetilde{g_1}(v), \ldots, \widehat{\widetilde{g_{\tilde{j}(v)}}(v)}, \ldots, \widetilde{g_{s+1}}(v))\}_{k(v)/k}$$

This is a relation of type **M2** corresponding to the data $K \in \mathcal{T}_1(k)$, $h = g_{s+1} \in K^*$, $[(\tilde{f})'_K] \in CH_0(X_K)$ and $g_1, \ldots, g_s \in \mathbf{G}_m(K) = K^*$, where $\tilde{f}_K \in X_K$ is the point naturally obtained from the composed morphism Spec $K \longrightarrow \tilde{C} \xrightarrow{\tilde{f}} X$.

Thus $\alpha(d_{s+1}(C)) = 0 \in K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$, which completes the verification that α is well-defined.

Before defining the inverse map, we will need to derive two localization sequences for the higher Chow groups; they are both generalizations of the localization sequence for Bloch's higher Chow groups stated in [Bl 86] and proven in [Bl 94]. A useful consequence of the existence of localization sequences is a reciprocity law for higher Chow groups, which we state and prove below.

7.4 Localization Sequences and the Reciprocity Law

Theorem 7.4.1. *(Localization theorem) Let k be a field and X a quasiprojective variety over k. Suppose C is a smooth projective curve over k, and let $c \in C$ be any point. Then there exists a canonical quasi-isomorphism:*

$$\frac{z^*(X \times_k C, s)}{\bigoplus_{z \in C^1 : z \neq c} z^{*-1}(X \times_k k(z), s)} \longrightarrow z^*(X \times_k \mathcal{O}_{c,C}, s)$$

By taking the associated long exact sequence, we obtain:

Theorem 7.4.2. *(Single localization sequence)*
Let k be a field and X a quasiprojective variety over k. Suppose C is a smooth projective curve defined over k; let $c \in C$ be any point. Then there exists a long exact sequence:

$$\ldots \longrightarrow \bigoplus_{z \in C^1 : z \neq c} CH^{*-1}(X \times_k k(z), s+1) \longrightarrow CH^*(X \times_k C, s+1) \longrightarrow$$

$$CH^*(X \times_k \mathcal{O}_{c,C}, s+1) \xrightarrow{\partial} \bigoplus_{z \in C^1 : z \neq c} CH^{*-1}(X \times_k k(z), s) \longrightarrow\longrightarrow CH^*(X \times_k C, s) \longrightarrow$$

$$\ldots \longrightarrow \bigoplus_{z \in C^1 : z \neq c} CH^{*-1}(X \times_k k(z), 0) \longrightarrow CH^*(X \times_k C, 0) \longrightarrow CH^*(X \times_k \mathcal{O}_{c,C}, 0) \longrightarrow 0$$

Remark.

Note that the point c of Theorems 7.4.2 and 7.4.1 need not be a closed point.

Lemma 7.4.3. *With notation as above, let $\mathcal{I}$ denote the category whose objects are the open subsets $U \subseteq C$ containing c; for $V, U \in Ob\,\mathcal{I}$, Mor (V, U) is 0 unless $V \subseteq U$, in which case Mor (V, U) is the set containing the inclusion map $V \hookrightarrow U$. Let $\mathcal{J}$ denote the full subcategory of $\mathcal{I}$ whose objects are the open affine subsets $U \subseteq C$ containing c. Then $\mathcal{I}$ and $\mathcal{J}$ are filtered categories (cf. [Wei 95]), and*

$$\varinjlim_{U \in \mathcal{I}} z^*(X \times_k U, -) = \varinjlim_{U \in \mathcal{J}} z^*(X \times_k U, -)$$

Proof.

Since every open subset of a scheme contains an open affine subset, $\mathcal{J}$ is a final subcategory of $\mathcal{I}$ in the sense of [Ar 62], Definition 1.5. Lemma 7.4.3 then follows from [Ar 62], Proposition 1.6..

Lemma 7.4.4. *Let k, X, C and c be as in the statement of Theorem 7.4.1. Let $\{U_i\}_{i \in \mathcal{I}}$ denote the directed system of open neighborhoods of c in C. Then there is a natural map* *inducing an isomorphism:*

$$\xrightarrow{\lim}_{i\in\mathcal{I}} z^*(X \times_k U_i, \cdot) \cong z^*(X \times_k \mathcal{O}_{c,C}, \cdot)$$

Proof.

We may view the assignments $U \mapsto z^*(U \times_k U_i, \cdot)$ and $U \mapsto z^*(U \times_k \mathcal{O}_{c,C}, \cdot)$ as sheaves on (the Zariski site of) X, it suffices to prove Lemma 7.4.4 locally on X; in particular, we may assume that X is affine, say $X = \text{Spec } A$.

By Lemma 7.4.3 we may assume that $\mathcal{I}$ is the collection of open affine neighborhoods of c in C. Suppose $V, U \in \mathcal{I}$ and $V \hookrightarrow U$. The inclusion $V \hookrightarrow U$ is flat ([Ha 83], III. Theorem 9.7(a); thus, by [Ha 83], III. Theorem 9.7(b), the base extension $X \times_k V \times_k \square^n_k \longrightarrow X \times_k U \times_k \square^n_k$ is also flat. Since a direct limit of flat modules is flat, the morphism $X \times_k \text{Spec } \mathcal{O}_{c,C} \times_k \square^n_k \longrightarrow X \times_k U \times_k \square^n_k$ (for any fixed $U \in \mathcal{I}$) is also flat. By the discussion of flat pull-back maps in Section 5.1, we obtain (compatible) maps $z^*(X \times_k U \times_k \square^n_k, \cdot) \longrightarrow z^*(X \times_k \text{Spec } \mathcal{O}_{c,C} \times_k \square^n_k, \cdot)$ and hence a map

$$f : \xrightarrow{\lim}_{i\in\mathcal{I}} z^*(X \times_k U_i, \cdot) \longrightarrow z^*(X \times_k \mathcal{O}_{c,C}, \cdot)$$

We assert that the map f preserves our notion of dimension. To see this, choose a generator $[Z] \in z^*(X \times_k U_i, s)$ of the direct limit group on the left. We may view Z as a U_i-scheme $Z \longrightarrow U_i$ where $U_i = \text{Spec } B_i$. Since $Z \times_{U_i} \text{Spec } O_v$ is Noetherian, it has finitely many irreducible components; let n denote the number of components. Furthermore, since $O_v = \xrightarrow{\lim} B_i$, we may choose j sufficiently large so that $Z \times_{U_i} U_j$ has n irreducible components. Clearly each component Z_k of $Z \times_{U_i} U_j$ has the same dimension as Z; furthermore, each $Z_k \times_{U_j} (\text{Spec } O_v)$ is irreducible. Thus, to show that f maps Z to a cycle of the same dimension as Z, we may assume without loss of generality that $f(Z) = Z \times_{U_i} (\text{Spec } O_v)$ is irreducible.

Let η denote the image of the generic point of Z under the morphism $Z \longrightarrow U_i$. Since Z is actually of finite type over k, the last remark of Section 7.1 implies that $\dim Z$ is equal to Krull dim $(Z \times_k k(\eta))$ if η is a closed point of U_i and Krull dim $(Z \times_k k(\eta)) + 1$ if η is not a closed point (i.e. is the generic point) of U_i. If η is a closed point of U_i, then $\dim Z =$ Krull dim $(Z \times_{U_i} k(\eta))$. If $\eta \neq v$, then, choosing a neighborhood V of v which avoids η, the restriction $Z \times_{U_i} V$ is zero in $z^*(X \times_k V, \cdot)$, as is the cycle corresponding to the image $f(Z) = Z \times_{U_i} (\text{Spec } O_v)$. If $\eta = v$, then

$$\dim f(Z) = \text{Krull dim } (Z \times_{U_i} O_v) \times_{O_v} k(v) = \text{Krull dim } (Z \times_{U_i} k(v)) = \dim Z$$

as desired.

Now assume that η is the generic point of U_i. Thus $\dim Z = \text{Krull dim } (Z \times_k K) + 1$, where K is the function field of U_i. Since U_i is regular; therefore, by [Ha 83](III.9.7), $Z \longrightarrow U_i$ is flat. Since flatness is preserved under base extension, the morphism $Z \times_{U_i} (\text{Spec } O_v) \longrightarrow \text{Spec } O_v$ is also flat; since O_v is regular, by [Ha 83](III. 9.7) we have that $Z \times_{U_i} (\text{Spec } O_v)$ maps onto the generic point of O_v. Since the residue fields of the generic points of U_i and O_v are both K, we have

$$\dim(f(Z)) = \text{Krull dim } (f(Z) \times_{O_v} K) + 1 = \text{Krull dim } ((Z \times_{U_i} O_v) \times_{O_v} K) + 1$$

$$= \text{Krull dim } (Z \times_{U_i} K) + 1 = \dim Z$$

To construct an inverse map g, note that a generator $Z \in z^*(X \times_k \mathcal{O}_{c,C}, s) = z^*(\text{Spec } (A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s])$ may be identified with a closed subscheme of the affine scheme Spec $(A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s]$, hence with some prime ideal $\mathfrak{p}$ of $(A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s]$. Since $\mathcal{O}_{c,C}$ is a discrete valuation ring, it is Noetherian, and since A is itself Noetherian, the ring $(A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s]$ is Noetherian. Thus $\mathfrak{p}$ is finitely generated. Since $\mathcal{O}_{c,C} = \varinjlim_{U \in \mathcal{I}} \mathcal{O}_C(U)$, choose V such that all the generators of $\mathfrak{p}$ are contained in $(A \otimes_k \mathcal{O}_C(V))[t_1, \ldots, t_s]$. We then specify that g send Z to the class of $Z' \in z^*(\text{Spec } (A \otimes_k \mathcal{O}_C(V))[t_1, \ldots, t_s])$ defined by the prime ideal $\mathfrak{q} = \mathfrak{p}^c$, where contraction is defined with respect to the natural map $(A \otimes_k \mathcal{O}_C(V))[t_1, \ldots, t_s] \longrightarrow (A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s]$.

Finally, we show that the maps f and g are inverse to each other. To ease notation, set $R = (A \otimes_k \mathcal{O}_{c,C})[t_1, \ldots, t_s]$ and for each open neighborhood V of c in C, $R_V = (A \otimes_k \mathcal{O}_C(V))[t_1, \ldots, t_s]$. Then

$$(f \circ g)(Z) = (f \circ g)(\text{Spec } R/\mathfrak{p})$$

$$= f(\text{Spec } R_V/\mathfrak{p}^c)$$

$$= \text{Spec } R/\mathfrak{p}^{ce}$$

But since all the generators of $\mathfrak{p}$ were chosen to be in R_V, this is simply

$$= \text{Spec } R/\mathfrak{p} = Z$$

To show that $g \circ f = id$, it is sufficient to show that f is injective. Choose $Z \in \underset{\longrightarrow}{\lim}_{i \in \mathcal{I}} z^*(X \times_k U_i, \cdot)$, and represent it by some element $Z' \subseteq X \times_k U_i$, where $U_i = \text{Spec } B_i$. Thus Z' may be identified with the scheme Spec $(A \otimes_k B_i)/I$ for some ideal I. Then $f(Z)$ corresponds to the ring $(A \otimes_k O_v)/I^e$, where extension is defined with respect to the natural map $B_i \longrightarrow O_v$. If $f(Z) = 0$, then I^e generates $A \otimes_k O_v$; but since $A \otimes_k O_v$ is Noetherian, I^e is finitely generated, which implies that there is a neighborhood $U_j = \text{Spec } B_j$ of c such that the extension of I under the map $A \otimes_k B_i \longrightarrow A \otimes_k B_j$ generates $A \otimes_k B_j$. This enables us to conclude that the representing element Z' for Z maps to 0 under some morphism in the directed set, so $Z = 0$ as an element of the direct limit.

This concludes the proof of Lemma 7.4.4.

We now resume the proof of Theorem 7.4.2
Since homology commutes with direct limits, we have the following:

Corollary 7.4.5. *With notation as in Lemma 7.4.4, there is a natural map inducing an isomorphism*

$$\underset{\longrightarrow}{\lim}_{i \in I} CH^*(X \times_k U_i, \cdot) \cong CH^*(X \times_k \mathcal{O}_{c,C}, \cdot)$$

Now given $U \in \text{Ob } \mathcal{J}$, write $Z_U = C - U$.
For any $s \geq 0$, there exists an exact sequence (cf. [Bl 86],[Bl 94]):

$$0 \longrightarrow z^{*-1}(X \times_k Z_U, s) \longrightarrow z^*(Z_U \times_k C, s) \longrightarrow z^*(X \times_k U, s)$$

Since Z_U is a finite (in particular, discrete) set of a points; as schemes we have $Z_U \cong \coprod_{z \in Z_U} \text{Spec } k(z)$ and an exact sequence:

$$0 \longrightarrow \bigoplus_{z \in Z_U} z^{*-1}(X \times_k k(z), s) \longrightarrow z^*(Z \times_k C, s) \longrightarrow z^*(X \times_k U, s)$$

On the other hand, given a scheme S and an open immersion $i : T \hookrightarrow S$, pullback with respect to the (flat) morphism i gives a map

$$\text{res}_{S,T} : z^*(S, s) \longrightarrow z^*(T, s)$$

Thus, given $V, U \in \mathcal{J}$ such that $V \subseteq U$, we have a commutative diagram of complexes:

$$\begin{array}{ccc} z^*(X \times_k C, s) & \xrightarrow{\text{res}_{C,U}} & z^*(X \times_k U, s) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \text{res}_{U,V}} \\ z^*(X \times_k C, s) & \xrightarrow{\text{res}_{C,V}} & z^*(X \times_k V, s) \end{array}$$

This induces a map on the kernels of the horizontal maps; thus, we have a diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigoplus_{z \in Z_U} z^{*-1}(X \times_k k(z), s) & \longrightarrow & z^*(X \times_k C, s) & \xrightarrow{\text{res}_{C,U}} & z^*(X \times_k U, s) \\ & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \text{res}_{U,V}} \\ 0 \longrightarrow & \bigoplus_{z \in Z_V} z^{*-1}(X \times_k k(z), s) & \longrightarrow & z^*(X \times_k C, s) & \xrightarrow{\text{res}_{C,V}} & z^*(X \times_k V, s) \end{array}$$

Passing to the limit and noting that $\overset{\lim}{\rightarrow}$ is exact, we obtain an exact sequence:

$$0 \longrightarrow \bigoplus_{z \in Z - \{c\}} z^{*-1}(X \times_k k(z), s) \longrightarrow z^*(X \times_k C, s) \longrightarrow \overset{\lim}{\rightarrow}_{U \in \mathcal{J}} z^*(X \times_k \mathcal{O}_C(U), s)$$

Applying Lemma 7.4.4 to the last term in the sequence, we get:

$$0 \longrightarrow \bigoplus_{z \in Z - \{c\}} z^{*-1}(X \times_k k(z), s) \longrightarrow z^*(X \times_k C, s) \longrightarrow z^*(X \times_k \mathcal{O}_{c,C}, s)$$

By Property 5 of Section 7.2, the restriction map

$$\frac{z^*(X \times_k C, s)}{z^{*-1}(X \times_k Z_U, s)} \longrightarrow z^*(X \times_k U, s)$$

is a quasi-isomorphism for all U.

