

Representation Numbers of Complete Multipartite Graphs

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Abstract

A graph G has a representation modulo r if there exists an injective map $f : V(G) \rightarrow \{0, 1, \dots, r-1\}$ such that vertices u and v are adjacent if and only if $f(u) - f(v)$ is relatively prime to r . The representation number $rep(G)$ is the smallest r such that G has a representation modulo r . Following earlier work on stars, we study representation numbers of complete bipartite graphs and more generally complete multipartite graphs.

1 Introduction

A finite graph G is said to be *representable modulo r* if there exists an injective map $f : V(G) \rightarrow \{0, 1, \dots, r-1\}$ such that vertices u and v of G are adjacent if and only if

$\gcd(f(u) - f(v), r) = 1$ or equivalently if there exists an injective map $f : V(G) \rightarrow \mathbb{Z}_r$ such that u and v are adjacent if and only if $f(u) - f(v)$ is a unit of (the ring) \mathbb{Z}_r . The *representation number* of G , denoted $\text{rep}(G)$, is the smallest positive integer r modulo which G is representable. If we define the unitary Cayley graph $\text{Cay}(n)$ on n vertices to be the graph with vertex set $\{0, 1, \dots, n - 1\}$, two of whose vertices are adjacent if and only if their difference (as integers) is relatively prime to n , then $\text{rep}(G)$ is the smallest positive integer n such that G is isomorphic to an induced subgraph of $\text{Cay}(n)$. Representation numbers first appeared in [4] and were used by Erdős and Evans used to give a simpler proof of a result of Lindner et. al. [13] that any finite graph can be realized as an orthogonal Latin square graph. Since then, representation numbers have been studied for various classes of graphs (see for example [5], [6], [7], and [16]).

In [1], the authors studied the representation number of the stars $K_{1,n}$ from two vantage points, the first being the development of a formula for $\text{rep}(K_{1,n})$ and the second a study of the prime factors of $\text{rep}(K_{1,n})$. In this paper, we extend this two-pronged approach to study representation numbers of complete multipartite graphs. Quite a bit can be said in the case of complete bipartite graphs, so we treat these separately from the general case. After reviewing some results from number theory concerning the distribution of primes, we begin in Section 3 with some general results on the representation numbers of complete bipartite graphs. These are refined further in Section 4 and in Section 5 we study the same for complete multipartite graphs. Not surprisingly, the proofs become increasingly technical along with the complexity of the graph structure; nevertheless, the statements of the various results bear a striking similarity to one other.

2 Results on the distribution of primes

We begin by reviewing some results from number theory on the distribution of primes. Consider the statement

$P(\theta)$: For sufficiently large values of x , there exists a prime between x and $x + x^\theta$.

The statement $P(1)$ is Bertrand's postulate. This was first proved by Chebyshev; simpler proofs were given later by Ramanujan [18] and Erdős [3]. The first proof of the validity of $P(\theta)$ for $\theta < 1$ was given by Hoheisel [10], for $\theta = 32999/33000$. This result was improved upon by Heilbronn [9] for $\theta = 249/250$ and Tchudakoff [20] for $\theta = 3/4 + \varepsilon$; a major breakthrough was made by Ingham [11], who established the validity of $P(5/8)$. The best result to date is due to Baker, Harman, and Pintz:

Theorem 2.1. [2] *There exists N_0 such that for all $x > N_0$, there is a prime between x and $x + x^{.525}$.*

For the balance of the article, we will use N_0 to denote the smallest such integer which makes Theorem 2.1 true. In practice, we will not need the full strength of the theorem; for most of our applications, the case $\theta = 2/3$ is sufficient.

The following hypothesis is of interest in that its validity would imply interesting results about representation numbers:

Hypothesis 2.2. *For sufficiently large x , there is a prime between x and $x + x^{1/2}$.*

There are situations, however, when a weaker but somewhat more widely applicable result of Nagura suffices:

Proposition 2.3. [14]

If $n \geq 25$, then there exists a prime between n and $\frac{6}{5}n$.

Of course, Proposition 2.3 can be deduced from (now) well-known bounds on the prime counting function [19]. These bounds can be used to derive stronger results, including the following:

Proposition 2.4.

If $n \geq 640$, there exists a prime between n and $\frac{15}{14}n$.

These same bounds may be used to prove the next result, which is much weaker than Theorem 2.1 but nevertheless useful in its own right.

Proposition 2.5. *For all $\varepsilon > 0$ there exists $N > 0$ such that when $n > N$, the interval $(n, (1 + \varepsilon)n)$ contains a prime number.*

3 Generalities on $rep(K_{m,n})$

Throughout this section, $K_{m,n}$ denotes the complete bipartite graph with partite sets A, B of respective size m and n . We always assume $m \leq n$ and set $N = m + n$; we use $\phi(n)$ for the Euler totient function.

