Connectivity of the zero-divisor graph for finite rings

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Abstract

We study the vertex-connectivity and edge-connectivity of the zero-divisor graph Γ_R associated to a finite commutative ring R. We show that the edge-connectivity of Γ_R always coincides with the minimum degree, and that vertex-connectivity also equals the minimum degree when R is nonlocal. When R is local, we provide conditions for the equality of all three parameters to hold, give examples showing that the vertex-connectivity can be much smaller than minimum degree, and prove a general lower bound on the vertex-connectivity.

1 Introduction

Let R be a commutative ring with unit element $1 \neq 0$. The set of zero-divisors in R does not in general possess a convenient algebraic structure; hence, non-algebraic methods are often needed to study this set. One attempt in this direction involves the so-called *zero-divisor graph* Γ_R, whose definition was first given by Beck in [5] and later adjusted slightly by Anderson and Livingston [2]. The vertices of Γ_R are precisely the nonzero zero-divisors of R, with two vertices adjacent if and only if the product of the ring elements they represent is zero. The idea is that by studying combinatorial properties of Γ_R, one might hope to draw conclusions about the structure of the set of zero-divisors in R. Since the paper citeal, considerable work has been done on this topic; for details, see the recent survey articles [1] and [7].

One of the first results proved was that for any R, Γ_R is connected, and in fact has diameter at most 3 [2, Theorem 2.3]. A more refined combinatorial notion than connectedness is that of *connectivity*. For a graph G, the *vertex-connectivity*, denoted $\kappa(G)$, is the size of the smallest subset of vertices whose removal renders the graph disconnected or leaves a single vertex, while the *edge-connectivity*, denoted $\lambda(G)$, is the size of the smallest subset of edges whose removal renders the graph disconnected. In general, connectivity of either type is rather difficult to compute; however, when graphs have a lot of symmetry – as is the case with zero-divisor graphs – it is sometimes possible to perform calculations, or at least give meaningful bounds.

The vertex connectivity of $\Gamma(\mathbb{Z}_n)$, $n \geq 2$ was studied by Aaron Lauve [8], who later discovered a mistake in his proof of the key formula in Section 4. The present article started as a project to correct this mistake, but later developed into a more comprehensive study of both the vertex- and edge-connectivity of $\Gamma(R)$ for arbitrary finite rings. An obvious starting point is the set of bounds $\kappa(G) \leq \lambda(G) \leq \delta(G)$ (see Proposition 2.2), valid for any graph G; here $\delta(G)$ is the minimum degree of a vertex in G. In this article, we show that for all finite rings R, $\lambda(\Gamma_R) = \delta(\Gamma_R)$, and for nonlocal R, we also have $\kappa(\Gamma_R) = \delta(\Gamma_R)$. When R is local, however, $\kappa(\Gamma_R)$ is not nearly as well-behaved. For example, if R is a principal ideal domain, we always have $\kappa(\Gamma_R) = \delta(\Gamma_R)$; however, one can construct infinite families of rings for which $\kappa(\Gamma_R)$ is of order $\delta(\Gamma_R)^{3/4}$. We give more precise conditions under which $\kappa(\Gamma_R) = \delta(\Gamma_R)$ holds, and show that for any R, $\kappa(\Gamma_R)$ must at least be of order $\delta(\Gamma_R)^{1/3}$.

Problems related to the focus of the present article have been studied in the recent literature. The structure of minimal vertex cuts in Γ_R was studied in [6]; however, that article does not investigate the *size* of such cuts, as is the focus of the present article. Our results are of a distinctly different flavor and thus complement rather than duplicate those of [6]. The papers [4] and [9] are more focused in scope, and study graphs whose vertex-connectivity is 1.

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2 Preliminaries

Throughout this paper, all rings are finite and commutative with $1 \neq 0$, and all graphs are finite, with no loops or multiple edges.

If R is a ring, we denote by $Z(R)$ the set of zero-divisors in R.

Definition 2.1. *Let R be a ring. The* zero-divisor graph *of R, denoted Γ_R, is the graph whose vertex set is the set $Z(R) - \{0\}$, and in which $\{x, y\}$ is an edge if x and y are distinct zero-divisors of R such that $xy = 0$.*

By abuse of notation, we blur the distinction between elements of $Z(R) - \{0\}$ and elements of $V(\Gamma_R)$. For $x \in Z(R) - \{0\}$ we denote by $\operatorname{ann} x$ the annihilator of x. Hence, the degree of x (viewed as a vertex of Γ_R) is $|\operatorname{ann} x - \{0, x\}|$.