Since direct limits commute with homology, we pass to the limit and conclude that:

$$\frac{z^*(X \times_k C, s)}{\bigoplus_{z \in C^1 : z \neq c} z^{*-1}(X \times_k k(z), s)} \longrightarrow z^*(X \times_k \mathcal{O}_{c,C}, s)$$

is a quasi-isomorphism.

This concludes the proof of Theorem 7.4.1.

Theorem 7.4.6. *(Double localization on curves)*

Let k be a field and X a quasiprojective variety over k. Suppose C is a smooth projective curve over k; let $c \in C$ be any closed point. Then there exists a long exact sequence:

$$\ldots \longrightarrow CH^{*-1}(X \times_k k(c), s+1) \longrightarrow CH^*(X \times_k \mathcal{O}_{c,C}, s+1) \longrightarrow CH^*(X \times_k k(C), s+1) \xrightarrow{\partial}$$

$$CH^{*-1}(X \times_k k(c), s) \longrightarrow CH^*(X \times_k \mathcal{O}_{c,C}, s) \longrightarrow CH^*(X \times_k k(C), s) \xrightarrow{\partial}$$

$$\ldots \longrightarrow CH^{*-1}(X \times_k k(c), 0) \longrightarrow CH^*(X \times_k \mathcal{O}_{c,C}, 0) \longrightarrow CH^*(X \times_k k(C), 0) \longrightarrow 0$$

Proof.

The proof is done exactly as for Theorem 7.4.2; one shows the existence of a canonical quasi-isomorphism:

$$\frac{z^*(X \times_k \mathcal{O}_{c,C}, s)}{z^{*-1}(X \times_k k(c), s)} \longrightarrow z^*(X \times_k k(C), s)$$

by taking the limit over quasi-isomorphisms:

$$\frac{z^*(X \times_k U, s)}{z^{*-1}(X \times_k k(c), s)} \longrightarrow z^*(X \times_k (U - \{c\}), s)$$

coming from [Bl 94], Theorem (0.1), where $U \subseteq X$ runs through the collection of open neighborhoods of z in C.

We use our localization sequence to prove a "reciprocity law" for higher Chow groups, loosely following an idea of Gillet ([Gi 81],p.275). We are indebted to Wayne Raskind for bringing this method to our attention.

Theorem 7.4.7. *(Reciprocity law for Higher Chow Groups) Let k be a field, X a quasiprojective variety over k and C a smooth curve over k with function field K. Let $m, n \geq 0$ be integers. Let $\partial = (\partial_c) : CH^{m+1}(X \times_k K, n+1) \longrightarrow \bigoplus_{c \in C^1} CH^m(X \times_k k(c), n)$ denote the connecting homomorphism of the localization theorem 7.4.2 and for each $c \in C^1$, let $N_{k(c)/k} : CH^m(X \times_k k(c), n) \longrightarrow CH^m(X, n)$ denote the covariant map induced functorially by the canonical morphism $f_c : X \times_k k(c) \longrightarrow X$. Let $\sigma : C \longrightarrow Spec\ k$ denote the structure morphism. Then*

$$\sum_{c \in C^1} N_{k(c)/k} \circ \partial_c = 0$$

Proof.

Consider the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{c \in C^1} z^m(X \times_k k(c), \cdot) & \longrightarrow & z^{m+1}(X \times_k C, \cdot) & \longrightarrow & \frac{z^{m+1}(X \times_k C, \cdot)}{\bigoplus_{c \in C^1} z^m(X \times_k k(c), \cdot)} & \longrightarrow & 0 \\
\downarrow & & \downarrow {\scriptstyle \sum (f_c)_*} & & \downarrow {\scriptstyle \sigma_*} & & \downarrow & & \downarrow \\
0 & \longrightarrow & z^m(X, \cdot) & \xrightarrow{id} & z^m(X, \cdot) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

If we examine the long exact homology sequences associated to the top and bottom short exact sequences, we obtain a commutative diagram, taking into account Theorem 7.4.1, as follows:

$$\begin{array}{ccccc}
CH^{m+1}(X \times_k K, n+1) & \xrightarrow{\partial} & \bigoplus_{c \in C^1} CH^m(X \times_k k(c), n) & \longrightarrow & CH^m(X \times_k C, n) \\
\downarrow & & \downarrow {\scriptstyle \sum_c N_{k(c)/k}} & & \downarrow \\
0 & \xrightarrow{\partial} & CH^m(X, n) & \xrightarrow{id} & CH^m(X, n)
\end{array}$$

By commutativity of the left square, it is clear that

$$\sum_{c \in C^1} N_{k(c)/k} \circ \partial_c = 0$$

7.5 Conclusion of the Proof of Theorem 7.3.2

We now conlcude the proof of Theorem 7.3.2 by exhibiting inverses to the maps α and $\tilde{\alpha}$, respectively. Define

$$\beta : K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow CH^{d+s}(X, s)$$

and

$$\tilde{\beta} : \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow CH^{d+s}(X, s)$$

(both) by

$$\{x, a_1, \ldots, a_s\}_{E/k} \mapsto (\phi_E)_*(x \cdot a_1 \cdot \ldots \cdot a_s)$$

if none of the a_i are equal to 1, and by 0 if at least one a_i is equal to 1. Here $\phi_E : \mathrm{Spec}\ E \longrightarrow \mathrm{Spec}\ k$ is the canonical map; x is interpreted as being in $CH_0(X_E) \cong CH^d(X_E, 0)$; $a_1, \ldots, a_s$ are interpreted as being in $E^* \cong CH^1(X_E, 1)$ and $\cdot$ represents the product structure on the higher Chow groups. Since the $\tilde{K}$-groups have more relations that the K-groups, it suffices (to check well-definition of both maps) to verify that $\tilde{\beta}$ is well-defined.

The fact that $\tilde{\beta}$ kills relations of type **M1** is an immediate consequence of the projection formula for higher Chow groups ([Bl 86], Ex. 5.8(i)). As in the proof of Theorem 6.2.1, the proof that $\tilde{\beta}$ kills relations of type **M2** is virtually identical to the proof that it kills relations of type **S2**, so we assume the former and only treat the latter case. Consider the relation of type **S2** corresponding to some $K \in \mathcal{T}_1(k)$, $h \in K^*$, $[P] \in CH_0(X_K)$ and $g_1, \ldots, g_s \in \mathbf{G}_m(K) = K^*$. Such a relation r looks like this:

$$\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v([P])}(g_1, \ldots, g_s, h)$$

We now present two technical lemmas:

Lemma 7.5.1. *Let F be a field, X a smooth quasiprojective variety over F, $d = \dim X$ and $s \geq 0$. Then for any $x \in CH_0(X)$ and $a, b_1, \ldots, b_{s-2} \in F^*$ and $1 \leq i \leq j \leq s-2$, the class of the element $y = x \cdot b_1 \cdot \ldots \cdot b_i \cdot a \cdot b_{i+1} \cdot \ldots \cdot b_j \cdot (1-a) \cdot b_{j+1} \cdot \ldots \cdot b_{s-2}]$ of*

$CH^{d+s}(X, s)$ *is zero.*

Proof.

The argument here is essentially the same as in [To 92], p.7. Consider the rational curve:

$$t \mapsto (x, b_1, \ldots, b_i, t, b_{i+1}, \ldots, b_j, 1-t, b_{j+1}, \ldots, b_{s-2}, \frac{a-t}{1-t})$$

in $X \times_k \square_k^{s+1}$. Its only intersection with a codimension 1 face of the cube is the point $(x, b_1, \ldots, b_i, a, b_{i+1}, \ldots, b_j, 1-a, b_{j+1}, \ldots, b_{s-2}, 0)$, thus it defines an element of $z^{d+s}(X, s+1)$, and its boundary is (up to sign) equal to y. Therefore the class of y in $CH^{d+s}(X, s)$ is indeed zero.

Lemma 7.5.2. *Let F be a field, X a smooth quasiprojective variety defined over F, $d = \dim X$ and $s \geq 0$. Then for $x \in CH_0(X)$ and $a_1, \ldots, a_i, \ldots, a_s, b_i \in F^*$, the elements*

$$(x \cdot a_1 \cdot \ldots \cdot (a_i b_i) \cdot \ldots \cdot a_s) - (x \cdot a_1 \cdot \ldots \cdot a_i \cdot \ldots \cdot a_s) - (x \cdot a_1 \cdot \ldots \cdot b_i \cdot \ldots \cdot a_s)$$

and

$$(x \cdot a_1 \cdot \ldots \cdot a_i \cdot \ldots \cdot a_s) - (x \cdot a_1 \cdot \ldots \cdot a_i^{-1} \cdot \ldots \cdot a_s)$$

are zero in $CH^{d+s}(X, s)$.

Proof.

In the course of constructing a homomorphism $K_s^M(k) \longrightarrow CH^s(k, s)$, Totaro ([To 92], p.6-7) proves Lemma 7.5.2 for $X = \text{Spec } k$. Thus, he shows that each of the elements above is the image of some 1-cycle C under the boundary map $z^s(k, s+1) \stackrel{d_{s+1}}{\longrightarrow} z^s(k, s)$. Our cycles are then recovered as the image of the 1-cycle $\{x\} \times_k C \in z^{d+s}(X, s+1)$ under the corresponding boundary map $z^{d+s}(X, s+1) \stackrel{d_{s+1}}{\longrightarrow} z^{d+s}(X, s)$

The next lemma is the fundamental link between Milnor K-groups and higher Chow groups:

Lemma 7.5.3. *For any valuation $v \in \mathcal{P}(K/k)$ (viewed as a point on the smooth projective model of K), let*

$$\partial = \partial_v : CH^{d+s+1}(X \times_k K, s+1) \longrightarrow CH^{d+s}(X \times_k k(v), s)$$

denote the boundary map coming from Theorem 7.4.6 or Theorem 7.4.2.
Then

$$N_{k(v)/k}\partial([P]\cdot g_1\cdot\ldots\cdot g_s\cdot h)=\beta(S_{k(v),s_v([P])}(g_1,\ldots,g_s,h))$$

Proof.
Since both sides of the equation in Lemma 7.5.3 are linear in each of $g_1,\ldots,g_s,h$, we may assume, by Proposition 2.1.3, that $v(g_1)=\ldots=v(g_s)=0$ and $v(h)\geq 0$; in this case, the lemma reduces to showing the assertion:

$$\partial([P]\cdot g_1\cdot g_s\cdot h)=\mathrm{ord}_v(h)(s_v([P])\cdot g_1(v)\cdot\ldots\cdot g_s(v))$$

We give a proof by examining the diagram chase which gives us the definition of the map ∂. First set $y=[P]\cdot g_1\cdot\ldots\cdot g_s\cdot h\in CH^{d+s+1}(X_K,d+s+1)$. Thus y is represented by a cycle in $z^{d+s+1}(X_K,s+1)$. By the second remark of Section 7.1, we may view K as the base field, and thus interpret y as a closed point of $(X_K)\times_K\square_k^{s+1}$. As such, it is represented by a closed point $y_0\in X_K$ and a (dimension 0) closed subscheme y_1 of $\square_k^{s+1}$. If we let t_i denote the ith coordinate on $\mathbf{A}_k^1$, we have $\square_k^{s+1}\cong\mathrm{Spec}\ K[\frac{1}{1-t_1},\ldots,\frac{1}{1-t_{s+1}}]$. Since we want to define the subscheme y_1 by $t_i=g_i$ for $i=1,\ldots,s$ and $t_{s+1}=h$, we see that y_1 is defined by the ideal

$$(\frac{1}{1-t_1}-g_1\frac{1}{1-t_1}-1,\ldots,\frac{1}{1-t_s}-g_s\frac{1}{1-t_s}-1,\frac{1}{1-t_{s+1}}-h\frac{1}{1-t_{s+1}}-1)$$

of $K[\frac{1}{1-t_1},\ldots,\frac{1}{1-t_{s+1}}]$.
For convenience of notation, write g_{s+1} in place of h. Since we are dealing with zero-cycles, we may also assume without loss of generality that X is affine, say $X=\mathrm{Spec}\ A$. Hence we may view y_0 as determined by some ideal $J\subseteq A\otimes_k K$. By clearing denominators, we may assume that the generators of J lie in the subring $A\otimes_k O_v$.

Consider the following diagram of complexes coming from the exact sequence of Lemma 7.4.1:

$$\begin{array}{ccccc}
\uparrow{\scriptstyle d_s} & & \uparrow{\scriptstyle d_s} & & \uparrow{\scriptstyle d_s} \\
z^*(X \times_k O_v, s) & \xrightarrow{\pi} & \frac{z^*(X\times_k O_v,s)}{z^{*-1}(X\times_k k(v),s)} & \overset{\rho}{\hookrightarrow} & z^*(X \times_k K, s) \\
\uparrow{\scriptstyle d_{s+1}} & & \uparrow{\scriptstyle d_{s+1}} & & \uparrow{\scriptstyle d_{s+1}} \\
z^*(X \times_k O_v, s+1) & \xrightarrow{\pi} & \frac{z^*(X\times_k O_v,s+1)}{z^{*-1}(X\times_k k(v),s+1)} & \overset{\rho}{\hookrightarrow} & z^*(X \times_k K, s+1) \\
\uparrow{\scriptstyle d_{s+2}} & & \uparrow{\scriptstyle d_{s+2}} & & \uparrow{\scriptstyle d_{s+2}}
\end{array}$$

Let $i : O_v \hookrightarrow K$ denote the canonical map and let $\tilde{y}$ denote the closure of the image of $y \subseteq X_K \times_K \square_k^{s+1}$ under the naturally induced map $X_K \times_K \square_k^{s+1} \longrightarrow X_{O_v} \times_{O_v}$ Spec $O_v[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_{s+1}}]$. Then it is easy to see that $\rho(\pi(\tilde{y})) = y$. In accordance with this description, we may interpet $\tilde{y} \subseteq X_{O_v} \times_{O_v}$ Spec $O_v[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_{s+1}}]$ as determined by the subscheme $\tilde{y}_0$ of $X_{O_v} =$ Spec $(A \otimes_k O_v)$ corresponding to the ideal J^c (where contraction is taken with respect to $A \otimes_k O_v \longrightarrow A \otimes_k K$ and the subscheme $\tilde{y}_1$ of $\square_{O_v}^s$ corresponding to the ideal

$$(\frac{1}{1-t_1} - g_1\frac{1}{1-t_1} - 1, \dots, \frac{1}{1-t_s} - g_s\frac{1}{1-t_s} - 1, \frac{1}{1-ts+1} - h\frac{1}{1-t_{s+1}} - 1)$$

of $O_v[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_{s+1}}]$.

To compute the image of $\tilde{y}$ under d_{s+1}, we need to calculate the intersection of each of the faces $t_i = 0, \infty$ with $\tilde{y}$.

For the purpose of calculating this intersection, note that we may replace X by an affine neighborhood Spec R of $\tilde{y}_0 \in X_{O_v}$; hence R is some finitely generated O_v-algebra, and $\tilde{y}_0$ corresponds to some (prime) ideal I of R.