The following often allows us to make simplifying assumptions about representative labelings:

Lemma 3.1. *If G is any graph and $\ell : V(G) \rightarrow \mathbb{Z}_k$ is any representative labeling, then the following are also representative labelings:*

- For any $a \in \mathbb{Z}_k$, $\tau_a \circ \ell$, where $\tau_a : \mathbb{Z}_k \rightarrow \mathbb{Z}_k$, is the translation map $x \mapsto x + a$.
- $\psi \circ \ell$, where $\psi : \mathbb{Z}_k \rightarrow \mathbb{Z}_k$ is any group automorphism.

We review some elementary bounds on $\text{rep}(K_{m,n})$.

Proposition 3.2. [1, Proposition 2.4]

Let p be the smallest prime greater than N . Then $2n \leq \text{rep}(K_{m,n}) \leq \min\{4n - 4, 2p\}$. In particular, given any $\varepsilon > 0$, $\text{rep}(K_{m,n}) \leq 2(1 + \varepsilon)N$ when N is sufficiently large.

Elementary considerations give the following necessary condition:

Lemma 3.3. [1, Lemma 2.5] $\phi(\text{rep}(K_{m,n})) \geq n$.

The next result is useful in estimating the size of the smallest prime factor of $\text{rep}(G)$ when one has an upper bound for the latter.

Lemma 3.4. [1, Lemma 2.6] Let G be any graph, and p the smallest prime divisor of $\text{rep}(G)$. Then $\omega(G) \leq p \leq \frac{\text{rep}(G)}{\alpha(G)}$.

We proceed to develop more precise upper and lower bounds for $\text{rep}(K_{m,n})$.

Suppose $r = \text{rep}(K_{m,n})$ has prime factorization $r = p_1^{e_1} \dots p_s^{e_s}$. Note that giving a labeling of $K_{m,n}$ modulo r amounts to specifying s pairs $(A_1, B_1), \dots, (A_s, B_s)$ called the *associated label partition*, where for each i , $1 \leq i \leq s$, A_i and B_i are disjoint subsets of $\mathbb{Z}_{p_i^{e_i}}$. The vertices of A are labeled using elements of $A_1 \times \dots \times A_s$ and the vertices of B using elements of $B_1 \times \dots \times B_s$. Conversely, a label partition as above determines (up to permutation of vertices within A or within B) a representation of $K_{m,n}$ modulo r . As we will see in the proofs of Lemmas 3.6 and 3.8 below, label partitions are useful for studying representations of complete bipartite graphs in giving a lower bound on the representation number.

Proposition 3.5. Suppose $r = \text{rep}(K_{m,n})$ has prime factorization $p_1^{e_1} \dots p_s^{e_s}$.

- r is always divisible by 2 or 3.
- If $m = n$, then $s = 1$; that is, $\text{rep}(K_{n,n}) = \min\{r \geq 2n : r = 2^a \text{ or } r = 3^b\}$.
- If $N \geq 25$ and $n < \frac{9}{10}N$, then $s \leq 2$.

Proof.

Observe first that $r \leq 4n - 4 < 4$ by Proposition 3.2, so by Lemma 3.4, r is divisible by some prime less than 4. For convenience, let $m = \gamma N$, where $\gamma \leq \frac{1}{2}$. Considering the associated label partition, we have:

$$m = \gamma N \leq \prod_{i=1}^s |A_i| \text{ and } n = (1 - \gamma)N \leq \prod_{i=1}^s |B_i|$$

Multiplying these inequalities, and noting that A_i and B_i are disjoint subsets of $\mathbb{Z}_{p_i^{e_i}}$,

$$\gamma(1 - \gamma)N^2 \leq \prod_{i=1}^s |A_i||B_i| \leq \prod_{i=1}^s \left(\frac{|A_i| + |B_i|}{2} \right)^2 \leq \frac{1}{4^s} \prod_{i=1}^s p_i^{2e_i} = \frac{1}{4^s} r^2$$

If $m = n$, then $\gamma = \frac{1}{2}$ and $N = 2n$. Since $r < 4n$, the above inequality simplifies to $4^s < 16$, which holds only when $s = 1$.

For the second statement, we use the bound $r < \frac{12}{5}N$ coming from Propositions 2.3 and 3.2 and observe that the above inequality then becomes

$$\gamma(1 - \gamma)N^2 < \frac{1}{4^s} \cdot \frac{144}{25} N^2$$

or $4^s < \frac{144}{25\gamma(1 - \gamma)}$. If $\frac{144}{25\gamma(1 - \gamma)} < 64$, this will imply $s \leq 2$. Solving, this is true when $\gamma > \frac{1}{10}$, which gives $n = (1 - \gamma)N < \frac{9}{10}N$. \square

Remark.

We will see in Section 4 that in fact $s \leq 3$ always.

Lemma 3.6. *Suppose $r = \text{rep}(K_{m,n})$ is even but not a power of 2. Then $r \geq 2N$.*

Proof.