We also recall various conventions and definitions from graph theory; see [10] or any reference on graph theory for further details. For a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. For a vertex v, we denote by $N_G(v)$ (or simply $N(v)$ if the context is clear) the set of neighbors of v in G. We denote by $\delta(G)$ the minimum vertex degree in G.

If $S \subseteq V(G)$, we write $G - S$ to denote the graph with vertex set $\bar{S} = V(G) - S$ and edge set $E(G) - \{\{x, y\} : \{x, y\} \cap S \neq \emptyset\}$. If $T \subseteq E(G)$ is any subset, we denote by $G - T$ the graph with vertex set $V(G)$ and edge set $E(G) - T$. A *vertex cut* is a subset $S \subseteq V(G)$ such that $G - S$ is disconnected, and a *disconnecting set of edges* of G is a subset $T \subseteq E(G)$ such that the graph $G - T$ is disconnected; an *edge cut* is a disconnecting set of edges which is minimal (with respect to inclusion). Writing $[A, B]$ for the set of edges in G with one endpoint in each of the subsets A, B of $V(G)$, it is easily shown (cf. [10, Remark 4.1.8]) that any edge cut in G must be of the form $[S, \bar{S}]$ for some subset $S \subseteq V(G)$. The *vertex-connectivity* of G, denoted $\kappa(G)$, is the size of smallest set $S \subseteq V(G)$ such that S is a vertex cut or $G - S$ has only one vertex. Similarly, the *edge-connectivity* of G, denoted $\lambda(G)$, is the size of the smallest edge cut in G. For convenience, we write κ_R (respectively, λ_R, δ_R) instead of $\kappa(\Gamma_R)$ (respectively, $\lambda(\Gamma_R)$, $\delta(\Gamma_R)$). A well-known result relating these parameters is the following statement, due to Whitney.

Proposition 2.2. *([10, Theorem 4.1.9]) For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

3 Results

Theorem 3.1. *Let R be a finite nonlocal ring. Then $\kappa_R = \lambda_R = \delta_R$.*

Proof.
By the structure theorem for Artin rings, $R \cong R_1 \times \cdots \times R_k$, where $k \geq 2$ and each R_i is a finite local ring. In light of Proposition 2.2, it suffices to show $\kappa_R \geq \delta_R$. To this end, let $S \subseteq V(\Gamma_R)$ be a subset with $|S| < \delta_R$; we will show that $H = \Gamma_R - S$ is connected. For i, $1 \leq i \leq k$, define

$$C_i = \{(0, \ldots, 0, a_i, 0, \ldots, 0) \in R_1 \times \cdots \times R_k : a_i \in Z(R_i) - \{0\}\}.$$

We claim that every vertex in H is adjacent to a vertex in $C_i \cap V(H)$ for some $1 \leq i \leq k$. Since vertices of C_i are clearly adjacent to vertices of C_j when $i \neq j$, it will then follow that H is connected. Toward this goal, suppose $b = (b_1, \ldots, b_k) \in V(H)$, and fix i, $1 \leq i \leq k$ such that $b_i \neq 0$. If we define $b' = (1, \ldots, 1, b_i, 1, \ldots, 1)$, then clearly $N_{\Gamma_R}(b') \subseteq C_i$. In particular, this implies $|C_i| \geq \delta > |S|$, so H must contain some vertex $v \in N_{\Gamma_R}(b')$. Since $N_{\Gamma_R}(b) \supseteq N_{\Gamma_R}(b')$, we see that $v \in N_{\Gamma_R}(b) \cap C_i$, as desired. □