Hence $\tilde{y}$ is isomorphic to

$$\text{Spec } R/I \times_{O_v} \text{Spec } \frac{O_v[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1} - g_1\frac{1}{1-t_1} - 1, \dots, \frac{1}{1-t_{s+1}} - g_{s+1}\frac{1}{1-t_{s+1}} - 1)}$$

$$\cong \text{Spec } \frac{R/I[\frac{1}{1-t_1}, \dots, \frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1} - g_1\frac{1}{1-t_1} - 1, \dots, \frac{1}{1-t_{s+1}} - g_{s+1}\frac{1}{1-t_{s+1}} - 1)}$$

Intersecting with the face $t_i = 0$; that is, $\frac{1}{1-t_i} = 1$, gives:

$$\cong \text{Spec } \frac{R/I[\frac{1}{1-t_1},\dots,\frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1}-g_1\frac{1}{1-t_1}-1,\dots,\frac{1}{1-t_{s+1}}-g_{s+1}\frac{1}{1-t_{s+1}}-1,\frac{1}{1-t_i}-1)}$$

$$\cong \text{Spec } \frac{R/I[\frac{1}{1-t_1},\dots,\frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1}-g_1\frac{1}{1-t_1}-1,\dots,\frac{1}{1-t_{s+1}}-g_{s+1}\frac{1}{1-t_{s+1}}-1,g_i)}$$

However, Spec $(O_v/(g_i)) \cong \bigoplus_{j=1}^{v(g_i)}$ Spec $k(v)$; thus, the above expression is zero unless $i = s+1$ (a similar argument shows that intersecting with any of the face maps $t_i = \infty$ leaves nothing), in which case we have:

$$\cong \bigoplus_{j=1}^{\text{ord}_v(h)} \text{Spec } \frac{(R/I \otimes_{O_v} k(v))[\frac{1}{1-t_1},\dots,\frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1}-g_1\frac{1}{1-t_1}-1,\dots,\frac{1}{1-t_s}-g_s\frac{1}{1-t_s}-1)}$$

$$\cong \bigoplus_{j=1}^{\text{ord}_v(h)} \text{Spec } (R/I \otimes_{O_v} k(v)) \times_{k(v)} \text{Spec } \frac{k(v)[\frac{1}{1-t_1},\dots,\frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1}-g_1(v)\frac{1}{1-t_1}-1,\dots,\frac{1}{1-t_s}-g_s(v)\frac{1}{1-t_s}-1)}$$

We refer to this element of $z^*(X \times_k O_v, s)$ as a.

Returning to the commutative diagram, our original cycle y satisfies $d_{s+1}(y) = 0$ (since it represents an element of $CH^{d+s+1}(X, s+1)$). Since $\rho(\pi(\tilde{y})) = y$, we must have $d_{s+1}(\pi(\tilde{y})) = 0$, too. Since $\pi(a) = d_{s+1}(\pi(\tilde{y})) = 0$, a must come from some element $\tilde{a}$ of $z^s(X \times_k k(v), s)$.

Evidently, $\tilde{a}$ must also defined (this time as an element of $z^s(X \times_k k(v), s)$ by

$$\cong \bigoplus_{j=1}^{\text{ord}_v(h)} \text{Spec } (R/I \otimes_{O_v} k(v)) \times_{k(v)} \text{Spec } \frac{k(v)[\frac{1}{1-t_1},\dots,\frac{1}{1-t_{s+1}}]}{(\frac{1}{1-t_1}-g_1(v)\frac{1}{1-t_1}-1,\dots,\frac{1}{1-t_s}-g_s(v)\frac{1}{1-t_s}-1)}$$

Hence $\tilde{a}$ is determined by the zero-cycle Spec $(R/I \otimes_{O_v} k(v))$, interpreted as a cycle of $X \times_k k(v)$, and by the ideal $(\frac{1}{1-t_1} - g_1(v)\frac{1}{1-t_1} - 1, \dots, \frac{1}{1-t_s} - g_s(v)\frac{1}{1-t_s} - 1)$ of the ring Spec $k(v)[\frac{1}{1-t_1},\dots,\frac{1}{1-t_s}]$.

Recall that the closed set $\{\tilde{y}_0\}$ was by construction isomorphic (as a scheme) to Spec R/I; but $\{\tilde{y}_0\}$ is also the closure of $\{y_0\}$. Hence the cycle of $z^*(X \times_k k(v), s)$ defined by Spec $R/I \times_{O_v} k(v)$ is (cf. Section 5.1) exactly the specialization of the cycle defined by

y_0; that is, $s_v([P])$. Thus, by the product structure on the cubical complex, the residue of $\tilde{a}$ in $CH^{d+s}(X \times_k k(v), s)$ corresponds to the point $s_v([P]) \cdot g_1(v) \cdot \ldots \cdot g_s(v)$. Thus,

$$\partial([P] \cdot g_1 \cdot \ldots \cdot g_{s+1} \cdot h) = [\tilde{a}] = \mathrm{ord}_v(h)(s_v([P]) \cdot g_1(v) \cdot \ldots \cdot g_s(v))$$

as desired. This concludes the proof of Lemma 7.5.3.

Returning to the proof of Theorem 7.3.2, we have:

$$\tilde{\beta}(r) = \tilde{\beta}(\sum_{v \in \mathcal{P}(K/k)} S_{k(v), s_v([P])}(g_1, \ldots, g_s, h))$$

Finally, Lemma 7.5.3 implies:

$$\tilde{\beta}(r) = \sum_{v \in \mathcal{P}(K/k)} N_{k(v)/k}(\partial([P] \cdot g_1 \cdot \ldots \cdot g_s \cdot h))$$

which is zero by Theorem 7.4.7.

This concludes the verification that $\tilde{\beta}$ (and hence β) is well-defined.

It remains to show that the four compositions $\beta \circ \alpha$, $\alpha \circ \beta$, $\tilde{\beta} \circ \tilde{\alpha}$ and $\tilde{\alpha} \circ \tilde{\beta}$ are (each) the identity. Since the proof is the same for the latter two, we only consider the former two.

Examine a generator of $CH^{d+s}(X, s)$. As described during the construction of the map α, $CH^{d+s}(X, s)$ is generated by the classes of closed points $P : \mathrm{Spec}\ k(P) \longrightarrow X \times_k (\Box_k)^s$. Thus P is determined by $y : \mathrm{Spec}\ k(P) \longrightarrow X$ and elements $a_1, \ldots, a_s \in k(P)^*$. Let $i : \mathrm{Spec}\ k(P) \longrightarrow \mathrm{Spec}\ k$ be the canonical map. To keep the notation clear, we make the following distinction: if $f : \mathrm{Spec}\ k(P) \longrightarrow S$ is a morphism of schemes, we let $\iota^*(f)$ denote the naturally associated morphism $\mathrm{Spec}\ k(P) \longrightarrow S \times_k k(P)$. We reserve the notation i^* for the pullback map on higher Chow groups.

Then by the product structure on the cubical complex, we have:

$$[\iota^*(P)] = [\iota^*(y)] \cdot [a_1] \cdot \ldots \cdot [a_s]$$

viewing $[\iota^*(y)]$ as an element of $CH^d(X \times_k k(P), 0)$ and $a_1, \ldots, a_s$ as members of the group $CH^1(k(P), 1) \cong k(P)^*$.

Then

$$\beta(\alpha([P])) = \beta(\{[\iota^*(y)], a_1, \ldots, a_s\}_{k(P)/k})$$

$$= i_*([\iota^*(y)] \cdot [a_1] \cdot \ldots \cdot [a_s])$$

$$= i_*([\iota^*(P)])$$

$$= [P]$$

Thus $\beta \circ \alpha = id$.

To show that $\alpha \circ \beta = id$, it suffices to show that α is surjective. To this end, fix a generator $\{[P], a_1, \ldots, a_s\}_{E/k} \in K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m)$, where $P \in X_E$ and $a_1, \ldots, a_s \in E^*$. We may assume that E is the smallest extension of k containing all of $a_1, \ldots, a_s$, for if $F \overset{i}{\hookrightarrow} E$ is a smaller extension, then $\{[P], i^*(a_1), \ldots, i^*(a_s)\}_{E/k} = \{i_*([P]), a_1, \ldots, a_s\}_{F/k}$ by a relation of type **M1**. We may interpret P as a morphism Spec $L \longrightarrow X_E$; this, together with the points $a_i \in E^*$, determine a map Spec $L \longrightarrow X_E \times_E (\mathbf{P}^1_E - \{1\})^s$. Let x be the image of Spec L under the composition of P with the canonical map: Spec $L \longrightarrow X_E \longrightarrow X$. Now consider the point of $X \times_k (\square_k)^s$ determined by x and $a_i \in E^*$. This defines a morphism $Q : \text{Spec } k(Q) \longrightarrow X \times_k (\square_k)^s$ making the diagram below commute:

$$\begin{array}{ccc} \text{Spec } L & \longrightarrow & X_E \times_E (\mathbf{P}^1_E - \{1\})^s \\ \downarrow & & \downarrow \\ \text{Spec } k(Q) & \longrightarrow & X \times_k (\mathbf{P}_k - \{1\})^s \end{array}$$

Since E is the smallest extension of k containing all of the a_i, we have $k(Q) \supseteq E$. Let $E \overset{j}{\hookrightarrow} k(Q)$ be the inclusion. If $f : \text{Spec } E \longrightarrow S$ is an E-morphism of schemes, we denote by $\gamma^*(f)$ the naturally associated morphism Spec $k(Q) \longrightarrow S \times_E k(Q)$. Thus we have a commutative diagram:

$$\begin{array}{ccc}
\text{Spec } k(Q) & \xrightarrow{Q_{k(Q)}} & X_{k(Q)} \times_{k(Q)} (\mathbf{P}^1_{k(Q)} - \{1\})^s \\
\downarrow & & \downarrow \\
\text{Spec } L & \longrightarrow & X_E \times_E (\mathbf{P}^1_E - \{1\})^s \\
\downarrow & & \downarrow \\
\text{Spec } k(Q) & \xrightarrow{Q} & X \times_k (\mathbf{P}^1_k - \{1\})^s
\end{array}$$

Therefore $L = \text{Spec } k(Q)$. The above diagram shows that

$$\alpha([Q]) = \{[\gamma^*(P)], j^*(a_1), \dots, j^*(a_s)\}_{k(Q)/k}$$

By a relation of type **M1**, this is equal to

$$\{j_*([\gamma^*(P)]), a_1, \dots, a_s\}_{E/k}$$

$$= \{[P], a_1, \dots, a_s\}_{E/k}$$

Thus α is surjective.

This concludes the proof of Theorem 7.3.2.

The statement of Theorem 7.3.2 also clarifies the relationship between the K and $\tilde{K}$-groups:

Corollary 7.5.4. *Let k be a field, X a smooth projective variety defined over k, and $s \geq 0$ an integer. The natural map*

$$K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow \tilde{K}_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$$

is an isomorphism.

By combining Theorems 6.2.1 and 7.3.2 we obtain the following important result:

Theorem 7.5.5. *Let k be a field, X a smooth variety of dimension d defined over k, and $s \geq 0$. Then there is a canonical isomorphism:*

$$CH^{d+s}(X, s) \xrightarrow{\cong} H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

We remark in passing that Theorem 7.5.5 could have been proved directly, without the mediation of mixed K-groups. However, it was in the interest of resolving the issues surrounding the relations **S2** that we chose not to adopt this approach.

We can extend Theorem 7.5.5 as follows:

Corollary 7.5.6. *Let k be a field, and U a quasiprojective variety such that $U = X - Z$, where X, Z are smooth projective varieties. Let $d = \dim Z$ and $D = \dim X$. Then for every $s \geq 0$ there is a natural isomorphism:*

$$CH^{D+s}(U, s) \xrightarrow{\cong} H^{D}_{Zar}(U, \mathcal{K}^{M}_{D+s})$$

Proof.

We compare the localization sequence for higher Chow groups (Property 5 of Section 7.2) and the localization sequence for cohomology of Milnor K-sheaves on the Zariski site ([Rost 96], p.323). We have a commutative diagram as shown below; the two leftmost vertical arrows are isomorphisms by Theorem 7.5.5:

$$\begin{array}{ccccccc}
CH^{d+s}(Z,s) & \longrightarrow & CH^{D+s}(X,s) & \longrightarrow & CH^{D+s}(U,s) & \longrightarrow & CH^{d+s}(Z,s-1) = 0 \\
\downarrow \cong & & \downarrow \cong & & \vdots & & \\
H^{d}_{Zar}(Z, \mathcal{K}^{M}_{d+s}) & \longrightarrow & H^{D}_{Zar}(X, \mathcal{K}^{M}_{D+s}) & \longrightarrow & H^{D}_{Zar}(U, \mathcal{K}^{M}_{D+s}) & \longrightarrow & H^{d+1}_{Zar}(Z, \mathcal{K}^{M}_{d+s}) = 0
\end{array}$$

By a simple diagram chase, the (induced) dotted arrow is an isomorphism, which proves Corollary 7.5.6.

Remark.

A different proof of Corollary 7.5.6 will be given in Section 10.3; furthermore, the assertion will hold for all smooth quasiprojective U, without reference to an ambient projective scheme.

7.6 Naturality of the Totaro isomorphism

In this section, we summarize various naturality properties of the Totaro isomorphism (cf. Theorem 7.3.1) deduced from results of this chapter.

Proposition 7.6.1. *The Totaro isomorphism respects covariant functoriality in the sense that the following diagram commutes for every extension* $E \overset{i}{\hookrightarrow} F$ *of fields and all integers* $s \geq 0$*:*

$$\begin{array}{ccc} K_s^M(E) & \xrightarrow{T_E\ \cong} & CH^s(E,s) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i^*} \\ K_s^M(F) & \xrightarrow{T_F\ \cong} & CH^s(F,s) \end{array}$$

The proof is obvious and follows directly from the definitions.

Proposition 7.6.2. *Let* $E \overset{i}{\hookrightarrow} F$ *be a finite extension of fields. Then the Totaro isomorphism respects contravariant functoriality; in precise terms, the following diagram commutes:*

$$\begin{array}{ccc} K_s^M(F) & \xrightarrow{T_F\ \cong} & CH^s(F,s) \\ \downarrow{\scriptstyle N_{F/E}} & & \downarrow{\scriptstyle i_*} \\ K_s^M(E) & \xrightarrow{T_E\ \cong} & CH^s(E,s) \end{array}$$

Proof.

If M is any field, note that a closed point Spec $L \longrightarrow {\square_M}^s$ may be identified by a map $M[t_1,\ldots,t_s] \longrightarrow L$, and hence by an s-tuple $(b_1,\ldots,b_s)$, where b_i is the image of t_i under the above map.

Given $\{a_1,\ldots,a_s\} \in K_s^M(F)$, choose elements $\nu_{i_1},\ldots,\nu_{i_s}$ such that $N_{F/E}\{a_1,\ldots,a_s\} = \sum_i \{\nu_{i_1},\ldots,\nu_{i_s}\}$. Then $T_E(N_{F/E}(\{a_1,\ldots,a_s\}))$ is the element of $CH^s(E,s)$ corresponding to the morphism Spec $E \longrightarrow {\square_E}^s$ described above.

On the other hand, $i_*(T_F(\{a_1,\ldots,a_s\}))$ is the element of $CH^s(E,s)$ corresponding to the (composed) morphism

$$\text{Spec } F \longrightarrow {\square_F}^s \longrightarrow {\square_E}^s$$

where the first morphism is as described above and the second morphism is the canonical map.