The hypothesis implies that $\mathbb{Z}_r \cong \mathbb{Z}_{2^e} \times \mathbb{Z}_k$ for some odd $k > 1$. Then there exist pairs of disjoint subsets $C_1, C_2 \subseteq \mathbb{Z}_{2^e}$ and $D_1, D_2 \subseteq \mathbb{Z}_k$ such that the vertices of A are labeled using elements of $C_1 \times D_1$ and vertices of B are labeled using elements of $C_2 \times D_2$. Now consider the set $L = \{(c_1 + 1, d_1) : c_1 \in C_1, d_1 \in D_1\} \cup \{(c_2 + 1, d_2) : c_2 \in C_2, d_2 \in D_2\}$. By construction, none of the labels in L appear as a label on any vertex of $K_{m,n}$. Yet, $|L| = |C_1 \times D_1| + |C_2 \times D_2| = N$; thus, we must have $r \geq N + |L| \geq 2N$. \square

Proposition 3.7. *Suppose $N \geq 640$. If $n \geq \frac{5}{7}N$, then $\text{rep}(K_{m,n})$ is divisible by 2. If $n \geq \frac{6}{7}N$, then $\text{rep}(K_{m,n})$ is not divisible by 3.*

Proof.

Suppose $N \geq 640$. By Proposition 2.4 there exists a prime between N and $\frac{15}{14}N$, so by Proposition 3.2, $r < \frac{15}{7}N$. If $n \geq \frac{5}{7}N$, then $r < 3n$, so 2 divides r by Lemma 3.4. Now suppose $n \geq \frac{6}{7}N$. Fix a representative labeling of $K_{m,n}$ modulo r and let $0 \leq b_1 < \dots < b_n < r$ be the integers used on vertices of B in this labeling. For convenience, let $g_i = b_{i+1} - b_i$, $i = 1, \dots, n-1$ and let $g_n = r - b_n + b_1$. Then $g_i > 1$ for all i and $\sum_{i=1}^n g_i = r$, so $r < \frac{15}{7}N < \frac{5}{2}n$. Let $q = |\{i : 1 \leq i \leq n : g_i = 2\}|$. Then $2q + 3(n-q) \leq r < \frac{5}{2}n$, so $q > \frac{1}{2}n$. This implies that $g_n = g_1 = 2$ or there exists i , $1 \leq i \leq n-1$ such that $g_i = g_{i+1} = 2$. In any case, either $\{a_n, a_1, a_2\}$, $\{a_{n-1}, a_n, a_1\}$, or $\{a_i, a_{i+1}, a_{i+2}\}$ (for some i , $1 \leq i \leq n-2$) is a complete set of residues modulo 3, all of which appear on labels of B . This implies that the label of any vertex of A is congruent modulo 3 to the label on some vertex of B , so 3 cannot divide r . \square

Remark.

The bounds $n \geq \frac{5}{7}N$ and $n \geq \frac{6}{7}N$ given above are not in fact sharp and are only stated as such so as to phrase the result for a specified range of values. We will make this more precise in Proposition 3.9.

Lemma 3.8. *Suppose $r = \text{rep}(K_{m,n})$ is odd. If $N \geq 640$, then r is a power of 3.*

Proof.

Since r is odd, Proposition 3.5 implies that r must be divisible by 3, and by Proposition 3.7 we may assume $n < \frac{6}{7}N$. By Proposition 3.5, r has at most two prime factors, so $r = 3^e p^f$ for some prime $p \geq 5$. If $f = 0$, we are done, so we assume henceforth that $f \geq 1$. Interpreting labels modulo r as ordered pairs in $\mathbb{Z}_{3^e} \times \mathbb{Z}_{p^f}$, there exist pairs of disjoint subsets $C_1, C_2 \subseteq \mathbb{Z}_{3^e}$ and $D_1, D_2 \subseteq \mathbb{Z}_{p^f}$ such that the vertices of A are labeled using elements of $C_1 \times D_1$ and vertices of B are labeled using elements of $C_2 \times D_2$. Since $3|r$, no label on a vertex of A can be congruent modulo 3 to a label on a vertex of B . Moreover, since there are only three residue classes modulo 3, at least one of the following two statements is true: (a) all labels used on vertices of A are congruent modulo 3, (b) all labels used on vertices of B are congruent modulo 3.