From this point on, R will denote a finite local ring with maximal ideal $\mathfrak{m}$. Since R is Artinian, it follows from Nakayama's Lemma (cf. [3, Proposition 8.6]) that $\mathfrak{m}^n = 0$ for some positive integer n. We will reserve the symbol r for the smallest $n > 0$ satisfying this property. If $r = 1$, then R is a field and Γ_R is the empty graph. If $r = 2$, then Γ_R is a complete graph; so clearly $\kappa_R = \lambda_R = \delta_R = |\mathfrak{m}| - 2$. For the balance of the article, we assume $r \geq 3$, so in particular $\mathfrak{m}^2 \neq 0$. Since $\mathfrak{m}^{r-1} \subseteq \text{ann } \mathfrak{m}$, it follows immediately that $A_R = \text{ann } \mathfrak{m} - \{0\}$ is nonempty, and also that Γ_R is not complete. Viewed as a subset of $V(\Gamma_R)$, A_R is a dominating set in Γ_R. Clearly any vertex cut in Γ_R must contain A_R; thus, writing $\alpha_R = |A_R|$ and using Proposition 2.2, we have the following elementary bounds:

$$\alpha_R \leq \kappa_R \leq \lambda_R \leq \delta_R. \tag{1}$$

The following condition is important in that it presence forces all the inequalities in (1) to be equalities, but its absence typically has the opposite effect.

$$\text{There exists } x \in \mathfrak{m} \text{ such that ann } x = \text{ann } \mathfrak{m}. \tag{2}$$

Proposition 3.2. *Suppose condition (2) holds. Then* $\alpha_R = \kappa_R = \lambda_R = \delta_R$.

Proof.
If $x^2 = 0$, then $x \in \text{ann } x = \text{ann } \mathfrak{m}$. Thus, $\mathfrak{m} = \text{ann } x = \text{ann } \mathfrak{m}$, and so $\mathfrak{m}^2 = 0$. Hence, we may assume $x^2 \neq 0$. In this case, $\delta_R \leq \deg(x) = |\text{ann } x - \{x, 0\}| = |\text{ann } x - \{0\}| = |\text{ann } \mathfrak{m} - \{0\}| = \alpha_R$. □

If R is a principal ideal ring, condition (2) is certainly satisfied; therefore, we have:

Corollary 3.3. *Let p be a prime number and $n \geq 3$. Then* $\kappa(\mathbb{Z}/p^n\mathbb{Z}) = \lambda(\mathbb{Z}/p^n\mathbb{Z}) = p - 1$.

It turns out that for local rings, the edge-connectivity is much better behaved than the vertex-connectivity. Recalling that vertices of A_R are dominant in Γ_R, the determination of λ_R is strictly graph-theoretic and follows immediately from the following easily verified fact:

Proposition 3.4. *Let G be a graph with a dominant vertex. Then* $\lambda(G) = \delta(G)$.

Proof.
Choose $S \subseteq V(\Gamma_R)$ such that $T = [S, \bar{S}] \subseteq E(\Gamma_R)$ is an edge cut. We may assume without loss of generality that $\bar{S}$ contains a dominant vertex v. Since v is adjacent to all vertices of S, we must have $|T| \geq |S|$. On the other hand, every vertex in S has at least $\delta - |S| + 1$ neighbors in $\bar{S}$; so $\delta \geq |T| \geq |S|(\delta - |S| - 1)$. Rearranging the inequality $|S|(\delta - |S| + 1) \leq \delta$ gives $\delta(|S| - 1) \leq |S|(|S| - 1)$. If $|S| > 1$, then cancellation gives $\delta \leq |S|$ and so $|S| = |T| = \delta$. If $|S| = 1$, then all edges incident at the sole vertex in S must be in T, so $|T| = \delta$ in this case also. □

Corollary 3.5. *Let R be a local ring with $\mathfrak{m}^2 \neq 0$. Then $\lambda_R = \delta_R$.*

We now turn our attention to the vertex-connectivity of Γ_R. It is natural to ask how tight the bounds $\alpha_R \leq \kappa_R \leq \delta_R$ are. In the absence of condition (2), the lower bound is usually not met.

Proposition 3.6. *Let R be a local ring with $r \geq 4$ such that condition (2) fails. Then $\kappa_R > \alpha_R$.*

Proof.
First suppose $r \geq 5$. Any vertex cut must contain A_R, so it suffices to show that $H = \Gamma_R - A_R$ is connected. Because $\mathfrak{m}^{r-1} = \mathfrak{m}^{r-2}\mathfrak{m} \neq 0$, there exists some $x \in \mathfrak{m}^{r-2}$ such that $x \notin A_R$. Moreover, x is a finite sum of products of the form uv, where $u \in \mathfrak{m}^{r-3}$ and $v \in \mathfrak{m}$. Since $x \neq 0$ and $A_R \cup \{0\}$ is an ideal (hence closed under addition), at least one of these products must not be in A_R. Thus, we may assume without loss of generality that $x = uv$, where $u \in \mathfrak{m}^{r-3}$ and $v \in \mathfrak{m}$. Clearly u and v are also vertices of H, and because $r \geq 5$, $ux \in \mathfrak{m}^{2r-5} \subseteq \mathfrak{m}^r = 0$, so u is adjacent to x in H.