As Totaro states in his paper, the proof of Theorem 7.3.1 (cf. [To 92], Theorem 1) enables one to identify the class of the F-rational point $i_*(T_F(\{a_1,\ldots,a_s\}))$ of ${\square_E}^s$ in $CH^s(E,s)$

defined by the s-tuple $(a_1, \ldots, a_s)$ with that of the formal sum of E-rational points defined by $(\nu_{i_1}, \ldots, \nu_{i_s})$, that is, with $T_E(N_{F/E}(\{a_1, \ldots, a_s\}))$. Thus $T_E \circ N_{F/E} = i_* \circ T_F$ and the diagram commutes.

As a special case of Lemma 7.5.3, we have:

Proposition 7.6.3. *Let k be a field, and $s \geq 0$ an integer. Suppose $K \in \mathcal{T}_1(k)$ and $v \in \mathcal{P}(K/k)$. Then the Totaro isomorphism is natural with respect to the boundary maps in the sense that the following diagram commutes:*

$$\begin{array}{ccc} K^M_{s+1}(K) & \xrightarrow{T_K \,\cong} & CH^{s+1}(K, s+1) \\ \Big\downarrow{\scriptstyle \partial^M_v} & & \Big\downarrow{\scriptstyle \partial_v} \\ K^M_s(k(v)) & \xrightarrow{T_{k(v)} \,\cong} & CH^s(k(v), s) \end{array}$$

(Here ∂^M_v is the boundary map on Milnor K-theory and ∂_v is the boundary map coming from the localization sequence for higher Chow groups)

Chapter 8

Families of Abelian Varieties

In this section we give several isolated results concerning the rationalized Somekawa K-groups $K(k; G_1, \ldots, G_r) \otimes \mathbf{Q}$ in the case that the G_i are all abelian varieties.

8.1 Rationalized K-groups

Our first observation allows us to reduce certain questions about the groups
$K(k; A_1, \ldots, A_r) \otimes \mathbf{Q}$ to the case that A_i are Jacobians of smooth curves defined over k.

Proposition 8.1.1. *Suppose k is a field and $A_1, \ldots, A_r$ are abelian varieties defined over k. Then the group*

$$K(k; A_1, \ldots, A_r) \otimes \mathbf{Q}$$

is a quotient of

$$K(k; J_1, \ldots, J_r) \otimes \mathbf{Q}$$

where J_i are Jacobians of smooth curves defined over k, each containing a k-rational point. If k is algebraically closed, the above result holds without tensoring with $\mathbf{Q}$.

Proof.

We may assume that our field is infinite, for in the case of a finite field, the K-groups under consideration are torsion and thus become trivial when tensored with $\mathbf{Q}$.

By [Mi2 86], Theorem 10.1, for each A_i, there exists a Jacobian J_i and a homomorphism $J_i \longrightarrow A_i$. The first statement follows from iterated application of Proposition 3.5.3 and the second from iterated application of Corollary 3.5.4.

The next result shows that rationalized K-groups are invariant under isogenies of their (abelian variety) arguments.

Proposition 8.1.2. *Suppose k is a field, A_1, B_1 are abelian varieties defined over k, and $G_2, \ldots, G_r$ semi-abelian varieties defined over k. Let $A_1 \stackrel{f}{\longrightarrow} B_1$ be an isogeny.*
Then the canonically induced map

$$K(k; A_1, G_2, \ldots, G_r) \otimes \mathbf{Q} \stackrel{f_*}{\longrightarrow} K(k; B_1, G_2, \ldots, G_r) \otimes \mathbf{Q}$$

is an isomorphism.

Proof.
Since isogenies are surjective ([Mi1 86], Prop. 8.1), surjectivity of f_* follows from Proposition 3.5.3. Let $n = \deg f$, and let n_{A_1} denote the morphism $a \mapsto na$ defined on on A_1. Clearly $\mathrm{Ker}\ f \subseteq \mathrm{Ker}\ n_{A_1}$, so $A_1 \stackrel{n_{A_1}}{\longrightarrow} A_1$ factors as

$$A_1 \stackrel{f}{\longrightarrow} B_1 \stackrel{g}{\longrightarrow} A_1$$

where g is an isogeny. Then the composed map

$$K(k; A_1, G_2, \ldots, G_r) \otimes \mathbf{Q} \stackrel{f_*}{\longrightarrow} K(k; B_1, G_2, \ldots, G_r) \otimes \mathbf{Q} \stackrel{g_*}{\longrightarrow} K(k; A_1, G_2, \ldots, G_r) \otimes \mathbf{Q}$$

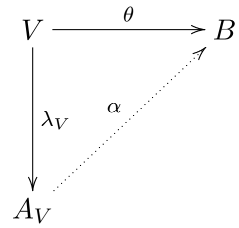
is clearly bijective, which in turn implies that f_* is injective.

8.2 The Albanese variety

Our primary reference for matters concerning the Albanese variety is [Lang 59]. We admit that the definition given below is not by any means the most general definition possible; however, it is sufficient for our needs and simplifies matters considerably.

Let k be a field, and V a smooth projective variety defined over k such that $V(k) \neq \emptyset$. Given a k-rational point P, there exists a pair (A_V, λ_V), where A_V is an abelian variety and $\lambda_V : V \longrightarrow A_V$ is a morphism satisfying the following universal property.

- $\lambda_V(P) = 0 \in A_V$
- Given an abelian variety B and a morphism $\theta : V \longrightarrow B$ such that $\theta(P) = 0$, there exists a unique homomorphism $\alpha : A_V \longrightarrow B$ such that the diagram below commutes:



The pair (A_V, λ_V) is called an *Albanese variety* for V. Clearly the pair (A_V, λ_V) is unique up to unique isomorphism. By abuse of terminology, we often refer to A_V as *the* Albanese variety of V. Usually we use the more suggestive notation $Alb(V)$ in place of A_V.

The Albanese variety possesses the following properties::

- If A is an abelian variety, then (A, id) is an Albanese variety for V.
- If V, W are smooth projective varieties, then $Alb(V \times_k W) \cong Alb(V) \times_k Alb(W)$
- Let V be a smooth projective curve, and $\iota : V \longrightarrow J$ the canonical map of V into its Jacobian (cf. [Mi2 86]). Then (J, ι) is an Albanese variety for V (cf. [Mi2 86], Proposition 6.1).
- The morphism $\lambda_V : V \longrightarrow Alb(V)$ induces a map $(\lambda_V)_* : Z_0(V) \longrightarrow Z_0(Alb(V))$. This may be composed with the map $Z_0(Alb(V)) \longrightarrow Alb(V)$ defined by the rule $\sum_{i=1}^t n_i[P_i] \mapsto \sum_{i=1}^t n_i P_i$, where the first sum is formal and the second sum is the group law on the abelian variety $Alb(V)$, to obtain a homomorphism $Z_0(V) \longrightarrow Alb(V)(k)$. This factors through a homomorphism

  $$\text{alb} : A_0(V) \longrightarrow Alb(V)(k)$$

  called the *Albanese homomorphism.* We refer to the kernel of alb as the *Albanese kernel.*

We will make use of the following important theorem, commonly known as Roitman's theorem, concerning the Albanese variety.

Theorem 8.2.1. *([Roj 80], [Bl 79]) Suppose k is algebraically closed. Then the map $alb : A_0(V) \longrightarrow Alb(V)(k)$ restricts to an isomorphism*

$$A_0(V)_{tors} \xrightarrow{\cong} Alb(V)(k)_{tors}$$

on torsion subgroups.

The original form of Theorem 8.2.1, due to Roitman, stated that the map alb restricts to an isomorphism on torsion prime to char k. The stronger form which appears above is due to Bloch.

Remark.

The Albanese map is in general far from being an isomorphism; a much-quoted counterexample is given by Mumford in ([Mum 69]). However, when k is the algebraic closure of a finite field, the group $A_0(X)$ is a direct limit of finite groups, hence torsion (cf. Theorem 9.1.4); thus Theorem 8.2.1 implies that the Albanese map is in fact an isomorphism. (see also [Mi 82]). A more detailed (and very readable) explanation of these matters may be found in [Ra 89].

Chapter 9

Zero-cycles over finite fields

In this section, we give an (almost) complete description of the higher Chow groups of zero-cycles of smooth quasiprojective varieties defined over a finite field or an algebraic extension of a finite field. Our calculations will make use of the work of Kahn ([Ka 92]) and also Bloch's localization sequence (see Section 7.2).

9.1 Background results

First we review some of the main results concerning Milnor-type K-groups associated to finite fields and function fields.

Theorem 9.1.1. *(Steinberg, [Mil 71]) Let k be a finite field. Then for $s \geq 2$,*

$$K_s^M(k) = 0$$

Theorem 9.1.2. *(Bass-Tate, [BT 73], II. Theorem 2.1) Let K be a global field, and let r_1 be the number of real archimedean places of K. Then $K_2^M(K)$ is an infinite torsion group, and for $s \geq 3$,*

$$K_s^M(K) = (\mathbf{Z}/2\mathbf{Z})^{r_1}$$

In particular, if K is a function field, then for $s \geq 3$,

$$K_s^M(K) = 0$$

The following generalization of Theorem 9.1.1 holds concerning the Somekawa groups:

Theorem 9.1.3. *(Kahn, [Ka 92])*
Let k be a finite field, and $G_1, \ldots, G_r$ semi-abelian varieties defined over k. If $r \geq 2$, then

$$K(k; G_1, \ldots, G_r) = 0$$

Remark.
As we have seen in Section 4, Theorem 9.1.3 is false for $r = 0, 1$ except for the trivial case $r = 1$, $G_1 = 0$.
Remark.
Theorems 9.1.1 and 9.1.3 are also valid (via limit arguments) if k is an algebraic extension of a finite field. We sketch an explanation of this for Theorem 9.1.3: given an element $\{g_1, \ldots, g_s\}_{k/k} \in K(k; G_1, \ldots, G_r)$, let F denote the prime field of k, and let F'/F be some finite extension large enough that $G_1, \ldots, G_r$ are all defined over F' and such that $g_1, \ldots, g_s$ are points defined over F'. Then our element is in the image of the base change homomorphism

$$0 = K(F'; (G_1)_{F'}, \ldots, (G_s)_{F'}) \longrightarrow K(k; (G_1)_k, \ldots, (G_s)_k)$$

(cf. 3.4.1) and hence is equal to zero. (Subscripts are used to indicate the field of definition for the varieties)
The third result we will make use of is a consequence of some (nontrivial) work of Kato and Saito:

Theorem 9.1.4. *(Kato-Saito, [KS 83])*
Let k be a finite field, and X a smooth projective variety defined over k. Then $A_0(X)$ is finite.

Finally, we need the following Proposition:

Proposition 9.1.5. *([Sou 84], 1.5.3. Lemme 1) Let X be a a smooth projective variety defined over a finite field k. Then X admits a zero-cycle of degree* 1.

9.2 Some calculations

By using our previous results, we note the following:

Proposition 9.2.1. *Let k be a finite field and $C_1, \dots, C_r$ smooth projective curves defined over k such that $C_i(k) \neq \emptyset$ for all i. Let J_i denote the Jacobian of C_i. Then the Albanese map*

$$A_0(C_1 \times_k \dots \times_k C_r) \longrightarrow J_1(k) \bigoplus \dots \bigoplus J_r(k)$$

is an isomorphism.

Proof.

Since each curve C_i has a k-rational point, there is an isomorphism:

$$CH_0(C_1 \times_k \dots \times_k C_r) \longrightarrow \mathbf{Z} \bigoplus A_0(C_1 \times_k \dots \times_k C_r)$$

On the other hand, Corollary 2.4 of [RS 97] yields an isomorphism:

$$CH_0(C_1 \times_k \dots \times_k C_r) \cong K(k; \mathcal{CH}_0(C_1), \dots, \mathcal{CH}_0(C_r))$$

Finally, Corollary 5.4.5 yields an isomorphism

$$K(k; \mathcal{CH}_0(C_1), \dots, \mathcal{CH}_0(C_r)) \cong \mathbf{Z} \oplus \bigoplus_{\nu=1}^{r} \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq r} K(k; J_{i_1}, \dots, J_{i_\nu})$$

Comparing the last expression above to the first in this discussion, and observing that (in either case) the $\mathbf{Z}$ term comes from the degree filtration, we conclude that

$$A_0(C_1 \times_k \dots \times_k C_r) \cong \bigoplus_{\nu=1}^{r} \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq r} K(k; J_{i_1}, \dots, J_{i_\nu})$$

By Theorem 4.2.1, we may rewrite the right hand side as

$$\bigoplus_{i=1}^{r} J_i(k) \oplus \bigoplus_{\nu=2}^{r} \bigoplus_{1 \leq i_1 < \dots < i_\nu \leq r} K(k; J_{i_1}, \dots, J_{i_\nu})$$

Since

$$Alb(C_1 \times_k \dots \times_k C_r) \cong Alb(C_1) \times_k \dots \times_k Alb(C_r)$$

we have

$$Alb(C_1 \times_k \ldots \times_k C_r)(k) \cong J_1(k) \bigoplus \ldots J_r(k)$$

It is easy to see that the Albanese map is surjective. Furthermore, the Albanese kernel is isomorphic to

$$\bigoplus_{\nu=2}^{r} \bigoplus_{1 \le i_1 < \ldots < i_\nu \le r} K(k; J_{i_1}, \ldots, J_{i_\nu})$$

Each group in the above direct sum is zero by Theorem 9.1.3; thus, the Albanese map is an isomorphism.

From Theorem 9.1.1, we deduce the following:

Proposition 9.2.2. *Let k be a finite field or an algebraic extension thereof and X a smooth projective variety defined over k. Then*

$$K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) = 0$$

when $s \ge 2$.

Proof.

Fix $s \ge 2$, and suppose k is a finite field. Let $d = \dim X$. By Theorem 6.2.1, we have $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) \cong H^d(X, \mathcal{K}^M_{s+d})$; the latter expression is a quotient of the group $\bigoplus_{x \in X^d} K^M_s(k(x))$. By Theorem 9.1.1, $K^M_s(k(x)) = 0$, whence the desired result. If k is an algebraic extension of a finite field, the result follows by taking limits.

Proposition 9.2.2 does not cover the case $s = 1$. Following the method of [Ka 92], we prove the following:

Theorem 9.2.3. *Let k be a finite field, and X a smooth projective variety defined over k. Then $K_s(k; \mathcal{A}_0(X), \mathbf{G}_m) = 0$ for all $s \ge 1$. In particular, there is a natural isomorphism:*

$$K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) \xrightarrow{\cong} K^M_s(k)$$

for all $s \ge 1$.

Remark.

Before embarking on the proof of Theorem 9.2.3, consider the case in which X is a curve. Let J denote the Jacobian of X. Then Corollary 5.4.5 gives an isomorphism $K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) \cong K_s^M(k) \bigoplus K_s(k; J, \mathbf{G}_m)$. By Kahn's Theorem 9.1.3, the right hand term vanishes, which proves Theorem 9.2.3 in this case.

Proof.

Consider the second assertion of Theorem 9.2.3. The existence of a zero-cycle of degree 1 on X (Proposition 9.1.5) and Proposition 5.4.3 give an isomorphism:

$$K_s(k; \mathcal{CH}_0(X), \mathbf{G}_m) \xrightarrow{\cong} K_s(k; \mathbf{G}_m) \bigoplus K_s(k; \mathcal{A}_0(X), \mathbf{G}_m)$$

Since the first term on the right is isomorphic to $K_s^M(k)$ by Theorem 3.6.1 we see that it suffices, in light of Proposition 9.2.2, to prove that $K(k; \mathcal{A}_0(X), \mathbf{G}_m) = 0$.