If both are true, then consider

$$L = \{(c + j, d) : c \in C_1, d \in D_1, j = 1, 2\} \cup \{(c + j, d) : c \in C_2, d \in D_2, j = 1, 2\}$$

Clearly $|L| = 2N$ and no label of L appears on any vertex of $K_{m,n}$. This implies $r \geq 3N$. Now suppose (a) is true and (b) is false. Then there exist two vertices $u, v \in B$ whose respective labels are (c, d) and (c', d') where c and c' are not congruent modulo 3. Since these two vertices are not adjacent, it must be the case that d and d' are congruent modulo p . Now suppose w is *any* vertex of B , and let (c'', d'') be its label; we will show that $d'' \equiv d' \equiv d \pmod{p}$ and hence that all labels on B are congruent modulo p . To this end, we may assume without loss of generality that c'' is congruent to c (and hence not to c') modulo 3. Now w is not adjacent to v , so it must be the case that d'' is congruent to d' (and hence also to d) modulo p . Thus, $|B| \leq \frac{2}{3} \cdot \frac{1}{p} r \leq \frac{2}{15} r$, so $N = |A| + |B| \leq 2|B| \leq \frac{4}{15} r$. and so $r \geq \frac{15}{4} N$. Finally, if (a) is false but (b) is true, then by analogous reasoning it must be the case that all labels on the vertices of A are congruent to some fixed value modulo p . This implies that $|A| \leq \frac{2}{3} \cdot \frac{1}{p} r \leq \frac{2}{15} r$. Moreover, since (b) is true, $|B| \leq \frac{1}{3} r$. Hence $N = |A| + |B| \leq \frac{7}{15} r$ or $r \geq \frac{15}{7} N$.

In all cases, $r \geq \frac{15}{7} N$. However, $N \geq 640$, so by Proposition 2.4 there is a prime between N and $\frac{15}{14} N$, and Proposition 3.2 implies $r < \frac{15}{7} N$, a contradiction. \square

Proposition 3.9. *Fix any $\varepsilon > 0$. If $n > (2/3 + \varepsilon)N$ and N is sufficiently large, $r = \text{rep}(K_{m,n})$ is divisible by 2 but not by 3.*

Proof.

For N sufficiently large, Proposition 2.5 implies the existence of a prime between N and $(1 + 3\varepsilon/2)N$, so if $n > (2/3 + \varepsilon)N$, then $r < 2(1 + 3\varepsilon/2)N < 3n$ by Proposition 3.2 and thus $2|r$ by Lemma 3.4. By Proposition 3.7, we may assume $n < \frac{6}{7}N$; then Proposition 3.5 implies that r has at most two prime factors. Hence if $3|r$, then $r = 2^a 3^b$ for some $a, b \geq 1$. Now fix a representative labeling $f : K_{m,n} \rightarrow \mathbb{Z}_{2^a} \times \mathbb{Z}_{3^b}$ and let $\alpha : \mathbb{Z}_{2^a} \rightarrow \mathbb{Z}_2$ and $\beta : \mathbb{Z}_{3^b} \rightarrow \mathbb{Z}_3$ be the obvious quotient maps. Then we may assume without loss of generality that $\alpha(f(A)) = 0$ and $\alpha(f(B)) = 1$; informally this means that the labels used on vertices of A are even and those on B are odd. Furthermore, we may assume either $\beta(f(A)) = 0$ or $\beta(f(B)) = 0$. In the first case, there are no vertices which receive labels of the form (x, y) where $\alpha(x) = 0$ and $\beta(y) \neq 0$ or $\alpha(x) = 1$ and $\beta(y) = 0$. Thus $r \geq (|A| + |B|) + (2|A| + |B|) = 3m + 2n =$

$m + 2N \geq \frac{15}{7}N$. In the second case, there are no vertices which receive labels of the form (x, y) where $\alpha(x) = 1$ and $\beta(y) \neq 0$ or $\alpha(x) = 0$ and $\beta(y) = 0$. Thus $r \geq (|A| + |B|) + (|A| + 2|B|) = 2m + 3n = 2N + n \geq \frac{5}{2}N$. In either case, Proposition 3.5 provides a contradiction when N is sufficiently large. \square

Remark.

The lower bound of Proposition 3.9 cannot be improved. Indeed, consider the graph $G = K_{3^{a-1}, 2 \cdot 3^{a-1}}$ where a is chosen so that $3^a \geq 640$ and there is no power of 2 in the interval $\left(\frac{4}{3} \cdot 3^a, 2 \cdot 3^a\right)$; for example, one may choose $a = 8$. Then G is representable modulo $\ell = 2 \cdot 3^a$ simply by assigning labels congruent to 0 modulo 6 on vertices of A and labels congruent to 1 or 5 modulo 6 on vertices of B . Now if G is representable modulo 2^k , then $2^k \geq 2n = \frac{4}{3} \cdot 3^a$, so by our hypothesis on a , $r = \text{rep}(G)$ is not a power of 2. Likewise, if G is representable modulo 3^k , then all the labels on vertices of A must be congruent modulo 3, and similarly for vertices of B ; hence $3^{k-1} \geq |B| = 2 \cdot 3^{a-1}$, so $3^k \geq 2 \cdot 3^a = \ell$. Thus, $3^k > \ell \geq r$ and r is not a power of 3. By Lemma 3.8, r must be even; then Lemma 3.6 implies that $r \geq 2N = 2 \cdot 3^a$.

4 Complete Bipartite Graphs

In this section, we derive bounds for the representation number of complete bipartite graphs and study its prime factorization further.

