Connectivity of the zero-divisor graph for finite rings

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Abstract

We study the vertex-connectivity and edge-connectivity of the zero-divisor graph $\Gamma_R$ associated to a finite commutative ring $R$. We show that the edge-connectivity of $\Gamma_R$ always coincides with the minimum degree, and that vertex-connectivity also equals the minimum degree when $R$ is nonlocal. When $R$ is local, we provide conditions for the equality of all three parameters to hold, give examples showing that the vertex-connectivity can be much smaller than minimum degree, and prove a general lower bound on the vertex-connectivity.

1 Introduction

Let $R$ be a commutative ring with unit element $1 \neq 0$. The set of zero-divisors in $R$ does not in general possess a convenient algebraic structure; hence, non-algebraic methods are often needed to study this set. One attempt in this direction involves the so-called zero-divisor graph $\Gamma_R$, whose definition was first given by Beck in [5] and later adjusted slightly by Anderson and Livingston [2]. The vertices of $\Gamma_R$ are precisely the nonzero zero-divisors of $R$, with two vertices adjacent if and only if the product of the ring elements they represent is zero. The idea is that by studying combinatorial properties of $\Gamma_R$, one might hope to draw conclusions about the structure of the set of zero-divisors in $R$. Since the paper cites, considerable work has been done on this topic; for details, see the recent survey articles [1] and [7].

One of the first results proved was that for any $R$, $\Gamma_R$ is connected, and in fact has diameter at most 3 [2, Theorem 2.3]. A more refined combinatorial notion than connectedness is that of connectivity. For a graph $G$, the vertex-connectivity, denoted $\kappa(G)$, is the size of the smallest subset of vertices whose removal renders the graph disconnected or leaves a single vertex, while the edge-connectivity, denoted $\lambda(G)$, is the size of the smallest subset of edges whose removal renders the graph disconnected. In general, connectivity of either type is rather difficult to compute; however, when graphs have a lot of symmetry – as is the case with zero-divisor graphs – it is sometimes possible to perform calculations, or at least give meaningful bounds.
The vertex connectivity of $\Gamma(\mathbb{Z}_n)$, $n \geq 2$ was studied by Aaron Lauve [8], who later discovered a mistake in his proof of the key formula in Section 4. The present article started as a project to correct this mistake, but later developed into a more comprehensive study of both the vertex- and edge-connectivity of $\Gamma(R)$ for arbitrary finite rings. An obvious starting point is the set of bounds $\kappa(G) \leq \lambda(G) \leq \delta(G)$ (see Proposition 2.2), valid for any graph $G$; here $\delta(G)$ is the minimum degree of a vertex in $G$. In this article, we show that for all finite rings $R$, $\lambda(\Gamma_R) = \delta(\Gamma_R)$, and for nonlocal $R$, we also have $\kappa(\Gamma_R) = \delta(\Gamma_R)$. When $R$ is local, however, $\kappa(\Gamma_R)$ is not nearly as well-behaved. For example, if $R$ is a principal ideal domain, we always have $\kappa(\Gamma_R) = \delta(\Gamma_R)$; however, one can construct infinite families of rings for which $\kappa(\Gamma_R)$ is of order $\delta(\Gamma_R)^{3/4}$. We give more precise conditions under which $\kappa(\Gamma_R) = \delta(\Gamma_R)$ holds, and show that for any $R$, $\kappa(\Gamma_R)$ must at least be of order $\delta(\Gamma_R)^{1/3}$.

Problems related to the focus of the present article have been studied in the recent literature. The structure of minimal vertex cuts in $\Gamma_R$ was studied in [6]; however, that article does not investigate the size of such cuts, as is the focus of the present article. Our results are of a distinctly different flavor and thus complement rather than duplicate those of [6]. The papers [4] and [9] are more focused in scope, and study graphs whose vertex-connectivity is 1.