We prove the stronger statement that $M(k; \mathcal{A}_0(X), \mathbf{G}_m) = 0$. (see Section 5.2 for the definition of $M(k; \mathcal{A}_0(X), \mathbf{G}_m)$) Fix an algebraic closure $\bar{k}$ of k. We use the notation $\bar{X}$ to denote the variety $X \times_k \bar{k}$.

We begin by listing five key facts.

Proposition 9.2.4. *1. For every finite extension E/k, $A_0(X_E)$ and $\mathbf{G}_m(E) = E^*$ are finite.*

2. *$A_0(\bar{X})$ and $\mathbf{G}_m(\bar{k}) = \bar{k}^*$ are divisible and torsion.*

3. *For every diagram of finite extensions $k \hookrightarrow E \hookrightarrow F$, the norm map $N_{F/E} : F^* \longrightarrow E^*$ is surjective.*

4. *For every (rational) prime $l \neq$ char k, the associated Frobenius automorphism ϕ of Gal $(\bar{k}/k)$ acts (diagonally) without fixed points on $\mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m)$*

   Here $\mathbf{T}_l(A_0(\bar{X})) = \overset{\lim}{\longleftarrow} A_0(\bar{X})[l^n]$ and $\mathbf{T}_l(\mathbf{G}_m) = \overset{\lim}{\longleftarrow} \mu_{l^n}$, the ordinary Tate module of $\bar{k}$.

5. *$M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is a quotient of $\bigoplus_{E/k \text{ finite}} H_0(\text{Gal } (E/k), A_0(X_E) \otimes \mathbf{G}_m(E))$.*

Proof.

Write $G = \text{Gal }(\bar{k}/k)$ and $G_{F/E} = \text{Gal }(F/E)$. The finiteness assertion of Property 1 is obvious for $\mathbf{G}_m(E)$; for $A_0(X_E)$, it follows from Theorem 9.1.4.

Property 2 is obvious for $\mathbf{G}_m(\bar{k})$; for $A_0(\bar{X})$ it follows from [Bl 80], Lemma 1.3.

Property 3 may be seen as follows; there are, of course, much easier proofs, but the one given below also works for semi-abelian varieties (cf. [Ka 92]).

By Hilbert's Theorem 90 $H^1(G_{F/E}, F^*) = 0$. Since $G_{F/E}$ is (finite) cyclic and F^* is finite by Property 1, we have $\#\hat{H}^0(G_{F/E}, F^*) = \#H^1(G_{F/E}, F^*) = 0$ by Chap. VIII, Proposition 8, [Se 62]. (Here, $\hat{H}^n$ denotes Tate cohomology as defined in Chap VIII, [Se 62]). By definition of Tate cohomology, this implies that $N_{F/E} : F^* \longrightarrow (F^*)^{G_{F/E}} = E^*)$ is surjective.

Fix a prime $l \neq \text{char } k$. That ϕ acts on $\mathbf{T}_l(\mathbf{G}_m)$ with no fixed points follows from the fact that ϕ acts as multiplication by l on the torsionfree group $\mathbf{T}_l(\mathbf{G}_m) \cong \mathbf{Z}_l$. To interpret the action of ϕ on $\mathbf{T}_l(A_0(\bar{X}))$, we apply Roitman's Theorem (Theorem 8.2.1) to conclude that the Albanese map

$$A_0(\bar{X}) \longrightarrow Alb_{\bar{X}/\bar{k}}(\bar{k})$$

induces an isomorphism

$$A_0(\bar{X})[l^n] \xrightarrow{\cong} Alb_{\bar{X}/\bar{k}}(\bar{k})[l^n]$$

for each n; hence (by taking inverse limits) an isomorphism

$$\mathbf{T}_l(A_0(\bar{X})) \xrightarrow{\cong} \mathbf{T}_l(Alb_{\bar{X}/\bar{k}})$$

where $\mathbf{T}_l$ on the right hand side is the ordinary Tate module. By the Weil conjectures, the eigenvalues of the action of ϕ on $\mathbf{T}_l(Alb_{\bar{X}/\bar{k}}) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ have absolute value $\sqrt{l} \neq 1$; therefore, ϕ must act without fixed points on $\mathbf{T}_l(Alb_{\bar{X}/\bar{k}})$, hence also on $\mathbf{T}_l(A_0(\bar{X}))$. Property 4 follows from the above and the fact that ϕ acts diagonally on $\mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m)$.

Property 5 follows from Remark 5.2.

Lemma 9.2.5. *$H_0(G, \mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m))$ is a finite group.*

Let $G = \text{Gal }(\bar{k}/k)$ as above. Property 4 implies that the automorphism $\phi - id$ of

$(\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m))\otimes_{\mathbf{Z}_l}\mathbf{Q}_l$ is invertible; hence $H_0(G,(\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m)))\otimes_{\mathbf{Z}_l}\mathbf{Q}_l$ is a quotient of

$$\frac{\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m)}{[\phi - id](\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m))}\otimes_{\mathbf{Z}_l}\mathbf{Q}_l = 0$$

and must itself be 0.

Thus $H_0(G,\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m))$, being the quotient of a finitely generated $\mathbf{Z}_l$-module, is itself finitely generated; by the above, it must be a torsion $\mathbf{Z}_l$-module. However, all torsion $\mathbf{Z}_l$-modules are finite direct sums of modules of the form $\mathbf{Z}_l/a\mathbf{Z}_l$ for some $a \in \mathbf{Z}_l$, hence are finite. Therefore, $H_0(G,\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m))$ is a finite group.

Let $m' = \#H_0(G,\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m))$.

Lemma 9.2.6. *For every finite extension E/k, $(A_0(X_E)\otimes E^*)\{l\}$ is a quotient of a finite number of copies of $\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m)$.*

Proof.

Since both $A_0(X_E)$ and E^* are finite by Property 1 of Proposition 9.2.4, it follows that $(A_0(X_E)\otimes E^*)\{l\} \cong A_0(X_E)\{l\}\otimes E^*\{l\}$. Write $A_0(X_E)\{l\}$ as a finite direct sum of cyclic groups. Each such group is contained in a cyclic group of order l^a for some (fixed) $a \geq 1$; thus $A_0(X_E)\{l\}$ is isomorphic to a subgroup of some power of $T_a = A_0(\bar{X})[l^a] \cong Alb(\bar{X})[l^a] \cong (\mathbf{Z}/l^a\mathbf{Z})^{2g}$, where $g = \dim Alb(\bar{X})$. Since these groups are all finite, $A_0(X_E)\{l\}$ may also be realized (cf. [Lang 93], Corollary 9.3) as a quotient of a power of T_a. Finally, from the construction of the Tate module (inverse limit), T_a is itself a quotient of $\mathbf{T}_l(A_0(\bar{X}))$; hence $A_0(X_E)\{l\}$ is a quotient of some power of $\mathbf{T}_l(A_0(\bar{X}))$. An analogous argument shows that $E^*\{l\}$ is a quotient of some power of $\mathbf{T}_l(\mathbf{G}_m)$; hence we may conclude that $(A_0(X_E)\otimes_{\mathbf{Z}_l}E^*)\{l\}$ is a quotient of some power of $\mathbf{T}_l(A_0(\bar{X}))\otimes_{\mathbf{Z}_l}\mathbf{T}_l(\mathbf{G}_m)$.

Now observe that since all the groups involved are finite abelian groups, we have an isomorphism $A_0(X_E)\{l\}\otimes_{\mathbf{Z}_l}\mathbf{G}_m(E)\{l\} \cong (A_0(X_E)\otimes_{\mathbf{Z}_l}\mathbf{G}_m(E))\{l\}$, and since both $A_0(X_E)$ and $\mathbf{G}_m(E)$ are finite abelian groups, hence direct sums of cyclic groups, we see that $(A_0(X_E)\otimes_{\mathbf{Z}_l}\mathbf{G}_m(E))\{l\} \cong (A_0(X_E)\otimes\mathbf{G}_m(E))\{l\}$, obtaining the desired result.

From Lemma 9.2.6 and right exactness of the functor H_0, we deduce the existence of a surjective map

$$H_0(G, (\mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m))^n) \longrightarrow H_0(G, (A_0(X_E) \otimes E^*)\{l\})$$

where $n \geq 1$ is some (sufficiently large) integer.

We compose this with the natural isomorphism

$$H_0(G, (A_0(X_E) \otimes E^*)\{l\}) \xrightarrow{\cong} H_0(G_{E/k}, (A_0(X_E) \otimes E^*)\{l\})$$

of Shapiro's Lemma (cf. [Wei 95], 6.3) to obtain a surjective map

$$H_0(G, (\mathbf{T}_l(A_0(\bar{X})) \otimes_{\mathbf{Z}_l} \mathbf{T}_l(\mathbf{G}_m))^n) \longrightarrow H_0(G_{E/k}, (A_0(X_E) \otimes E^*)\{l\})$$

Since $H = A_0(X_E) \otimes E^*$ is finite abelian, we may decompose H into its q-primary parts: $H = \oplus_q H\{q\}$. Since H_0 commutes with direct sums, we have $H_0(G_{E/k}, (A_0(X_E) \otimes E^*)\{l\}) = H_0(G_{E/k}, A_0(X_E) \otimes E^*)\{l\}$. Finally, since m' kills the finite group $H_0(G_{E/k}, (\mathbf{T}_l(A_0(\bar{X})) \otimes \mathbf{T}_l(\mathbf{G}_m))^n)$, it must also kill $H_0(G_{E/k}, A_0(X_E) \otimes E^*)\{l\}$ for any finite extension E/k. In particular, any element of $\bigoplus_{E/k \text{ finite}} H_0(G_{E/k}, A_0(X_E) \otimes E^*)\{l\}$ has order dividing m'. Let m_l be the largest power of l less than or equal to m'. Then m_l kills

$\bigoplus_{E/k \text{ finite}} H_0(G_{E/k}, A_0(X_E) \otimes E^*)\{l\}$.

Before we proceed further, we need the following two lemmas:

Lemma 9.2.7. *$M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is torsion divisible.*

Proof.

Property 1 shows that $M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is torsion. To show that it is divisible, fix an integer m and $[x, y]_{E/k} \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)$. Using divisibility of $A_0(\bar{X})$ (Property 2), choose an extension $E \overset{i}{\hookrightarrow} F$ such that $i^*(x) = m \cdot x'$ for some $x' \in A_0(X_F)$. By property 3, there exists $y' \in F^*$ such that $N_{F/E}(y') = y$. Using relation **M1**, we have

$$[x, y]_{E/k} = [i^*(x), y']_{F/k} = m \cdot [x', y']_{F/k}$$

and hence $M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is m-divisible.

Lemma 9.2.8. *$M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\}$ is killed by m_l.*

Proof.

Fix $a \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\}$. By Property 5 of Proposition 9.2.4, a is represented by some element $a' \in H_0(G_{E/k}, \mathcal{A}_0(X_E) \otimes E^*)$ where E/k is some finite extension. If l does not divide the order of a', then the order of a' is prime to l; hence the order of a is prime to l. The assumption $a \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\}$ then forces $a = 0$, which is trivially killed by m_l. If l divides the order a', then some multiple $n \cdot a'$ (n prime to l) is in $H_0(G_{E/k}, \mathcal{A}_0(X_E) \otimes E^*)\{l\}$, which is killed by m_l. Hence $m_l \cdot n \cdot a' = 0$, and thus $m_l \cdot n \cdot a = 0$. Finally, the assumption that a is killed by l^b for some b enables to conclude that a is killed by $\gcd(l^b, m_l \cdot n)$; since this number divides l, we deduce that a itself is killed by m_l. This concludes the proof of Lemma 9.2.8.

To conclude the proof of Theorem 9.2.3, we show that $M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\} = 0$. Choose $x \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\}$. By Lemma 9.2.7, there exists $y \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ such that $m_l y = x$. Since m_l was defined to be a power of l, we have $y \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\}$. Hence $0 = m_l y = x$, and thus $M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{l\} = 0$ for all $l \neq \text{char } k$.

Finally, let $p = \text{char } k$ and suppose $z \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{p\}$. Then z is a sum of symbols of the form $[x, y]_{E/k}$. For each such symbol, the order of y in $\mathbf{G}_m(E) = E^*$ is prime to p; hence the order of $[x, y]_{E/k} \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is also prime to p, and the same is true of the product z of these symbols. Since $z \in M(k; \mathcal{A}_0(X), \mathbf{G}_m)\{p\}$, we conclude that $z = 0$. Finally, since $M(k; \mathcal{A}_0(X), \mathbf{G}_m)$ is torsion, we conclude that $M(k; \mathcal{A}_0(X), \mathbf{G}_m) = 0$. This brings to a close the proof of Theorem 9.2.3.

By taking direct limits, we may also prove:

Corollary 9.2.9. *Let k be an algebraic extension of a finite field and V a smooth projective variety defined over k. Then there are canonical isomorphisms*

$$K_s(k; \mathcal{CH}_0(V), \mathbf{G}_m) \xrightarrow{\cong} K_s^M(k)$$

for $s \geq 1$.

9.3 Applications

We give several applications of the results of Sections 7 and 9.2.

Theorem 9.3.1. *Let k be a finite field or an algebraic extension of a finite field and X a smooth projective variety defined over k. Let $d = \dim X$. Then*

$$CH^{d+s}(X,s) \cong \begin{cases} \mathbf{Z} \oplus A_0(X) & \text{if } s = 0 \\ k^* & \text{if } s = 1 \\ 0 & \text{if } s \geq 2 \end{cases}$$

Proof.

By Theorem 7.3.2, we have an isomorphism $CH^{d+s}(X,s) \cong K_s(k;\mathcal{CH}_0(X),\mathbf{G}_m)$. Furthermore, Proposition 9.1.5 allows us to apply the results of Section 5.4. Given this, the first statement then follows from Proposition 5.4.3, the second from Theorem 9.2.3 or Corollary 9.2.9, and the last from Proposition 9.2.2.

Corollary 9.3.2. *Let k be as in Theorem 9.3.1, and let X, $Z \subseteq X$ be smooth projective varieties. Let $U = X - Z$ and set $d = \dim U = \dim X$. Then*

$$CH^{d+s}(U,s) \cong \begin{cases} \text{finite} & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases}$$

Proof.

All assertions follow from Theorem 9.3.1 and the localization sequence (Property 5 of 7.2).

Chapter 10

Higher-dimensional cycles over finite fields

In this section, we investigate various higher Chow groups of cycles of dimension > 0 associated to varieties over finite fields.

10.1 The tame kernel

We begin by studying a specific group of 1-cycles.

Proposition 10.1.1. *Let k be a finite field, and let C be a smooth projective curve defined over k. Then $CH^2(C,2)$ is finite.*

Proof.

By the localization sequence (Theorem 7.4.2), there is an exact sequence:

$$\ldots \longrightarrow \bigoplus_{z\in C^1} CH^1(k(z),2) \longrightarrow CH^2(C,2) \longrightarrow CH^2(K,2) \xrightarrow{\partial} \bigoplus_{z\in C^1} CH^1(k(z),1) \longrightarrow \ldots$$

By Property 8 of Section 7.2, we have $CH^1(k(z),2) = 0$ for all $z \in C^1$, and by Theorem 7.3.1 we have $CH^2(K,2) \cong K_2^M(K)$ and $CH^1(k(z),1) \cong K_1^M(k(z))$. By Proposition 7.6.3, the map $\partial : CH^2(K,2) \longrightarrow \bigoplus_{z\in C^1} CH^1(k(z),1)$ may be interpreted as the (collection of) boundary maps $K_2^M(K) \xrightarrow{(\partial_z)} \bigoplus_{z\in C^1} K_1^M(k(z))$ of Milnor K-theory. Putting all of this together, we have

$$CH^2(C,2) \cong \text{Ker } (K_2^M(K) \xrightarrow{(\partial_z)} \bigoplus_{z \in C^1} K_1^M(k(z)))$$

By [BT 73], II. Theorem 2.1, the "tame kernel" on the right is a finite group.