For any integer $k \neq 0, \pm 1$, we define the *radical* of k (denoted $\text{rad } k$) to be the product of its distinct prime divisors. Note that if $i \in \mathbb{Z}_k$ is any nonzero element, then $k/\text{rad } k = |\{j \in \mathbb{Z}_k : i \equiv j \pmod{p} \text{ for all primes } p|k\}|$. For convenience, we define $\psi(k) = \phi(k) + k/\text{rad } k$. In the context of complete bipartite graphs, the function ψ will play a role similar to that played by ϕ for stars.

The next result gives a useful lower bound for $\psi(r)$. We offer two proofs, each of a different nature.

Lemma 4.1. *Let $r = \text{rep}(K_{m,n})$. Then $N \leq \psi(r)$.*

First proof.

Suppose r has prime factorization $r = p_1^{e_1} \dots p_s^{e_s}$. Given a labeling modulo r , let $(A_1, B_1), \dots, (A_s, B_s)$ be the associated label partition. Then we have

$$N \leq \prod_{i=1}^s |A_i| + \prod_{i=1}^s |B_i|.$$

For each $i = 1, \dots, s$, A_i and B_i are both nonempty subsets of $\mathbb{Z}_{p_i^{e_i}}$ and no two elements of A_i and B_i are congruent modulo p_i . Moreover, given a residue t_i modulo p_i , the number of elements of $\mathbb{Z}_{p_i^{e_i}}$ congruent to t_i modulo p_i is $p_i^{e_i-1}$. Thus to find an upper bound for $\prod_{i=1}^s |A_i| + \prod_{i=1}^s |B_i|$, it suffices to find the maximum value of $f(a_1, \dots, a_s) = a_1 \dots a_s + (p_1 - a_1) \dots (p_s - a_s)$ where $1 \leq a_i \leq p_i - 1$ and multiply this by $r/\text{rad } r = p_1^{e_1-1} \dots p_s^{e_s-1}$. It is easy to see that $f(a_1, \dots, a_s)$ is maximized when $a_1 = \dots = a_s = 1$. Thus, $f(a_1, \dots, a_s) \leq 1 + \prod_{i=1}^s (p_i - 1)$ and so

$$N \leq \prod_{i=1}^s |A_i| + \prod_{i=1}^s |B_i| \leq r/\text{rad } r + \phi(r)$$

Second proof.

Let $X \subseteq \mathbb{Z}_r$ be the labels used on vertices of A and $Y \subseteq \mathbb{Z}_r$ be the labels used on vertices of B . By Lemma 3.1 we may assume without loss of generality that $0 \in X$. This implies immediately that

$$U = X - Y \subseteq \{t : 1 \leq t \leq r - 1, \text{gcd}(t, r) = 1\}$$

so $|U| \leq \phi(r)$.

However, by Kneser's Theorem for abelian groups [17, Theorem 4.3],

$$|U| \geq |X + H| + |-Y + H| - |H| \geq |X| + |Y| - |H|$$

where $H = \{h \in \mathbb{Z}_r : h + u \in U \text{ for all } u \in U\}$ is the setwise stabilizer of $U \subseteq \mathbb{Z}_r$ for the translation action. We claim that if $r = p_1^{e_1} \dots p_s^{e_s}$ is a prime factorization, then H is contained in the subgroup of \mathbb{Z}_r generated by $p_1 \dots p_s$; in particular, $|H| \leq p_1^{e_1-1} \dots p_s^{e_s-1} = r/\text{rad } r$. Suppose towards a contradiction that there exists $h \in H$ and i , $1 \leq i \leq s$ such that $h \not\equiv 0 \pmod{p_i}$. Then by adding an appropriate multiple of h to any element of U , one obtains an element of U congruent to 0 modulo p_i ; this contradicts the fact that (by construction) elements of U are relatively prime to r .

Putting all this together,

$$\phi(r) \geq |U| \geq |X| + |Y| - |H| \geq N - \frac{r}{\text{rad } r}$$

as desired. □

The next result provides upper and lower bounds for $\text{rep}(K_{m,n})$.

Proposition 4.2.

$$\min\{k : \psi(k) \geq N\} \leq \text{rep}(K_{m,n}) \leq \min\{k : 2|k, \phi(k) \geq N\}$$

Proof.