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2 Preliminaries

Throughout this paper, all rings are finite and commutative with $1 \neq 0$, and all graphs are finite, with no loops or multiple edges.

If $R$ is a ring, we denote by $Z(R)$ the set of zero-divisors in $R$.

**Definition 2.1.** Let $R$ be a ring. The zero-divisor graph of $R$, denoted $\Gamma_R$, is the graph whose vertex set is the set $Z(R) - \{0\}$, and in which $\{x,y\}$ is an edge if $x$ and $y$ are distinct zero-divisors of $R$ such that $xy = 0$.

By abuse of notation, we blur the distinction between elements of $Z(R) - \{0\}$ and elements of $V(\Gamma_R)$. For $x \in Z(R) - \{0\}$ we denote by $\text{ann } x$ the annihilator of $x$. Hence, the degree of $x$ (viewed as a vertex of $\Gamma_R$) is $|\text{ann } x - \{0,x\}|$.

We also recall various conventions and definitions from graph theory; see [10] or any reference on graph theory for further details. For a graph $G$, we denote by $V(G)$ its
vertex set and by \(E(G)\) its edge set. For a vertex \(v\), we denote by \(N_G(v)\) (or simply \(N(v)\) if the context is clear) the set of neighbors of \(v\) in \(G\). We denote by \(\delta(G)\) the minimum vertex degree in \(G\).

If \(S \subseteq V(G)\), we write \(G - S\) to denote the graph with vertex set \(\tilde{S} = V(G) - S\) and edge set \(E(G) - \{\{x, y\} : \{x, y\} \cap S \neq \emptyset\}\). If \(T \subseteq E(G)\) is any subset, we denote by \(G - T\) the graph with vertex set \(V(G)\) and edge set \(E(G) - T\). A vertex cut is a subset \(S \subseteq V(G)\) such that \(G - S\) is disconnected, and a disconnecting set of edges of \(G\) is a subset \(T \subseteq E(G)\) such that the graph \(G - T\) is disconnected; an edge cut is a disconnecting set of edges which is minimal (with respect to inclusion). Writing \([A, B]\) for the set of edges in \(G\) with one endpoint in each of the subsets \(A, B\) of \(V(G)\), it is easily shown (cf. \([10, \text{Remark 4.1.8}]\)) that any edge cut in \(G\) is of the form \([S, \bar{S}]\) for some subset \(S \subseteq V(G)\). The vertex-connectivity of \(G\), denoted \(\kappa(G)\), is the size of smallest set \(S \subseteq V(G)\) such that \(S\) is a vertex cut or \(G - S\) has only one vertex. Similarly, the edge-connectivity of \(G\), denoted \(\lambda(G)\), is the size of the smallest edge cut in \(G\). For convenience, we write \(\kappa_R\) (respectively, \(\lambda_R, \delta_R\)) instead of \(\kappa(\Gamma_R)\) (respectively, \(\lambda(\Gamma_R), \delta(\Gamma_R)\)). A well-known result relating these parameters is the following statement, due to Whitney.

**Proposition 2.2.** ([10, Theorem 4.1.9]) For any graph \(G\), \(\kappa(G) \leq \lambda(G) \leq \delta(G)\).

### 3 Results

**Theorem 3.1.** Let \(R\) be a finite nonlocal ring. Then \(\kappa_R = \lambda_R = \delta_R\).

**Proof.**

By the structure theorem for Artin rings, \(R \cong R_1 \times \cdots \times R_k\), where \(k \geq 2\) and each \(R_i\) is a finite local ring. In light of Proposition 2.2, it suffices to show \(\kappa_R \geq \delta_R\). To this end, let \(S \subseteq V(\Gamma_R)\) be a subset with \(|S| < \delta_R\); we will show that \(H = \Gamma_R - S\) is connected. For \(i, 1 \leq i \leq k\), define

\[
C_i = \{(0, \ldots, 0, a_i, 0, \ldots, 0) \in R_1 \times \cdots \times R_k : a_i \in Z(R_i) - \{0\}\}.
\]