We claim that there is a path in H from every $y \in V(H)$ to x. If $y = u$ or $y = x$, this is clear, so assume otherwise. Since condition (2) fails, y has a neighbor z in H, so $yz = 0$. Now consider the product zu. If $zu = 0$, then y, z, u, x is a path. If $zu \neq 0$ but $zu \in A_R$, then $zx = (zu)v = 0$ and y, z, x is a path. Finally, if $zu \neq 0$ and $zu \notin A_R$, then zu is a vertex of H; moreover, $y(zu) = 0$ and $x(zu) = (xu)z = 0$, so y, zu, x is a path.

Now suppose $r = 4$. Then $\mathfrak{m}^4 = 0$ but $\mathfrak{m}^3 \neq 0$, so there exists $x \in \mathfrak{m}^2$ such that x is a vertex of $H = \Gamma_R - A_R$. It suffices to show that there is a path from any vertex of H to x. To this end, let y be a vertex of H distinct from x. Since condition (2) fails, y has a neighbor z in H, i.e. $yz = 0$. If $z\mathfrak{m} \subseteq A_R$, then $z\mathfrak{m}^2 = 0$ and z is adjacent to

x. If $z\mathfrak{m} \not\subseteq A_R$, then there exists $w \in \mathfrak{m}$ such that zw is a vertex of H. Now zw is a neighbor of y; however, $zw \in \mathfrak{m}^2$, so it is also a neighbor of x.

□

Remark.
The hypothesis $r \geq 4$ in Proposition 3.6 is necessary: when $r = 3$, there exist rings R not satisfying condition (2) for which $\kappa_R = \alpha_R$ and others for which $\kappa_R > \alpha_R$.

As an example of the former, let $\mathbb{F}_2$ be the field with two elements and consider $R = \dfrac{\mathbb{F}_2[x,y]}{(x^2,y^2)}$. By abuse of notation, we will use elements of $\mathbb{F}_2[x,y]$ to describe the cosets they represent in R. Then $\mathfrak{m} = (x,y)$ has eight elements and $\mathfrak{m}^2 = \operatorname{ann} \mathfrak{m} = \{0, xy\}$. Thus, Γ_R has seven vertices, with xy a dominant vertex; moreover, $\Gamma_R - \{xy\}$ is a graph on six vertices with three connected components $\{x, x+xy\}$, $\{y, y+xy\}$ and $\{x+y, x+y+xy\}$, so $\kappa_R = \alpha_R = 1$. Note also that for any $t \in R$, $\operatorname{ann} t$ contains (t). Since (t) has at least 4 elements for any $t \neq 0$, there is no way for the equality $\operatorname{ann} t = \operatorname{ann} \mathfrak{m}$ to hold for any $t \in V(\Gamma_R)$. Hence, condition (2) necessarily fails.