The lower bound follows from Lemma 4.1. For the upper bound, suppose $2|k$ and $\phi(k) \geq N$. If $k = 2^a$ for some a , then the condition $\phi(2^a) \geq N$ implies $2^{a-1} \geq N$; so certainly $K_{m,n}$ may be labeled by \mathbb{Z}_{2^a} , simply by assigning even integers to vertices of A and odd integers to vertices of B . If k is even but not a power of 2, then k is divisible by some other prime p . Hence, $N \leq \phi(k) \leq \frac{1}{2} \cdot \frac{p-1}{p} k$. We proceed to construct a representative labeling of $K_{m,n}$ by elements of $\mathbb{Z}_{2^a} \times \mathbb{Z}_{p^b}$. As before, we denote the obvious quotient maps by $\alpha : \mathbb{Z}_{2^a} \rightarrow \mathbb{Z}_2$ and $\beta_{p^b} : \mathbb{Z}_{p^b} \rightarrow \mathbb{Z}_p$. For each $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_p$, let $L(a, b) = \{(x, y) \in \mathbb{Z}_{2^a} \times \mathbb{Z}_{p^b} : \alpha(x) = a, \beta(y) = b\}$. Clearly $|L(a, b)| = \frac{k}{2p} \geq \frac{N}{p-1}$. Thus we construct our labeling by assigning (in any order) labels of the form $L(a, b)$, $a \equiv 0 \pmod{2}$, $b = 0, \dots, \lceil \frac{|A|}{2p} \rceil$ to vertices of A and labels of the form $L(a, b)$, $a \equiv 1 \pmod{2}$, $b = \lceil \frac{|A|}{2p} \rceil + 1, \dots, p-1$ to vertices of B . \square

Remark.

It is worth noting that both bounds are tight: if one takes a graph with representation number $r = 2^a p^b$, where p is an odd prime and $a, b \geq 1$ (see the example following Theorem 4.6), then $\psi(r) = \frac{r}{2} = N$. On the other hand, [1, Theorem 3.1] states $\text{rep}(K_{1,n}) = \min\{k : 2|k, \phi(k) \geq n\}$, so when n is odd, this equals $\min\{k : 2|k, \phi(k) \geq n+1 = N\}$.

We now turn towards results on the form of the representation number for complete bipartite graphs.

Lemma 4.3. *Given a prime $p \geq 5$, there are only finitely many n with the property that $2pq | \text{rep}(K_{m,n})$ for some odd prime $q \neq p$.*

Proof.

Fix a prime $p \geq 5$ and suppose $q \neq p$ is an odd prime such that $2pq | \text{rep}(K_{m,n})$. By Lemma 4.1, $N \leq \psi(r) = \phi(r) + \frac{r}{\text{rad } r} \leq r \left(\frac{(p-1)(q-1)}{2pq} + \frac{1}{2pq} \right)$. Thus $r \geq 2N \frac{pq}{(p-1)(q-1)+1} \geq 2N \left(1 + \frac{1}{2(p-1)} \right)$. However, when N is sufficiently large, Proposition 3.2 implies that this inequality cannot hold. \square

Lemma 4.4. *Suppose k is an even number divisible by at least two odd primes $p < q$. Then $\psi(k) \leq \frac{p-1}{2p} \cdot k$.*

Proof.

Suppose not. Then $\psi(k) = \phi(k) + \frac{k}{\text{rad } k} > \frac{p-1}{2p} \cdot k$. Since $\phi(k) \leq \frac{k}{2} \cdot \frac{p-1}{p} \cdot \frac{q-1}{q}$ and $\text{rad } k \geq 2pq$, we have $\frac{(p-1)(q-1)}{2pq} + \frac{1}{2pq} > \frac{p-1}{2p}$, which reduces to $p < 2$, a contradiction. \square

Lemma 4.5. *Suppose k is sufficiently large and not of the form 2^a , $2^a p^b$, or $2^a pq$, where p, q are odd primes and $a, b \geq 1$. Then there exists a prime in the interval $(\psi(k), k/2)$. If Hypothesis 2.2 holds, then this statement is true when k is of the form $2^a pq$.*

Proof.

Suppose not. By Theorem 2.1, it must be the case that $\psi(k) + [\psi(k)]^{2/3} > k/2$. Moreover, the hypothesis implies that $k = 2^a p^b q^c t$, where p is the smallest odd prime dividing k , q is the second smallest odd prime dividing k , $a, b, c \geq 1$, and $\text{gcd}(t, 2pq) = 1$. By Lemma 4.4, $\frac{p-1}{2p}k + [\frac{p-1}{2p}k]^{2/3} > \frac{k}{2}$, which simplifies to $(p-1)^2 > 2^{a-1} p^{b-1} q^c t$. If $b \geq 2$ or $c \geq 2$, we immediately arrive at a contradiction; so the only case left to consider is $b = c = 1$. In this case, $k = 2^a pqt$, and the hypothesis implies that $t > 1$. Since t is divisible only by primes greater than q , we again arrive at a contradiction. Assuming Hypothesis 2.2, the relevant inequality becomes $\psi(k) + [\psi(k)]^{1/2} > \frac{k}{2}$, which simplifies to $k < 2(p-1)p < 2^a pqt = k$, a contradiction. \square

Theorem 4.6. *For n sufficiently large, $\text{rep}(K_{m,n})$ is of one of the following forms: 2^a , 3^a , $2^a p^b$, or $2^a pq$, where $a, b \geq 1$, and p, q are distinct odd primes.*

If Hypothesis 2.2 holds, then the last possibility cannot occur.

Proof.