We claim that every vertex in \(H\) is adjacent to a vertex in \(C_i \cap V(H)\) for some \(1 \leq i \leq k\). Since vertices of \(C_i\) are clearly adjacent to vertices of \(C_j\) when \(i \neq j\), it will then follow that \(H\) is connected. Toward this goal, suppose \(b = (b_1, \ldots, b_k) \in V(H)\), and fix \(i, 1 \leq i \leq k\) such that \(b_i \neq 0\). If we define \(b' = (1, \ldots, 1, b_i, 1, \ldots, 1)\), then clearly \(N_{\Gamma_R}(b') \subseteq C_i\). In particular, this implies \(|C_i| \geq \delta > |S|\), so \(H\) must contain some vertex \(v \in N_{\Gamma_R}(b')\). Since \(N_{\Gamma_R}(b) \supseteq N_{\Gamma_R}(b')\), we see that \(v \in N_{\Gamma_R}(b) \cap C_i\), as desired. \(\square\)
From this point on, $R$ will denote a finite local ring with maximal ideal $\mathfrak{m}$. Since $R$ is Artinian, it follows from Nakayama’s Lemma (cf. [3, Proposition 8.6]) that $\mathfrak{m}^n = 0$ for some positive integer $n$. We will reserve the symbol $r$ for the smallest $n > 0$ satisfying this property. If $r = 1$, then $R$ is a field and $\Gamma_R$ is the empty graph. If $r = 2$, then $\Gamma_R$ is a complete graph; so clearly $\kappa_R = \lambda_R = \delta_R = |\mathfrak{m}| - 2$. For the balance of the article, we assume $r \geq 3$, so in particular $\mathfrak{m}^2 \neq 0$. Since $\mathfrak{m}^{r-1} \subseteq \text{ann} \mathfrak{m}$, it follows immediately that $A_R = \text{ann} \mathfrak{m} - \{0\}$ is nonempty, and also that $\Gamma_R$ is not complete. Viewed as a subset of $V(\Gamma_R)$, $A_R$ is a dominating set in $\Gamma_R$. Clearly any vertex cut in $\Gamma_R$ must contain $A_R$; thus, writing $\alpha_R = |A_R|$ and using Proposition 2.2, we have the following elementary bounds:

$$\alpha_R \leq \kappa_R \leq \lambda_R \leq \delta_R. \quad (1)$$

The following condition is important in that it presence forces all the inequalities in (1) to be equalities, but its absence typically has the opposite effect.

There exists $x \in \mathfrak{m}$ such that $\text{ann} x = \text{ann} \mathfrak{m}$. \hfill (2)

**Proposition 3.2.** Suppose condition (2) holds. Then $\alpha_R = \kappa_R = \lambda_R = \delta_R$.

**Proof.**
If $x^2 = 0$, then $x \in \text{ann} x = \text{ann} \mathfrak{m}$. Thus, $\mathfrak{m} = \text{ann} x = \text{ann} \mathfrak{m}$, and so $\mathfrak{m}^2 = 0$.

Hence, we may assume $x^2 \neq 0$. In this case, $\delta_R \leq \deg (x) = |\text{ann} x - \{x, 0\}| = |\text{ann} x - \{0\}| = |\text{ann} \mathfrak{m} - \{0\}| = \alpha_R.$ \hfill \Box

If $R$ is a principal ideal ring, condition (2) is certainly satisfied; therefore, we have:

**Corollary 3.3.** Let $p$ be a prime number and $n \geq 3$. Then $\kappa(\mathbb{Z}/p^n\mathbb{Z}) = \lambda(\mathbb{Z}/p^n\mathbb{Z}) = p - 1$.