10.2 The coniveau spectral sequence

The localization sequence of Theorem 7.4.2 is a special instance of a spectral sequence identified by Bloch in [Bl 86]. Here we introduce the spectral sequence in preparation of the results of the next section.

Our first task is to define a sheaf analogous to the Milnor K-sheaves of Section 6.1. Let k be a field and X an algebraic k-scheme. Given integers r, m, we define $\mathcal{CH}^r(m)$ to be the sheaf associated to the presheaf $U \mapsto CH^r(U, m)$, where $CH^r(U, m)$ is understood to be zero if $m < 0$.

Now let X be a smooth quasiprojective variety over k. Imitating Quillen's construction of a spectral sequence for algebraic K-theory (cf. [Qu 73], Theorem 5.4), Bloch derives a spectral sequence for higher Chow groups, associated to the coniveau filtration

$$F^n z^r(X, \cdot) = \{z \in z^r(X, \cdot) | \text{ the projection of Supp } z \text{ on } X \text{ has codimension } \geq n\}$$

as follows:

Proposition 10.2.1. *(Bloch, [Bl 86], Section 10) Fix $r \geq 0$. With notation as above, there exists a spectral sequence:*

$$E_1^{p,q} = \bigoplus_{x \in X^p} CH^{r-p}(\mathit{Spec}\ k(x), -p-q) \Longrightarrow CH^r(X, -p-q)$$

Remark.

If X is a curve, the spectral sequence reduces (cf. [Wei 95], Ex. 5.2.1) to the localization sequence of Theorem 7.4.2 with c taken to be the generic point of X.

Having established the spectral sequence above, Bloch proves the following "Gersten conjecture" (cf. Theorem 6.1.6) for the sheaf $\mathcal{CH}^r(-q)$:

Theorem 10.2.2. *(Bloch, [Bl 86], Theorem 10.1) Given a point $x \in X$ and an abelian group A, let $(i_x)_*A$ denote the direct image of the constant sheaf A under the map i_x : Spec $k(x) \longrightarrow X$. Consider the following sequence of sheaves, in which the maps are induced by the differentials of the spectral sequence 10.2.1:*

$$\bigoplus_{x \in X^0} (i_x)_* CH^r(\text{Spec } k(x), -q) \longrightarrow^{d_1^{0,q}} \bigoplus_{x \in X^1} (i_x)_* CH^{r-1}(\text{Spec } k(x), -q-1) \longrightarrow^{d_1^{1,q}} \ldots$$

$$\longrightarrow^{d_1^{q-1,q}} \bigoplus_{x \in X^{-q}} (i_x)_* CH^{r-q}(\text{Spec } k(x), 0) \longrightarrow 0$$

The above sequence is a flasque resolution of the sheaf $\mathcal{CH}^r(-q)$ on X.

Theorem 10.2.2 enables us to rewrite the spectral sequence 10.2.1 starting at the E_2 terms as:

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{CH}^r(-q)) \Longrightarrow CH^r(X, -p-q)$$

Now suppose $q = -r$. For such q, we have:

$$E_2^{p,-r} = H^p_{Zar}(X, \mathcal{CH}^r(r))$$

Note that in this case, the sheaf $\mathcal{CH}^r(r)$ has a resolution (10.2.2) in terms of constant sheaves of higher Chow groups (of zero cycles) of the form $CH^{-q-n}(\text{Spec } k(x), -q-n)$, each of which is isomorphic to $K^M_{-q-n}(k(x))$ via the Totaro isomorphism of Theorem 7.3.1. Even more is true:

Proposition 10.2.3. *(Müller-Stach, Elbaz-Vincent, [MSEV 98])*
Let $r \geq 0$ be any integer and let $d = \dim X$. The Totaro isomorphisms T induce a natural isomorphism of complexes:

$$\begin{array}{ccccc} \bigoplus_{x\in X^0}(i_x)_*CH^r(Spec\ k(x),r) & \xrightarrow{d_1^{0,-r}} \cdots & \xrightarrow{d_1^{d-1,-r}} & \bigoplus_{x\in X^d}(i_x)_*CH^{r-d}(Spec\ k(x),r-d) & \longrightarrow 0 \\ \downarrow T & & & \downarrow T & \\ \bigoplus_{x\in X^0}(i_x)_*K_r^M(k(x)) & \xrightarrow{\partial} \cdots & \xrightarrow{\partial} & \bigoplus_{x\in X^d}(i_x)_*(K_{r-d}^M(k(x))) & \longrightarrow 0 \end{array}$$

In particular, there is an isomorphism of sheaves

$$\mathcal{N}_{X,r} \xrightarrow{\cong} \mathcal{CH}^r(r)$$

where $\mathcal{N}_{X,r}$ is as defined in Theorem 6.1.6.

In the following sections, we will exploit this isomorphism to give an alternate proof of Theorem 7.5.5 and to give some results concerning groups of 1-cycles on varieties defined over finite fields.

10.3 Applications

Another proof of Theorem 7.5.5

Let k be a field, X a smooth quasiprojective variety over k of dimension d and $s \geq 0$ an integer. Then the spectral sequence 10.2.1 reads:

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{CH}^{d+s}(-q)) \Longrightarrow CH^{d+s}(X, -p-q)$$

Note that (for $t \geq 2$), $E_t^{d+t,-d-s-t+1} = 0$ for reasons of dimension and $E_t^{d-t,-d-s+t-1} = H^{d-t}(X, \mathcal{CH}^{d+s}(d+s+1-t)) = 0$ (the sheaf is zero by Theorem 10.2.2), so $E_{t+1}^{d,-d-s} \cong E_t^{d,-d-s}$ for all $t \geq 2$. Hence

$$E_\infty^{d,-d-s} \cong E_2^{d,d-s} = H^d_{Zar}(X, \mathcal{CH}^{d+s}(d+s)) \cong H^d_{Zar}(X, \mathcal{N}_{X,d+s}) \cong H^d_{Zar}(X, \mathcal{K}^M_{d+s})$$

where the first isomorphism comes from Proposition 10.2.3 and the latter from Theorem 6.1.6.

Let F^* denote the filtration on $CH^{d+s}(X, \cdot)$ associated to the spectral sequence. For convenience of notation, we write F^t as shorthand for $F^tCH^{d+s}(X, s)$.

We have:

$$E_{\infty}^{d,-d-s} \cong E_2^{d,d-s} = H^d_{Zar}(X, \mathcal{CH}^{d+s}(d+s)) \cong H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \cong \frac{F^d}{F^{d+1}}$$

We also have, for all integers u,

$$E_{\infty}^{d+u,-d-s-u} \cong \frac{F^{d+u}}{F^{d+u+1}}$$

However, if $u > 0$,

$$E_2^{d+u,-d-s-u} = H^{d+u}(X, \mathcal{CH}^{d+s}(d+s+u)) = 0$$

for reasons of dimension and if $u < 0$, the sheaf $\mathcal{CH}^{d+s}(d+s+u)$ is zero by Theorem 10.2.2. Since $E_2^{d+u,-d-s-u} = 0$ for all $u \neq 0$, it follows that $E_{\infty}^{d+u,-d-s-u} \cong \frac{F^{d+u}}{F^{d+u+1}} = 0$ for all $u \neq 0$. Since convergence of the spectral sequence implies that $F^t = 0$ for t sufficiently large and $F^t = CH^{d+s}(X, s)$ for t sufficiently small, we conclude that $F^{d+1} = F^{d+2} = \ldots = 0$ and $CH^{d+s}(X, s) = \ldots = F^{d-1} = F^d$. Thus

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \cong \frac{F^d}{F^{d+1}} \cong CH^{d+s}(X, s)$$

A crash course in the Quillen K-theory of schemes

Since we will use some properties of Quillen K-theory in the sequel, we digress briefly to present a (very superficial) account of these properties. For details, we refer the reader to Quillen's paper [Qu 73]. Rosenberg ([Ros 94]) also presents a readable account of some of the constructions involved.

For any scheme X and $n \geq 0$, Quillen defines groups $K_n(X)$, now known as "Quillen K-theory" as the homotopy groups of a certain topological space associated to the category of locally free sheaves on X. He also defines groups $G_n(X)$ as the homotopy groups of a topological space associated to the category of coherent sheaves on X, and proves that when X is regular, there is a natural isomorphism $K_n(X) \xrightarrow{\cong} G_n(X)$ induced by the obvious inclusion of categories. Further, one may define a product operation (cf. [Lo 76]) $K_i(X) \otimes K_j(X) \longrightarrow K_{i+j}(X)$, which makes $K_*(X) = \bigoplus_{n \geq 0} K_n(X)$ into a graded ring, and similarly for $G_*(X) = \bigoplus_{n\geq 0} G_n(X)$.

Relation to Milnor K-theory

If A is a ring and $X = \text{Spec } A$, we write $K_*(A)$ as shorthand for $K_*(\text{Spec } A)$. It can be shown that when F is a field, the isomorphism $F^* \xrightarrow{i_1} K_1(F)$ induces a (well-defined) homomorphism $K_*^M(F) \xrightarrow{i} K_*(F)$ of graded rings. The work of Mastumoto (cf. [Mil 71], Chapter 12) shows, furthermore, that the map $K_2^M(F) \xrightarrow{i_2} K_2(F)$ is an isomorphism. However, the map $K_n^M(F) \xrightarrow{i_n} K_n(F)$ is not in general an isomorphism for $n \geq 3$.
The case $n = 3$ has been studied in considerable detail, and has given rise to some standard terminology. The image of the homomorphism $K_3^M(F) \xrightarrow{i_3} K_3(F)$ is called the *decomposable* part of $K_3(F)$, denoted $K_3(F)_{dec}$ and the cokernel of i_3 is called the *indecomposable* part of $K_3(F)$, and is denoted $K_3(F)_{nd}$. We will see in the course of proving Proposition 10.3.6 that $K_3(F)_{nd}$ is never zero. Although i_3 is never surjective, the kernel of i_3 has been shown by Suslin ([Su 85]) to be killed by multiplication by 2; Shapiro ([Shap 81], Proposition 1) has shown that i_3 is actually injective for number fields. It has been conjectured that i_3 is always injective; however, any hope that the maps i_n might also be injective for $n > 3$ is dashed by another result of Shapiro ([Shap 81], Proposition 2) that the map i_4 is zero for $F = \mathbf{Q}$, whereas $K_4^M(\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z}$ by Theorem 9.1.2.

Bloch's theorem

Let k be a field, X/k a quasiprojective algebraic scheme, and $n \geq 0$ an integer. As Bloch himself states in [Bl 86], the primary motivation behind his development of the theory of higher Chow groups was to prove the following theorem, sometimes called the "Riemann-Roch Theorem for higher algebraic K-theory".

Theorem 10.3.1. *(Bloch, [Bl 86]) There exists an isomorphism:*

$$G_n(X) \otimes \mathbf{Q} \xrightarrow{\cong} \bigoplus_i CH^i(X, n) \otimes \mathbf{Q}$$

In particular, if X is regular, there is an isomorphism:

$$K_n(X) \otimes \mathbf{Q} \xrightarrow{\cong} \bigoplus_i CH^i(X, n) \otimes \mathbf{Q}$$

Sometimes it is possible to use properties of K-theory to draw conclusions about (rationalized) Chow groups. A key ingredient in our later results will be the following result of Quillen:

Theorem 10.3.2. *(Quillen, [Qu 72]) Let k be a finite field with q elements. Then $K_0(k) \cong \mathbf{Z}$, $K_n(k) \cong \mathbf{Z}/(q^m - 1)\mathbf{Z}$ if $n = 2m - 1$ is odd, and $K_n(k) = 0$ if n is even. In particular, $K_n(k)$ is torsion for $n \geq 1$.*

Theorems 10.3.1 and 10.3.2 together yield the following:

Corollary 10.3.3. *Let k be a finite field. Then for all $(r, s) \neq (0, 0)$, the group $CH^r(k, s)$ is torsion.*

10.3.1 Extensions of Theorem 7.5.5

Given the result of Theorem 7.5.5 linking Zariski cohomology of Milnor K-sheaves to higher Chow groups, it is natural to ask whether or not a similar result holds for higher-dimensional cycles. Phrased precisely, for which m, n is $CH^m(X, n)$ naturally isomoprhic to $H^{m-n}_{Zar}(X, \mathcal{K}^M_m)$ or even $H^{m-n}_{Zar}(X, \mathcal{CH}^m(m))$
(naturally) isomorphic to $CH^m(X, n)$ or even an isomorphism $H^{m-n}_{Zar}(X, \mathcal{CH}^m(m)) \cong CH^m(X, n)$?

Using the spectral sequence 10.2.1, it is easy to see that there is always a natural surjection $CH^m(X, n) \longrightarrow H^{m-n}_{Zar}(X, \mathcal{CH}^m(m))$. Furthermore, by using the spectral sequence in conjunction with specific results of Property 8, Section 7.2, Müller-Stach has proven the following extension of the results of Kato [Kato 86] and Landsburg [Land 91]:

Theorem 10.3.4. *(Müller-Stach, [MS 98], Corollary 5.2) Let k be a field and X a smooth quasiprojective variety defined over k. Then for any $0 \leq n \leq 2$ and any $m \geq 0$ there are isomorphisms:*

$$H^{m-n}_{Zar}(X, \mathcal{CH}^m(m)) \xrightarrow{\cong} CH^m(X, n)$$

Note that the map of complexes in Theorem 10.2.3 6.1.4 gives an isomorphism between the Gersten resolution for $\mathcal{CH}^m(m)$ and a complex which by Theorem 6.1.4 (and the coincidence of the groups K_m and K^M_m for fields when $m \leq 2$) happens to be a resolution of the sheaf $\mathcal{K}_m$. Thus $\mathcal{CH}^m(m)$ and $\mathcal{K}_m$ are isomorphic as sheaves and we have the following:

Corollary 10.3.5. *When $m \leq 2$, there are natural isomorphisms:*

$$H^{m-n}_{Zar}(X, \mathcal{K}_m) \xrightarrow{\cong} H^{m-n}_{Zar}(X, \mathcal{CH}^m(m)) \xrightarrow{\cong} CH^m(X, n)$$

Our contribution is to show that the statement of Theorem 10.3.4 is false for $m = 2, n = 3$. Since the group $H^{-1}_{Zar}(X, \mathcal{CH}^2(2))$ is always zero, it suffices to show that for appropriate choices of k and X, we have $CH^2(X, 3) \neq 0$. We prove a stronger statement.