If r is not one of the said forms, we may assume by Lemma 3.8 that (for sufficiently large n), $\text{rep}(K_{m,n})$ is even. By applying Lemma 4.3 iteratively, we may choose n to be large enough so that r is not divisible by any odd prime less than N_0 . Suppose $r = \text{rep}(K_{1,n})$ may be prime factored as $r = 2^a p^b q^c p_1^{e_1} \dots p_s^{e_s}$, where $b, c \geq 1$, $s \geq 0$ and $e_i \geq 0$ for all $i = 1, \dots, s$, and $b + c \geq 3$ or $(b = c = 1$ and some $e_i \geq 1)$. For convenience, let $t = p_1^{e_1} \dots p_s^{e_s}$. By making the identification $\mathbb{Z}_r \cong \mathbb{Z}_{2^a} \times \mathbb{Z}_{p^b} \times \mathbb{Z}_{q^c} \times \mathbb{Z}_t$, we may interpret labels modulo r as 4-tuples. Fix a labeling $\ell : V(K_{m,n}) \rightarrow \mathbb{Z}_r$ and let $L = \ell(V(K_{m,n}))$. For each odd prime divisor η of r , let

$$L_{\eta,A} = \{y \in \mathbb{Z}_\eta : \text{there exists a vertex of } A \text{ whose label is congruent to } y \pmod{\eta}\}$$

and define $L_{\eta,B}$ similarly. Clearly all these sets are nonempty. Then

$$|L| \leq \left(|L(p,A)| \cdot |L(q,A)| \cdot \prod_{i=1}^s |L(p_i,A)| + |L(p,B)| \cdot |L(q,B)| \cdot \prod_{i=1}^s |L(p_i,B)| \right) \frac{r}{rad\ r}.$$

The expression in parentheses is no larger than $1 + (p-1)(q-1) \prod_{i=1}^s (p_i-1)$; hence,

$$|L| \leq \left(1 + (p-1)(q-1) \prod_{i=1}^s (p_i-1) \right) \frac{r}{rad\ r} = \frac{r}{rad\ r} + \phi(r) = \psi(r). \text{ By Lemma 4.5,}$$

there exists a prime $\pi \in (\psi(r), r/2)$; thus, $N = |L| \leq \psi(r) < \pi$. We may then convert any representative labeling modulo r into a representative labeling mod $\tilde{r} = 2\pi < r$ as follows: first fix an injective function $\alpha : L \rightarrow \mathbb{Z}_\pi$; then convert the label (x, y, z, w) mod r into the label $(x \pmod{2}, \alpha(y, z, w)) \pmod{\tilde{r}}$. Thus, $K_{m,n}$ is representable by $\tilde{r} < r$, a contradiction. If Hypothesis 2.2 is assumed, an analogous argument can be made for the case $b = c = t = 1$. \square

Remark. Unlike the case of stars, it is possible for $rep(K_{m,n})$ to be of the form $2^a p^b$, where p is an odd prime and $b \geq 2$. This may be seen by generalizing the construction used in the example following Proposition 3.9. Choose positive integers a, b such that there is no power of 2 or power of 3 between $2^a(p-1)p^{b-1}$ and $2^a p^b$. Let $m = 2^{a-1} p^{b-1}$ and $n = (p-1)m = 2^{a-1}(p-1)p^{b-1}$. We claim that $rep(K_{m,n}) = 2N = 2^a p^b$.

To obtain an upper bound, one constructs a labeling of $K_{m,n}$ as follows: let

$$L_1 = \{(x, y) \in \mathbb{Z}_{2^a} \times \mathbb{Z}_{p^b} : \alpha(x) = 0, \beta(y) = 0\}$$

and

$$L_2 = \{(x, y) \in \mathbb{Z}_{2^a} \times \mathbb{Z}_{p^b} : \alpha(x) = 1, \beta(y) \neq 0\}$$

and assign labels in L_1 to vertices of A , and labels of L_2 to vertices of B . This shows that $rep(K_{m,n}) \leq 2^a p^b = 2N$. Now, $rep(K_{m,n})$ is neither a power of 2 nor a power of 3 since $rep(K_{m,n}) \geq 2n = 2^a(p-1)p^{b-1}$ and there is no power of 2 or 3 between $2^a(p-1)p^{b-1}$ and $2^a p^b = 2N$. Thus Lemmas 3.6 and 3.8 apply (when N is large enough) to show $2|N$ and thus $rep(K_{m,n}) \geq 2N$.

5 Complete Multipartite Graphs

In this section, we generalize Theorem 4.6 to the setting of complete multipartite graphs. Throughout this section, $G = K_{n_1, \dots, n_t}$ will denote a complete multipartite graph with partite sets A_1, \dots, A_t , $t \geq 2$, where $|A_i| = n_i$ for each $i = 1, \dots, t$; we let $N = n_1 + \dots + n_t$.

We begin with some elementary bounds analogous to those of Propositions 3.2 and 3.3.