It turns out that for local rings, the edge-connectivity is much better behaved than the vertex-connectivity. Recalling that vertices of $A_R$ are dominant in $\Gamma_R$, the determination of $\lambda_R$ is strictly graph-theoretic and follows immediately from the following easily verified fact:

**Proposition 3.4.** Let $G$ be a graph with a dominant vertex. Then $\lambda(G) = \delta(G)$.  

Proof.
Choose $S \subseteq V(\Gamma_R)$ such that $T = [S, \bar{S}] \subseteq E(\Gamma_R)$ is an edge cut. We may assume without loss of generality that $\bar{S}$ contains a dominant vertex $v$. Since $v$ is adjacent to all vertices of $S$, we must have $|T| \geq |S|$. On the other hand, every vertex in $S$ has at least $\delta - |S| + 1$ neighbors in $\bar{S}$; so $\delta \geq |T| \geq |S|(\delta - |S| - 1)$. Rearranging the inequality $|S|(\delta - |S| + 1) \leq \delta$ gives $\delta(|S| - 1) \leq |S|(|S| - 1)$. If $|S| > 1$, then cancellation gives $\delta \leq |S|$ and so $|S| = |T| = \delta$. If $|S| = 1$, then all edges incident at the sole vertex in $S$ must be in $T$, so $|T| = \delta$ in this case also. \hfill \square

Corollary 3.5. Let $R$ be a local ring with $m^2 \neq 0$. Then $\lambda_R = \delta_R$.

We now turn our attention to the vertex-connectivity of $\Gamma_R$. It is natural to ask how tight the bounds $\alpha_R \leq \kappa_R \leq \delta_R$ are. In the absence of condition (2), the lower bound is usually not met.

Proposition 3.6. Let $R$ be a local ring with $r \geq 4$ such that condition (2) fails. Then $\kappa_R > \alpha_R$.

Proof.
First suppose $r \geq 5$. Any vertex cut must contain $A_R$, so it suffices to show that $H = \Gamma_R - A_R$ is connected. Because $m^{r-1} = m^{r-2}m \neq 0$, there exists some $x \in m^{r-2}$ such that $x \notin A_R$. Moreover, $x$ is a finite sum of products of the form $uv$, where $u \in m^{r-3}$ and $v \in m$. Since $x \neq 0$ and $A_R \cup \{0\}$ is an ideal (hence closed under addition), at least one of these products must not be in $A_R$. Thus, we may assume without loss of generality that $x = uv$, where $u \in m^{r-3}$ and $v \in m$. Clearly $u$ and $v$ are also vertices of $H$, and because $r \geq 5$, $ux \in m^{2r-5} \subseteq m^r = 0$, so $u$ is adjacent to $x$ in $H$.

We claim that there is a path in $H$ from every $y \in V(H)$ to $x$. If $y = u$ or $y = x$, this is clear, so assume otherwise. Since condition (2) fails, $y$ has a neighbor $z$ in $H$, so $yz = 0$. Now consider the product $zu$. If $zu = 0$, then $y, z, u, x$ is a path. If $zu \neq 0$ but $zu \in A_R$, then $zx = (zu)v = 0$ and $y, z, x$ is a path. Finally, if $zu \neq 0$ and $zu \notin A_R$, then $zu$ is a vertex of $H$; moreover, $y(zu) = 0$ and $x(zu) = (zu)z = 0$, so $y, zu, x$ is a path.

Now suppose $r = 4$. Then $m^4 = 0$ but $m^3 \neq 0$, so there exists $x \in m^2$ such that $x$ is a vertex of $H = \Gamma_R - A_R$. It suffices to show that there is a path from any vertex of $H$ to $x$. To this end, let $y$ be a vertex of $H$ distinct from $x$. Since condition (2) fails, $y$ has a neighbor $z$ in $H$, i.e. $yz = 0$. If $zm \subseteq A_R$, then $zm^2 = 0$ and $z$ is adjacent to
Remark.

The hypothesis \( r \geq 4 \) in Proposition 3.6 is necessary: when \( r = 3 \), there exist rings \( R \) not satisfying condition (2) for which \( \kappa_R = \alpha_R \) and others for which \( \kappa_R > \alpha_R \).