As an example of the latter, consider $R = \dfrac{\mathbb{F}_2[x,y,z,w]}{(x^2,y^2,z^2,w^2,xy,yz,zw,wx)}$. It is easily seen that R is a local ring satisfying $t^2 = 0$ for all $t \in R$, whose maximal ideal $\mathfrak{m} = (x,y,z,w)$ satisfies $\mathfrak{m}^3 = 0$, $\mathfrak{m}^2 \neq 0$. Moreover, $\operatorname{ann} \mathfrak{m} = (xz, yw)$, so $\alpha_R = 3$. As in the previous example, $t \in \operatorname{ann} t$ for all $t \in R$, so it is easily seen that condition (2) is not satisfied. Now let $H = \Gamma_R - A_R$; we will show that H is connected, and hence that $\kappa_R > 3$. Observe first that every vertex of H is of the form $c_1x+c_2y+c_3z+c_4w+c_5xz+c_6yw$, where the c_i are elements of $\mathbb{F}_2$, and $c_1,\ldots,c_4$ are not all 0. Evidently each such vertex is adjacent to $c_1x+c_2y+c_3z+c_4w$. Since x,y,z,w,x is a cycle in H, it will suffice (to show that H is connected) to construct a path from any vertex of the form $c_1x+c_2y+c_3z+c_4w$ (with not all c_i equal to 0) to one of the vertices of the abovementioned cycle. If v_1, v_2 are distinct elements of $\{x,y,z,w\}$ which are adjacent in H, then v_1+v_2 is adjacent to v_1. If v_1, v_2 are not adjacent, then choose v_3 from this set, distinct from v_1 and v_2; then v_3 will be adjacent to v_1+v_2. If v_1,v_2,v_3 are distinct elements of $\{x,y,z,w\}$, then we may assume without loss of generality that v_2 is adjacent to both v_1 and v_3. It follows that $v_1+v_2+v_3$ is adjacent to v_2. Finally, $x+y+z+w$ is adjacent to $x+z$. Thus H is connected, and so $\kappa_R > 3 = \alpha_R$.

The next family of examples shows that both bounds $\alpha_R \leq \kappa_R \leq \delta_R$ can be quite loose.

Proposition 3.7. *Let F be a field of order $f = 2^s$ and $R = \dfrac{F[x,y,z]}{(x^2,y^2,z^2)}$. Then $\alpha_R = f-1$, $\kappa_R = f^3-1$, and $\delta_R = f^4-2$.*

Proof.
Observe that R is a local ring with maximal ideal $\mathfrak{m} = (x,y,z)$ such that $t^2 = 0$ for all $t \in R$. Moreover, $\mathfrak{m}^2 = (xy,xz,yz)$, $\mathfrak{m}^3 = (xyz)$, and $\mathfrak{m}^4 = 0$.

Clearly R is generated (as an F-vector space) by $\{1,x,y,z,xy,xz,yz,xyz\}$; from this description, it is easily seen that $|R| = f^8$, $|\mathfrak{m}| = f^7$, $|\mathfrak{m}^2| = f^4$, and $|\mathfrak{m}^3| = f$. Also, $\operatorname{ann} \mathfrak{m} = \mathfrak{m}^3$, so $\alpha_R = f-1$. Now since $t^2 = 0$ for all $t \in R$, it follows that $\operatorname{ann} t \supseteq (t)$; because $|\operatorname{ann} t| \cdot |(t)| = |R|$, we have $|\operatorname{ann} t| \geq |R|^{1/2} = f^4$ for all $t \in R$. Direct computation shows that $\operatorname{ann} x = (x)$, so x is a vertex in Γ_R of minimum degree $\delta_R = f^4-2$.

Let $S = (\operatorname{ann} x \cap \mathfrak{m}^2) - \{0\}$. Also, any element in $(x) - S - \{0\}$ is associate to x and hence has the same neighborhood in Γ_R; in fact, $(x) - S - \{0\}$ is a clique and a connected component of $\Gamma_R - S$. Thus there is no path in $\Gamma_R - S$ from x to y, and so $\kappa_R \leq |S| = f^3-1$.

Now suppose $T \subseteq V(\Gamma_R)$ is a set of vertices such that $|T| < f^3-1$. Given $t \in \mathfrak{m}$, consider the multiplication by t map $\mathfrak{m}^2 \to t\mathfrak{m}^2$. This is an R-module homomorphism whose kernel is $\operatorname{ann} t \cap \mathfrak{m}^2$; hence $|\mathfrak{m}^3| \geq |t\mathfrak{m}^2| = \dfrac{|\mathfrak{m}^2|}{|\operatorname{ann} t \cap \mathfrak{m}^2|}$ and so $|\operatorname{ann} t \cap \mathfrak{m}^2| \geq \dfrac{|\mathfrak{m}^2|}{|\mathfrak{m}^3|} = f^3$. Taking into account that 0 and possibly t itself are elements of $\operatorname{ann} t$, this implies that every vertex of $H = \Gamma_R - T$ has a neighbor (in H) lying in $\mathfrak{m}^2$. To show that H is connected, let a and b be vertices of H. Then a has a neighbor $c \in \mathfrak{m}^2$ in H and b has a neighbor $d \in \mathfrak{m}^2$ in H. Now $cd \in \mathfrak{m}^4 = 0$, so c and d are adjacent in H, proving that there exists a path from a to b.