Proposition 10.3.6. *Let k be any field and X/k any smooth quasiprojective variety. Then $CH^2(X, 3) \neq 0$.*

We begin with the following lemma:

Lemma 10.3.7. *With hypotheses as in Proposition 10.3.6, let $n \geq 3$ be any integer. Then there is an isomorphism $CH^2(X, n) \xrightarrow{\cong} CH^2(k(X), n)$.*

We analyze $CH^2(X, n)$ using the spectral sequence 10.2.1 with $r = 2$. As before we write F^* for the filtration on $CH^2(X, n)$.
Clearly for $u < 0$ we have $0 = E_2^{u,-n-u} \cong E_\infty^{u,-n-u} \cong \dfrac{F^u}{F^{u+1}}$.
Next we consider the case $u = 0$. The incoming differential $E_2^{-2,-n+1} \longrightarrow E_2^{0,-n}$ is zero since $E_2^{-2,-n+1} = 0$, and the outgoing differential $E_2^{0,-n} \longrightarrow E_2^{2,-n-1}$ has image $E_2^{2,-n-1} = H^2_{Zar}(X, \mathcal{CH}^2(n+1))$ which is isomorphic to a quotient of $\bigoplus_{x \in X^2} CH^0(k(x), n-1)$ by Theorem 10.2.2; the latter is obviously zero, which implies that the differential is zero. Using Theorem 10.2.2, we calculate

$$\begin{aligned} E_2^{0,-n} = H^0_{Zar}(X, \mathcal{CH}^2(n)) &\cong \mathrm{Ker}\ (CH^2(k(X), n) \to \bigoplus_{x \in X^1} CH^1(k(x), n-1)) \\ &\cong CH^2(k(X), n) \end{aligned}$$

by Property 8 of Section 7.2. Thus

$$E_\infty^{0,-n} \cong E_2^{0,-n} = H^0_{Zar}(X, \mathcal{CH}^2(n)) \cong CH^2(k(X), n) \cong \frac{F^0}{F^1}$$

Finally, for $u > 0$, Theorem 10.2.2 yields

$$E_2^{u,-n-u} \cong H^u_{Zar}(X, \mathcal{CH}^2(n+u)) \cong \frac{\bigoplus_{x\in X^u} CH^{2-u}(k(x),n)}{\text{image of boundary map}} = 0 \cong \frac{F^u}{F^{u+1}}$$

Putting the filtered pieces together, we conclude that there is an isomorphism:
$CH^2(X,n) \cong CH^2(k(X),n)$

Using the spectral sequence for motivic cohomology, Bloch and Lichtenbaum ([BlL 95], Theorem 7.2) have established that $CH^2(k(X),3) \cong K_3(k(X))_{nd}$. Thus it suffices to show that $K_3(k(X))_{nd}$ is nonzero.

Let Π be the prime field of $k(X)$. Merkur'ev and Suslin ([MeSu 91], Proposition 11.6) have shown that the natural map $K_3(\Pi)_{nd} \longrightarrow K_3(k(X))_{nd}$ induced by the inclusion $\Pi \hookrightarrow k(X)$ restricts to an isomorphism on torsion subgroups; therefore we reduce the problem to showing that $K_3(\Pi)_{nd}$ has nonzero torsion. If $\Pi = \mathbf{F}_p$ for some prime number p, we have $K_3(\mathbf{F}_p)_{nd} = \dfrac{K_3(\mathbf{F}_p)}{i_3(K_3^M(\mathbf{F}_p))}$. The numerator is isomorphic to $\mathbf{Z}/(p^2-1)\mathbf{Z}$ by [Qu 72] (see also Theorem 10.3.2), and $K_3^M(\mathbf{F}_p) = 0$ by Theorem 9.1.2; thus we have $K_3(\Pi)_{nd} \cong \mathbf{Z}/(p^2-1)\mathbf{Z}$. In the case $\Pi = \mathbf{Q}$, we have $K_3(\mathbf{Q})_{nd} \cong \mathbf{Z}/24\mathbf{Z}$ by [MeSu 91], Remark 11.12.

10.3.2 One-cycles

In this section, we use the spectral sequence and our knowledge of the K-theory of finite fields to make deductions concerning various groups of 1-cycles.

Proposition 10.3.8. *Let k be a finite field and X a smooth quasiprojective variety of dimension d over k. Then for all $s \geq 2$, the group $CH^{d+s-1}(X,s)$ of 1-cycles is a torsion group.*

Proof.

As above, we use the notation F^* for the filtration on $CH^{d+s-1}(X,s)$ associated to the spectral sequence 10.2.1, which reads

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{CH}^{d+s-1}(-q)) \Longrightarrow CH^{d+s-1}(-p-q)$$

In particular, we observe that $E_2^{d-1,1-d-s} = H^{d-1}(X, \mathcal{CH}^{d+s-1}(d+s-1))$ is a quotient of $\bigoplus_{x \in X^1} K_s^M(k(x))$ by Theorem 10.2.2 and Theorem 10.2.3. Since each field $k(x)$ appearing in the direct sum is in $\mathcal{T}_1(k)$ and $s \geq 2$, $K_s^M(k(x))$ is torsion by Theorem 9.1.2. Thus $E_2^{d-1,1-d-s}$ is torsion, and hence $E_\infty^{d-1,1-d-s} \cong \dfrac{F^{d-1}}{F^d}$ is torsion.

Next, consider the quotient $\dfrac{F^d}{F^{d+1}} \cong E_\infty^{d,-d-s}$. Theorem 10.2.2 implies that $E_2^{d,-d-s} = H^d_{Zar}(X, CH^{d+s-1}(d+s))$ is a quotient of a direct sum of groups $CH^{s-1}(k(x), s)$, where $k(x)$ is a finite extension of k. By Corollary 10.3.3, $CH^{s-1}(k(x), s)$ is torsion; hence the same is true of the E_2 term above, and thus also for the E_∞ term. Thus the quotient $\dfrac{F^d}{F^{d+1}}$ is torsion.

To analyze the remaining graded pieces, note that we always have $\dfrac{F_t}{F_{t+1}} \cong E_\infty^{t,-t-s}$. However, $E_2^{t,-t-s} = H^t_{Zar}(X, \mathcal{CH}^{d+s-1}(s+t))$, so if we have $t \leq d-2$, then Theorem 10.2.2 imply that the sheaf $\mathcal{CH}^{d+s-1}(s+t)$ is zero. If on the other hand we have $t > d$, then the cohomology group is zero for reasons of dimension. Thus $\dfrac{F_t}{F_{t=1}} = 0$ for all $t \leq d-2$ and $t > d$.

Collating our calculations, we see that all the quotients $\frac{F_t}{F_{d+1}}$ are torsion, and hence that $CH^{d+s-1}(X, s)$ is torsion.

10.3.3 Surfaces over a finite field

For the rest of this section, k is a finite field and X a smooth projective variety defined over k. We will use Theorem 10.3.1, together with our own work and results of Colliot-Thélène, Raskind, and Soulé to prove various facts about higher Chow groups of surfaces.

Soulé ([Sou 84]) has proved that when X belongs to a certain class $A(k)$ of varieties defined over k (which includes products of curves, abelian varieties, and unirational varieties of dimension ≤ 3), the groups $K_m(X)$ are of finite exponent (hence torsion) for $m > 0$. A possible strengthening of this result is the following:

Conjecture 10.3.9. *(Parshin's Conjecture, [Gei 98] 3.2) Let k be a finite field and Y a smooth projective variety over k. Then for all $s > 0$, $K_s(Y)$ is torsion.*

Proposition 10.3.10. *Let k be a finite field, and X a smooth projective surface defined over k. Then the isomorphism $K_2(X) \otimes \mathbf{Q} \xrightarrow{\cong} \bigoplus_i CH^i(X, 2) \otimes \mathbf{Q}$ induces an isomorphism:*

$$K_2(X) \otimes \mathbf{Q} \xrightarrow{\cong} CH^2(X,2) \otimes \mathbf{Q}$$

If furthermore $X \in A(k)$, then $CH^2(X,2)$ is finite. If Parshin's conjecture holds, then $CH^2(X,2)$ is finite for any surface X.

Proof.

To exhibit the first isomorphism, it suffices to show that all of the groups $CH^i(X,2)$ are torsion for $i \neq 2$. For $i \leq 0$ or $i \geq 5$, this is obvious for reasons of dimension. For $i = 1$, it follows from Property 8 of Section 7.2. For $i = 3$, it follows from Proposition 10.3.8. Finally, for $i = 4$, Theorem 9.3.1 implies that $CH^4(X,2) = 0$, which completes the necessary verification.

If we assume $X \in A(k)$ (or Parshin's conjecture), then [Sou 84], Théorème 4 implies that $K_2(X) \otimes \mathbf{Q} = 0$, and hence (by Theorem 10.3.1) that $CH^2(X,2)$ is torsion. However, $CH^2(X,2) \cong H^0_{Zar}(X, \mathcal{CH}^2(2))$ by Theorem 10.3.4 and using the isomorphism $\mathcal{CH}^2(2) \cong \mathcal{K}_2$ deduced by comparison of the resolutions of Theorem 6.1.4 and Theorem 10.2.2, we conclude that $CH^2(X,2) \cong H^0_{Zar}(X, \mathcal{K}_2)$. Since $H^0_{Zar}(X, \mathcal{K}_2)$ has finite torsion subgroup by Proposition 1.15 of [CTR 85], we conclude that $CH^2(X,2)$ is finite.

Chapter 11

Other fields

In this section we put forth some results concerning the higher Chow groups associated to varieties defined over an infinite base field.

11.1 Rank results

By using functorial properties of the higher Chow groups, we extend results of [BT 73] on the rank of Milnor K-groups:

Definition 11.1.1. *Let F be a field and F_0 its prime field. The Kronecker dimension $\delta(F)$ of F is tr $deg_{F_0}(F)$ if char $F_0 > 0$ and $1 + tr\ deg_{F_0}(F)$ if $F_0 = \mathbf{Q}$.*

Proposition 11.1.2. *Let F be a field and X a quasi-projective variety of dimension d, defined over F. Assume further that $X(F) \neq \emptyset$, and suppose $1 \leq s \leq \delta(F)$. Then*

1. *If X is proper over F, $CH^{d+s}(X, s)$ has rank Card(F).*

2. *If X is smooth over F, then $CH^s(X, s)$ has rank Card(F).*

Proof.

It is clear from the presentation of either group by generators and relations that it has rank at most Card(F), and by assumption there exists a section Spec $F \longrightarrow X$ to the structure morphism $X \longrightarrow$ Spec F. In the first case, covariant functoriality of the higher Chow groups realizes $CH^s(F, s)$ as a direct summand of $CH^{d+s}(X, s)$ and in the second, contravariant functoriality of the higher Chow groups realizes $CH^s(F,s)$ as a direct summand of $CH^s(X,s)$. Hence it suffices to show that $CH^s(F,s)$ has rank Card(F). However, $CH^s(F,s) \cong K_s^M(F)$ by Theorem 7.3.1, and the latter group has rank equal to Card(F) by Proposition 5.10 of [BT 73].

11.2 Global Fields

11.2.1 Zero-cycles over global fields

Let k be a global field, X/k a smooth quasiprojective variety of dimension d and $s \geq 0$ an integer. As in the proof of Theorem 9.3.1, we identify $CH^{d+s}(X,s)$ with $H^d_{Zar}(X, \mathcal{K}^M_{d+s})$ by means of Theorem 7.5.5 and observe that the latter is a quotient of the group $\bigoplus_{x \in X^d} K_s^M(k(x))$. Theorem 9.1.2 then yields the following:

Proposition 11.2.1. $CH^{d+s}(X,s) \cong \begin{cases} \textit{torsion} & \textit{if } s = 2 \\ 2-\textit{torsion} & \textit{if } s \geq 3 \textit{ and char } k = 0 \\ 0 & \textit{if } s \geq 3 \textit{ and char } k > 0 \end{cases}$

11.2.2 A problem of Bloch

The case $s = 1$ is conspicuously absent from the above proposition, and represents a significant gap in our knowledge, even in the simplest nontrivial case in which X is a projective curve. Clearly we have $CH^2(X,1) \cong k^* \oplus K(k; A_0(X), \mathbf{G}_m)$, so $CH^2(X,1)$ itself cannot be torsion; however, the question of whether or not $V(X) = K(k; A_0(X), \mathbf{G}_m) \cong K(k; J(X), \mathbf{G}_m)$ is torsion, first posed by Bloch in [Bl 81], remains open. (Here $J(X)$ is the Jacobian of X) When char $k > 0$, Raskind [Ra 90] has shown that if $V(X)$ is not torsion, then $V(X)$ must contain an infinite divisible subgroup. Furthermore, if C is a smooth projective model for k over its field of constants k_0, he notes that Abhyankar's theorem on resolution of singularities gives a smooth projective surface Y/k_0 together with a proper flat map $Y \longrightarrow C$ of k_0-schemes whose generic fiber is X. Raskind then proves that $V(X)$ is torsion if $H^1_{Zar}(Y, \mathcal{K}_2)$ is torsion. By comparing the Gersten resolutions of $\mathcal{CH}^2(2)$ and $\mathcal{K}_2$ (Theorems 10.2.2 and 6.1.4, respectively), a proof similar to that of Theorem 10.2.3 shows that these sheaves are isomorphic. Finally, identifying $H^1_{Zar}(Y, \mathcal{K}_2) \cong H^1_{Zar}(Y, \mathcal{CH}^2(2))$ with $CH^2(Y,1)$ by means of Theorem 10.3.4, we have:

Proposition 11.2.2. *Suppose char $k > 0$. If Parshin's conjecture 10.3.9 holds, then $V(X)$ is torsion.*

In the following, we give a proof that $V(X)$ is torsion when X is a constant curve with a point rational over the constant field. In this case, the surface Y is a product of curves, so the result is immediate from Raskind's work. However, our proof uses the language of higher Chow groups and yields a slightly stronger result.

Proposition 11.2.3. *Let C be a projective curve, defined and smooth over a field k_0 of positive characteristic, such that $C(k_0) \neq \emptyset$. Let $k \in \mathcal{T}_1(k_0)$ and set $X = C \times_{k_0} k$. Then $V(X)$ is torsion. If furthermore $A_0(C) \neq 0$, then $V(X)$ is infinite.*

Proof.
Let $p : C \longrightarrow \mathrm{Spec}\ k_0$ denote the structure map; fix some section $s : \mathrm{Spec}\ k_0 \longrightarrow C$ of p. Now let B be a smooth proper model for k over k_0. The maps p and s induce, by covariant functoriality, maps of exact sequences of complexes as shown below. Since $p \circ s = id$, the maps p_* above are (split) surjections; the cycle complexes $a^*(-,-)$ are thus defined by the following diagram, all of whose rows and columns are exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus_{z \in B^1} a^{*-1}(C \times_{k_0} k(z), -) & \longrightarrow & a^*(C \times_{k_0} B, -) & \longrightarrow & a^*(X, -) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus_{z \in B^1} z^{*-1}(C \times_{k_0} k(z), -) & \longrightarrow & z^*(C \times_{k_0} B, -) & \longrightarrow & z^*(X, -) \\
 & & s_* \uparrow \downarrow p_* & & s_* \uparrow \downarrow p_* & & s_* \uparrow \downarrow p_* \\
0 & \longrightarrow & \bigoplus_{z \in B^1} z^{*-2}(k(z), -) & \longrightarrow & z^{*-1}(B, -) & \longrightarrow & z^{*-1}(k, -) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

A diagram chase together with Theorem 7.4.1 shows that the naturally induced map

$$\frac{a^*(C \times_{k_0} B, -)}{\bigoplus_{z \in B^1} a^{*-1}(C \times_{k_0} k(z), -)} \hookrightarrow a^*(X, -)$$

is a quasi-isomorphism. Define $A^2(C \times_{k_0} B, 1) := \mathrm{Ker}\ (p_* : CH^2(C \times_{k_0} B, 1) \longrightarrow CH^1(B, 1))$ Then the associated long exact sequence in homology reads:

$$\ldots \longrightarrow A^2(C \times_{k_0} B, 1) \longrightarrow V(X) \xrightarrow{\partial} \bigoplus_{z \in B^1} A_0(C \times_{k_0} k(z)) \xrightarrow{\alpha} A_0(C \times_{k_0} B) \longrightarrow 0$$

Since $A^2(C \times_{k_0} B, 1)$ is a direct summand of $CH^2(C \times_{k_0} B, 1)$, we have $A^2(C \times_{k_0} B, 1) \otimes \mathbf{Q} \hookrightarrow CH^2(C \times_{k_0} B, 1) \otimes \mathbf{Q} \hookrightarrow K_1(C \times_{k_0} B) \otimes \mathbf{Q}$ by Theorem 10.3.1, and the latter is 0 by results of Soulé (cf. Section 10.3.3).