As an example of the former, let \( \mathbb{F}_2 \) be the field with two elements and consider \( R = \mathbb{F}_2[x, y] / (x^2, y^2) \). By abuse of notation, we will use elements of \( \mathbb{F}_2[x, y] \) to describe the cosets they represent in \( R \). Then \( \mathfrak{m} = (x, y) \) has eight elements and \( \mathfrak{m}^2 = \text{ann } \mathfrak{m} = \{0, xy\} \). Thus, \( \Gamma_R \) has seven vertices, with \( xy \) a dominant vertex; moreover, \( \Gamma_R - \{xy\} \) is a graph on six vertices with three connected components \( \{x, x + xy\}, \{y, y + xy\}, \{x + y, x + y + xy\} \), so \( \kappa_R = \alpha_R = 1 \). Note also that for any \( t \in R \), \( \text{ann } t \) contains \( (t) \). Since \( (t) \) has at least 4 elements for any \( t \neq 0 \), there is no way for the equality \( \text{ann } \mathfrak{m} = \text{ann } \mathfrak{m} \) to hold for any \( t \in V(\Gamma_R) \). Hence, condition (2) necessarily fails.

As an example of the latter, consider \( R = \mathbb{F}_2[x, y, z, w] / (x^2, y^2, z^2, w^2, xy, yz, zw, wx) \). It is easily seen that \( R \) is a local ring satisfying \( t^2 = 0 \) for all \( t \in R \), whose maximal ideal \( \mathfrak{m} = (x, y, z, w) \) satisfies \( \mathfrak{m}^3 = 0, \mathfrak{m}^2 \neq 0 \). Moreover, \( \text{ann } \mathfrak{m} = (xz, yw) \), so \( \alpha_R = 3 \). As in the previous example, \( t \in \text{ann } t \) for all \( t \in R \), so it is easily seen that condition (2) is not satisfied. Now let \( H = \Gamma_R - A_R \); we will show that \( H \) is connected, and hence that \( \kappa_R > 3 \). Observe first that every vertex of \( H \) is of the form \( c_1x + c_2y + c_3z + c_4w + c_5xz + c_6yw \), where the \( c_i \) are elements of \( \mathbb{F}_2 \), and \( c_1, \ldots, c_4 \) are not all 0. Evidently each such vertex is adjacent to \( c_1x + c_2y + c_3z + c_4w \). Since \( x, y, z, w, x \) is a cycle in \( H \), it will suffice (to show that \( H \) is connected) to construct a path from any vertex of the form \( c_1x + c_2y + c_3z + c_4w \) (with not all \( c_i \) equal to 0) to one of the vertices of the abovementioned cycle. If \( v_1, v_2 \) are distinct elements of \( \{x, y, z, w\} \) which are adjacent in \( H \), then \( v_1 + v_2 \) is adjacent to \( v_1 \). If \( v_1, v_2 \) are not adjacent, then choose \( v_3 \) from this set, distinct from \( v_1 \) and \( v_2 \); then \( v_3 \) will be adjacent to \( v_1 + v_2 \). If \( v_1, v_2, v_3 \) are distinct elements of \( \{x, y, z, w\} \), then we may assume without loss of generality that \( v_2 \) is adjacent to both \( v_1 \) and \( v_3 \). It follows that \( v_1 + v_2 + v_3 \) is adjacent to \( v_2 \). Finally, \( x + y + z + w \) is adjacent to \( x + z \). Thus \( H \) is connected, and so \( \kappa_R > 3 = \alpha_R \).

The next family of examples shows that both bounds \( \alpha_R \leq \kappa_R \leq \delta_R \) can be quite loose.
Proposition 3.7. Let \( F \) be a field of order \( f = 2^s \) and \( R = \frac{F[x, y, z]}{(x^2, y^2, z^2)} \). Then \( \alpha_R = f - 1, \kappa_R = f^3 - 1, \) and \( \delta_R = f^4 - 2 \).