This shows that $\kappa_R = f^3-1$.

□

In the example of Proposition 3.7, κ_R is roughly $\dfrac{1}{|F|}\delta_R$, so by taking F to be arbitrarily large, we see that there is no hope for a general upper bound on κ_R which is linear in δ_R; in fact, in this family, κ_R is roughly $\delta_R^{3/4}$. It is natural, then, to ask for the maximum value of a, $0 < a \leq 3/4$, such that κ_R can be bounded below (for all finite rings R) by a function of order δ_R^a. As a first step in this direction, we offer:

Proposition 3.8. *Let R be a finite ring. Then $\kappa_R \geq \left(\dfrac{\delta_R}{2}\right)^{1/3} - \dfrac{1}{\sqrt{3}}$.*

The proof relies crucially on the following observation:

Lemma 3.9. *Let R be a ring and S a vertex cut of Γ_R such that $V(G)$ is the disjoint union of two nonempty sets A and B with no edges between A and B. Suppose $|S| < \delta_R$. If $a \in A$ and $b \in B$, then $ab \in S$, $|ann\ a| \geq \dfrac{|B|}{|S|}$ and $|ann\ b| \geq \dfrac{|A|}{|S|}$.*

Proof.
The hypothesis $|S| < \delta_R$ implies that a has some neighbor $x \in A$ and that b has some neighbor $y \in B$. Then $ab \neq 0$, but ab is a neighbor of both $x \in A$ and $y \in B$; thus, $ab \in S$. Now let $B = \{b_1, \ldots, b_n\}$. Since each of the products $ab_1, \ldots, ab_n$ is an element of S, some element $s \in S$ appears at least $\dfrac{|B|}{|S|}$ times in this list; without loss of generality, we may assume that $ab_1 = \ldots = ab_k = s$, where $k \geq \dfrac{|B|}{|S|}$. Thus, $0, b_2 - b_1, \ldots, b_k - b_1$ are distinct elements of ann a and hence $|\text{ann } a| \geq k \geq \dfrac{|B|}{|S|}$. The proof of the remaining assertion is similar. □

Proof of Proposition 3.8.
If $\kappa_R = \delta$, there is nothing to prove, so assume $\kappa_R < \delta$ and let $S \subseteq V(\Gamma_R) = \mathfrak{m} - \{0\}$ be a minimal vertex cut. Partition the vertices of $H = \Gamma_R - S$ into two disjoint nonempty sets A and B such that there are no edges between A and B; we may assume without loss of generality that B is the larger of these two sets, i.e.
$|A| \leq \dfrac{|\mathfrak{m}| - |S|}{2} \leq |B|$.
Now if $x \in A$ and $y \in B$, Lemma 3.9 implies that H contains no vertices from ann $x \cap$ ann y. Since the zero element is not a vertex of Γ_R, we have, again using Lemma 3.9:

$$|S| \geq |\text{ann } x \cap \text{ann } y| - 1 = \frac{|\text{ann } x||\text{ann } y|}{|\text{ann } x + \text{ann } y|} - 1 \geq \frac{|B|/|S| \cdot |A|/|S|}{|\mathfrak{m}|} - 1.$$

Thus,

$$|S|^3 \geq \frac{|A||B|}{|\mathfrak{m}|} - |S|^2 \geq |A|\frac{|\mathfrak{m}| - |S|}{2|\mathfrak{m}|} - |S|^2 = \frac{|A|}{2} - \frac{|A||S|}{|\mathfrak{m}|} - |S|^2 \geq \frac{|A|}{2} - \frac{|S|}{2} - |S|^2.$$

However, the neighbors of $x \in A$ in Γ_R are all members of $A \cup S$. Thus, $|A| + |S| \geq \delta + 1$ and so, continuing the calculation from above, we have:

$$|S|^3 + |S|^2 + \frac{|S|}{2} \geq \frac{|A|}{2} - 1 \geq \frac{\delta - |S| - 1}{2}$$

which, upon rearrangement, gives

$$2(|S|^3 + |S|^2 + |S| + \frac{1}{2}) \geq \delta.$$

Hence, $2(|S| + \frac{1}{\sqrt{3}})^3 \geq \delta$, and rearranging the inequality gives the desired result. $\square$

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