For the second assertion, let J denote the Jacobian of C. If $A_0(C) \cong J(k_0) \neq 0$, then $A_0(C \times_{k_0} k(z)) \cong J(k(z)) \neq 0$ for all $z \in C^1$. Hence $\bigoplus_{z \in B^1} A_0(C \times_{k_0} k(z))$ is infinite. However, $A_0(C \times_{k_0} B)$ is finite by Theorem 9.1.4. Therefore $\mathrm{Ker}\ \alpha = \mathrm{Im}\ \partial$ is infinite, and so $V(X)$ must also be infinite.

11.3 A weak conjecture of Bass type

A well-known conjecture due to Hyman Bass asserts that the K-groups of a scheme of finite type over $\mathbf{Z}$ are finitely generated. The appropriate analogue for the higher Chow groups would be the following:

Conjecture 11.3.1. *Let X be a scheme of finite type over $\mathbf{Z}$. Then for all i and j, the group $CH^i(X, j)$ is finitely generated.*

In the following, we restrict attention to certain such schemes, namely varieties defined over a finite field k. Indeed, in the case of a projective variety, smooth over k, the above conjecture together with Parshin's conjecture 10.3.9 implies that the groups $CH^i(X, j)$ are actually *finite* for $j \geq 1$. Now consider the following consequence of Conjecture 11.3.1:

Conjecture 11.3.2. *Let X be a quasiprojective variety over a finite field k and $n \geq 2$ an integer. Then for all i, j, the group $CH^i(X, j)/nCH^i(X, j)$ is finite.*

Drawing upon our previous results, we prove the following:

Proposition 11.3.3. *Let k be a finite field and X a quasiprojective variety defined over k. Conjecture 11.3.1 holds (for such X) in the following cases*

1. *X is a point, $i = 0, 1$, j arbitrary*

2. *X is a point, $i = 2$, $j = 3$*

3. *X is a curve, $i = 0, 1$, j arbitrary*

4. *X is a curve, $i = 2$, $j \leq 2$*

5. *X is an abelian variety, $i = 0, 1$, j arbitary*

6. *X is projective and smooth over k, $i = j + \dim X$*

Proof.

Property 1 is trivial. Property 2 follows directly from the reasoning of Proposition 10.3.6: if $X = \text{Spec } L$ for some finite field L, we have $CH^2(L, 3) \cong K_3(L)_{nd} \cong \dfrac{K_3(L)}{i_3(K_3^M(L))} \cong K_3(L)$ which is finite by Theorem 10.3.2.

To show Property 3, note that we have $CH^0(X, 0) \cong \mathbf{Z}$, and $CH^0(X, j) = 0$ for $j > 0$. Now consider the case $i = 1$, $j = 0$. When X is projective, and smooth over k, $CH^1(X, 0)$ is finitely generated by Theorem 9.1.4. Now suppose X is an open (quasiprojective) subvariety of a smooth projective curve X_c. The localization sequence for higher Chow groups and Property 3 for X_c then shows that Property 3 also holds for X. Next, let X be an arbitrary projective curve, and let $\tilde{X} \xrightarrow{\nu} X$ denote the normalization map. Choose an smooth open subvariety $U \subseteq X$ such that ν restricts to an isomorphism from $\tilde{U} = \nu^{-1}(U)$ to U. Since Property 3 holds for $\tilde{U}$, it also holds for U; the localization sequence

$$\ldots \longrightarrow CH^0(X - U, 0) \longrightarrow CH^1(X, 0) \longrightarrow CH^1(U, 0) \longrightarrow 0$$

then shows that it also holds for X. Finally, given an arbitrary quasiprojective curve X, embed it in its projective closure X_c and apply localization to obtain the desired property for X.

Next, we consider the case $i = 1, j = 1$; the proof is similar to the previous case. When X is smooth and projective, the formula $CH^1(X, 1) \cong k^*$ of Section 7.2 implies Property 3 directly. Arguing as above, suppose next that X is an open subvariety of some smooth projective curve X_c. The localization sequence

$$\ldots \longrightarrow CH^1(X_c, 1) \longrightarrow CH^1(X, 1) \longrightarrow CH^0(X_c - X, 0) \longrightarrow \ldots$$

sandwiches $CH^1(X,1)$ between two finitely generated groups, so it must itself be finitely generated. Now let X be an arbitrary projective curve; the same argument with normalization allows us to conclude that it has an open subvariety for which Property 3 holds; Property 3 for X follows from localization. Finally, if X is an arbitrary quasiprojective curve, one may embed X in its projective closure X_c and apply localization to conclude.

For $i = 1$, $j \geq 2$ and X smooth, we have $CH^1(X,j) = 0$ by Property 8 of Section 7.2. When X is not smooth, induction on the dimension of X together with localization (excision of the singular locus) yields the desired result.

When $i = 2$, $j = 2$, and X is a smooth projective curve, Proposition 10.1.1 implies that $CH^2(X,2)$ is finite. Arguing with localization sequences as above, we may deduce that $CH^2(X,2)$ is finite even in the general case.

When X is an abelian variety, the only nontrivial case is $i = 1$, $j = 0$. In this case, we have $CH^1(X,0) \cong \text{Pic } X \cong \mathbf{Z} \oplus \text{Pic }^0(X) \cong \mathbf{Z} \oplus X^\vee(k)$, where $X^\vee$ denotes the dual abelian variety. Since $X^\vee(k)$ is obviously finite, the result follows.

Finally, when X is smooth and projective, the result follows from Theorem 9.3.1.

The remainder of this section is devoted to the proof of the following:

Proposition 11.3.4. *Let X be a projective variety, defined and smooth over a finite field k. Conjecture 11.3.2 holds for X in the cases $i = 2, j = 1$ and $i = 2, j = 2$.*

We will actually prove a stronger statement, for which it will be helpful to define (following Bloch [Bl 86]) higher Chow groups with coefficients.

11.3.1 Higher Chow groups with coefficients

Definition 11.3.5. *Let k be a field, X an algebraic scheme over k, and G any abelian group. The higher Chow groups with coefficients in G, denoted $CH^*(X, \cdot\,; G)$, are the homology groups of the complex $z^*(X, \cdot\,; G) := z^*(X, \cdot) \otimes G$.*

We shall be most interested in the case when $G = \mathbf{Z}/n\mathbf{Z}$ for some integer n. As the complexes $z^*(X,n)$ are free, tensoring with the exact sequence $0 \longrightarrow \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$

(viewed as complexes concentrated in degree 0) yields an exact sequence

$$0 \longrightarrow z^*(X,\cdot) \xrightarrow{n} z^*(X,\cdot) \longrightarrow z^*(X,\cdot\,;\mathbf{Z}/n\mathbf{Z}) \longrightarrow 0$$

of complexes which we will call the *Bockstein sequence.* (Sometimes we abuse terminology and refer to the associated long exact homology sequence as the Bockstein sequence). Many of the standard properties enjoyed by the usual higher Chow groups (cf. Section 7.2) carry over to the setting of higher Chow groups with coefficients:

Proposition 11.3.6. *1. The higher Chow groups with coefficients in $\mathbf{Z}/n\mathbf{Z}$ satisfy functoriality, homotopy, and localization properties analogous to those described in Section 7.2.*

2. Let X, Y be quasiprojective schemes and $p,q,r,s \geq 0$ integers. Then there exists an external product:

$$CH^p(X,q;\mathbf{Z}/n\mathbf{Z}) \otimes CH^r(Y,s;\mathbf{Z}/n\mathbf{Z}) \stackrel{\cdot}{\longrightarrow} CH^{p+r}(X\times_k Y, q+s;\mathbf{Z}/n\mathbf{Z})$$

If $U \subseteq X \times_k \square^q_k$ and $V \subseteq Y \times_k \square^s_k$ are cycles in good position, then the external product sends $([U]\otimes 1)\otimes([V]\otimes 1) \mapsto ([U\times_k V]\otimes 1)$

3. For any integer i,j, there is an exact sequence

$$0 \longrightarrow \frac{CH^i(X,j)}{nCH^i(X,j)} \longrightarrow CH^i(X,j;\mathbf{Z}/n\mathbf{Z}) \longrightarrow CH^i(X,j-1)[n] \longrightarrow 0$$

Proof.

Part (1) follows easily from the Bockstein sequence and the corresponding properties for (ordinary) higher Chow groups. To prove Part (2), given a generator $a\otimes b$ of $CH^p(X,q;\mathbf{Z}/n\mathbf{Z})\otimes CH^r(Y,s;\mathbf{Z}/n\mathbf{Z})$, choose representing cycles $\tilde{a}\in z^p(X,q;\mathbf{Z}/n\mathbf{Z})$ and $\tilde{b}\in z^q(Y,s;\mathbf{Z}/n\mathbf{Z})$ and lift them to cycles $\tilde{\alpha}\in z^p(X,q)$ and $\tilde{\beta}$ of $z^r(Y,s)$, respectively. Now let α denote the class of $\tilde{\alpha}$ in $CH^p(X,q)$ and β the class of $\tilde{\beta}$ in $CH^r(Y,s)$. Form the (ordinary higher Chow group) product $\gamma=\alpha\times\beta\in CH^{p+r}(X\times_k Y, q+s)$ and consider its image c under the natural map $CH^{p+r}(X\times_k Y,q+s)\longrightarrow CH^{p+r}(X\times_k Y,q+s;\mathbf{Z}/n\mathbf{Z})$. We then define $a\cdot b := c$. An elementary diagram chase shows that this predefinition is independent of any choices made, and the explicit description of the product map follows immediately from the construction. Part (3) follows immediately from the (homology) Bockstein sequence.

We include the following calculation as an illustration of how the Bockstein sequence may be used:

Proposition 11.3.7. *Let k be a field, and $n \geq 1$ an integer. Then*

$$CH^1(k,s;\mathbf{Z}/n\mathbf{Z}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } s=0 \\ \dfrac{k^*}{(k^*)^n} & \text{if } s=1 \\ \mu_n(k) & \text{if } s=2 \\ 0 & \text{if } s \geq 3 \end{cases}$$

Proof.

Taking into account the results of Property 8, Section 7.2, we note that the Bockstein sequence gives rise to a long exact homology sequence

$$\ldots \longrightarrow 0 \xrightarrow{n} 0 \longrightarrow CH^1(k,m;\mathbf{Z}/n\mathbf{Z}) \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \xrightarrow{n} 0 \longrightarrow$$

$$\longrightarrow CH^1(k,2;\mathbf{Z}/n\mathbf{Z}) \longrightarrow k^* \xrightarrow{n} k^* \longrightarrow CH^1(k,1;\mathbf{Z}/n\mathbf{Z}) \longrightarrow$$

$$\longrightarrow \mathbf{Z} \xrightarrow{n} \mathbf{Z} \longrightarrow CH^1(k,0,\mathbf{Z}/n\mathbf{Z}) \longrightarrow 0$$

from which the statement is clear.

11.3.2 Proof of Proposition 11.3.4

We will need the following result, due to Gros and Suwa, for the proof.

Theorem 11.3.8. *([GS 88], Théorèmes 4.11, 4.19) Let k be a finite field of characteristic p and X a projective variety, smooth over k. Then there are exact sequences:*

- $$0 \longrightarrow T \longrightarrow H^0(X,\mathcal{K}_2) \longrightarrow \bigoplus_{l\neq p} H^2_{\acute{e}t}(X,\mathbf{Z}_l(2)) \longrightarrow 0$$

  where T is uniquely divisible and the group on the right is finite

  and

- $$0 \longrightarrow D \longrightarrow H^1(X, \mathcal{K}_2) \longrightarrow \bigoplus_l H^3_{\acute{e}t}(X, \mathbf{Z}_l(2)) \longrightarrow 0$$

  where D is uniquely divisible and the group on the right is finite.

We verify Proposition 11.3.4 by proving the following stronger statement:

Proposition 11.3.9. *With hypotheses as in Proposition 11.3.4, $CH^2(X, 2; \mathbf{Z}/n\mathbf{Z})$ is a finite group. If X is a surface, then $CH^2(X, 1; \mathbf{Z}/n\mathbf{Z})$ is also finite.*

Proof.

We use Corollary 10.3.5 to make the identifications $H^0(X, \mathcal{K}_2) \cong CH^2(X, 2)$ and $H^1(X, \mathcal{K}_2) \cong CH^2(X, 1)$.

Lemma 11.3.10. *The kernel and cokernel of multiplication by n on $CH^2(X, 2)$ and $CH^2(X, 1)$ are finite groups.*

Proof.

Consider the action of multiplication by n on the exact sequence of Theorem 11.3.8; we obtain a diagram as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & CH^2(X,2) & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & T & \longrightarrow & CH^2(X,2) & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

where F is the finite group $\bigoplus_{l \neq p} H^2_{\text{ét}}(X, \mathbf{Z}_l(2))$. Since T is uniquely divisible, multiplication by n on T is an isomorphism; thus the snake lemma gives an exact sequence

$$0 \longrightarrow 0 \longrightarrow CH^2(X,2)[n] \longrightarrow F[n] \longrightarrow 0 \longrightarrow CH^2(X,2)/n \longrightarrow F/n \longrightarrow 0$$

from which the assertion follows. A similar argument yields the result for $CH^2(X, 1)$.

If X is any smooth projective variety, Assertion 3 of Proposition 11.3.6 gives an exact sequence

$$0 \longrightarrow CH^2(X,2)/n \longrightarrow CH^2(X,2;\mathbf{Z}/n\mathbf{Z}) \longrightarrow CH^2(X,1)[n] \longrightarrow 0$$

Lemma 11.3.10 implies that the outer two groups are finite; thus, the middle group $CH^2(X, 2; \mathbf{Z}/n\mathbf{Z})$ must also be finite.

If X is a surface, Assertion 3 gives an exact sequence

$$0 \longrightarrow CH^2(X, 1)/n \longrightarrow CH^2(X, 1; \mathbf{Z}/n\mathbf{Z}) \longrightarrow CH^2(X, 0)[n] \longrightarrow 0$$

Lemma 11.3.10 implies that $CH^2(X, 1)/n$ is finite; and $CH^2(X, 0)[n] = CH_0(X)[n] \cong A_0(X)[n]$ is finite by Theorem 9.1.4, so we conclude that $CH^2(X, 1; \mathbf{Z}/n\mathbf{Z})$ is also finite.

This concludes the proof of Proposition 11.3.9

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