Proof.

Observe that \( R \) is a local ring with maximal ideal \( \mathfrak{m} = (x, y, z) \) such that \( t^2 = 0 \) for all \( t \in R \). Moreover, \( \mathfrak{m}^2 = (xy, xz, yz), \mathfrak{m}^3 = (xyz), \) and \( \mathfrak{m}^4 = 0 \).

Clearly \( R \) is generated (as an \( F \)-vector space) by \( \{1, x, y, z, xy, xz, yz, xyz\} \); from this description, it is easily seen that \( |R| = f^8, |\mathfrak{m}| = f^7, |\mathfrak{m}^2| = f^4, \) and \( |\mathfrak{m}^3| = f \).

Also, \( \text{ann} \ \mathfrak{m} = \mathfrak{m}^3 \), so \( \alpha_R = f - 1 \). Now since \( t^2 = 0 \) for all \( t \in R \), it follows that \( \text{ann} \ t \supseteq (t) \); because \( |\text{ann} \ t| \cdot |(t)| = |R| \), we have \( |\text{ann} \ t| \geq |R|^{1/2} = f^4 \) for all \( t \in R \).

Direct computation shows that \( \text{ann} \ x = (x) \), so \( x \) is a vertex in \( \Gamma_R \) of minimum degree \( \delta_R = f^4 - 2 \).

Let \( S = (\text{ann} \ x \cap \mathfrak{m}^2) - \{0\} \). Also, any element in \( (x) - S - \{0\} \) is associate to \( x \) and hence has the same neighborhood in \( \Gamma_R \); in fact, \( (x) - S - \{0\} \) is a clique and a connected component of \( \Gamma_R - S \). Thus there is no path in \( \Gamma_R - S \) from \( x \) to \( y \), and so \( \kappa_R \leq |S| = f^3 - 1 \).

Now suppose \( T \subseteq V(\Gamma_R) \) is a set of vertices such that \( |T| < f^3 - 1 \). Given \( t \in \mathfrak{m} \), consider the multiplication by \( t \) map \( \mathfrak{m}^2 \to t\mathfrak{m}^2 \). This is an \( R \)-module homomorphism whose kernel is \( \text{ann} \ t \cap \mathfrak{m}^2 \); hence \( |\mathfrak{m}^3| \geq |t\mathfrak{m}^2| = \frac{|\mathfrak{m}^2|}{|\text{ann} \ t \cap \mathfrak{m}^2|} \) and so \( |\text{ann} \ t \cap \mathfrak{m}^2| \geq \frac{|\mathfrak{m}^2|}{|\mathfrak{m}^3|} = f^3 \).

Taking into account that \( 0 \) and possibly \( t \) itself are elements of \( \text{ann} \ t \), this implies that every vertex of \( H = \Gamma_R - T \) has a neighbor (in \( H \)) lying in \( \mathfrak{m}^2 \). To show that \( H \) is connected, let \( a \) and \( b \) be vertices of \( H \). Then \( a \) has a neighbor \( c \in \mathfrak{m}^2 \) in \( H \) and \( b \) has a neighbor \( d \in \mathfrak{m}^2 \) in \( H \). Now \( cd \in \mathfrak{m}^4 = 0 \), so \( c \) and \( d \) are adjacent in \( H \), proving that there exists a path from \( a \) to \( b \).

This shows that \( \kappa_R = f^3 - 1 \).

In the example of Proposition 3.7, \( \kappa_R \) is roughly \( \frac{1}{|F|} \delta_R \), so by taking \( F \) to be arbitrarily large, we see that there is no hope for a general upper bound on \( \kappa_R \) which is linear in \( \delta_R \); in fact, in this family, \( \kappa_R \) is roughly \( \frac{\delta_R^{3/4}}{2} \). It is natural, then, to ask for the maximum value of \( a, 0 < a \leq 3/4 \), such that \( \kappa_R \) can be bounded below (for all finite rings \( R \)) by a function of order \( \delta_R^n \). As a first step in this direction, we offer:

Proposition 3.8. Let \( R \) be a finite ring. Then \( \kappa_R \geq \left( \frac{\delta_R}{2} \right)^{1/3} - \frac{1}{\sqrt{3}} \).
The proof relies crucially on the following observation:

**Lemma 3.9.** Let $R$ be a ring and $S$ a vertex cut of $\Gamma_R$ such that $V(G)$ is the disjoint union of two nonempty sets $A$ and $B$ with no edges between $A$ and $B$. Suppose $|S| < \delta_R$. If $a \in A$ and $b \in B$, then $ab \in S$, $\left|\text{ann } a\right| \geq \frac{|B|}{|S|}$ and $\left|\text{ann } b\right| \geq \frac{|A|}{|S|}$.

**Proof.**

The hypothesis $|S| < \delta_R$ implies that $a$ has some neighbor $x \in A$ and that $b$ has some neighbor $y \in B$. Then $ab \neq 0$, but $ab$ is a neighbor of both $x \in A$ and $y \in B$; thus, $ab \in S$. Now let $B = \{b_1, \ldots, b_n\}$. Since each of the products $ab_1, \ldots, ab_n$ is an element of $S$, some element $s \in S$ appears at least $\frac{|B|}{|S|}$ times in this list; without loss of generality, we may assume that $ab_1 = \ldots = ab_k = s$, where $k \geq \frac{|B|}{|S|}$. Thus, $0, b_2 - b_1, \ldots, b_k - b_1$ are distinct elements of $\text{ann } a$ and hence $\left|\text{ann } a\right| \geq k \geq \frac{|B|}{|S|}$.

The proof of the remaining assertion is similar.

**Proof of Proposition 3.8.**

If $\kappa_R = \delta$, there is nothing to prove, so assume $\kappa_R < \delta$ and let $S \subseteq V(\Gamma_R) = m - \{0\}$ be a minimal vertex cut. Partition the vertices of $H = \Gamma_R - S$ into two disjoint nonempty sets $A$ and $B$ such that there are no edges between $A$ and $B$; we may assume without loss of generality that $B$ is the larger of these two sets, i.e. $|A| \leq \frac{|m| - |S|}{2} \leq |B|$.

Now if $x \in A$ and $y \in B$, Lemma 3.9 implies that $H$ contains no vertices from $\text{ann } x \cap \text{ann } y$. Since the zero element is not a vertex of $\Gamma_R$, we have, again using Lemma 3.9:

$$|S| \geq |\text{ann } x \cap \text{ann } y| - 1 = \frac{|\text{ann } x||\text{ann } y|}{|\text{ann } x + \text{ann } y|} - 1 \geq \frac{|B|/|S| \cdot |A|/|S|}{|m|} - 1.$$

Thus,

$$|S|^3 \geq \frac{|A||B|}{|m|} - |S|^2 \geq \frac{|A|m - |S|}{2|m|} - |S|^2 = \frac{|A|}{2} - \frac{|A||S|}{|m|} - |S|^2 \geq \frac{|A|}{2} - \frac{|S|}{2} - |S|^2.$$

However, the neighbors of $x \in A$ in $\Gamma_R$ are all members of $A \cup S$. Thus, $|A| + |S| \geq \delta + 1$ and so, continuing the calculation from above, we have:

$$|S|^3 + |S|^2 + \frac{|S|}{2} \geq \frac{|A|}{2} - 1 \geq \frac{\delta - |S| - 1}{2}.$$
which, upon rearrangement, gives

\[ 2(|S|^3 + |S|^2 + |S| + \frac{1}{2}) \geq \delta. \]

Hence, \( 2(|S| + \frac{1}{\sqrt{3}})^3 \geq \delta \), and rearranging the inequality gives the desired result. \( \square \)
References