Proposition 5.1. *Let ℓ be the smallest prime $\geq t$, p' the smallest prime $\geq N$, and p the smallest prime divisor of $\text{rep}(G)$. Then $\ell \leq p \leq \ell^2$ and $pn_t \leq \text{rep}(G) \leq \ell p'$.*

Proof.

Since $\omega(G) = t$, the lower bound first statement is a direct consequence of Lemma 3.4. By assigning (greedily) vertices in A_i nonnegative integers congruent to i modulo ℓ , we see that $\text{rep}(G)$ is no larger than the smallest power of ℓ greater than or equal to ℓn_t ; in particular, $\text{rep}(G) \leq \ell^2 n_t$. The upper bound then follows from the second statement of Lemma 3.4. Now given any representative labeling of G , the difference between any two labels on vertices in A_t must share a common factor with $\text{rep}(G)$, hence must be at least p . Thus, $\text{rep}(G) \geq pn_t$. Now for each $i = 1, \dots, t$ use labels $(i - 1, \sum_{j=1}^{i-1} n_j + m) \in \mathbb{Z}_\ell \times \mathbb{Z}_{p'}$, $m = 0, \dots, n_i - 1$ to label the vertices of A_i . This shows $\text{rep}(G) \leq \ell p'$, establishing the second statement. \square

Complete bipartite graphs have the pleasant property that in any representative labeling with respect to an even modulus, labels on vertices in the same partite set are congruent modulo 2. For a complete multipartite graph G with $t > 2$ partite sets, however, the smallest prime dividing $\text{rep}(G)$ may be greater than t , so the analogous property does not follow directly from definitions. Nevertheless, it is possible to circumvent this problem. For convenience, we call a representative labeling of a multipartite graph G *coherent* if for every partite set there exists a prime divisor of $\text{rep}(G)$ such that all the labels on vertices in that partite set are congruent modulo this prime; we then show that every complete multipartite graph G has a representation modulo $\text{rep}(G)$ which is described by a coherent labeling.

We begin with a basic lemma. In the following, $M(k, \ell)$ denotes a complete multipartite graph with k partite sets, each of size ℓ .

Lemma 5.2. *Suppose $H = \prod_{i=1}^s M(k_i, \ell_i)$, where $k_1 \leq \dots \leq k_s$. Then $\alpha(H) = \ell_1 \prod_{i=2}^s k_i \ell_i$.*

Proof.

If $s = 1$, the claim is trivial, so assume $s \geq 2$, and let P be any of the partite sets in $M(k_1, \ell_1)$. Then $P \times \prod_{i=2}^s M(k_i, \ell_i)$ is an independent set in H of size $\ell_1 \prod_{i=2}^s k_i \ell_i$. We may naturally identify vertices of $M(k_i, \ell_i)$ with pairs of integers (a, b) , where $1 \leq a \leq k_i$ and $1 \leq b \leq \ell_i$. Now define a coloring

$$f : V(\bar{H}) \rightarrow \mathbb{Z}_{\ell_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{\ell_2} \times \dots \times \mathbb{Z}_{k_s} \times \mathbb{Z}_{\ell_s}$$

by $f((a_1, b_1), \dots, (a_s, b_s)) = (b_1, a_2 - a_1, b_2, a_3 - a_1, b_3, \dots, a_s - a_1, b_s)$. Let $v = (a_1, b_1), \dots, (a_s, b_s)$ and $v' = f((a'_1, b'_1), \dots, (a'_s, b'_s))$ and suppose $f(v) = f(v')$. Then $b_i = b'_i$ and $a'_i = a_i + (a'_1 - a_1)$ for all i ; that is, $((a'_1, b'_1), \dots, (a'_s, b'_s)) = ((a_1, b_1), \dots, (a_s, b_s)) + ((c, 0), (c, 0), \dots, (c, 0))$ where $c = a'_1 - a_1$. If $c = 0$, then $v = v'$. If $c \neq 0$, then for each i , a_i and a'_i lie in different partite sets of $M(k_i, \ell_i)$, so v is adjacent to v' in H ; equivalently, v and v' are not adjacent in \bar{H} . Thus f is a proper vertex coloring of \bar{H} , and hence $\alpha(H) = \omega(\bar{H}) \leq \chi(\bar{H}) \leq \ell_1 \prod_{i=2}^s k_i \ell_i$. \square

Proposition 5.3. *Suppose a complete multipartite graph G has a representative labeling modulo r . Then G has a coherent representative labeling modulo r .*

Proof.

Since the proposition is immediate if $s = 1$, we assume henceforth $s \geq 2$. Suppose G is representable modulo $r = p_1^{e_1} \dots p_s^{e_s}$ and that $f : V(G) \rightarrow \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_s^{e_s}}$ is a representative labeling. Letting $\pi_j : \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_s^{e_s}} \rightarrow \mathbb{Z}_{p_j}$ denote the j th projection followed by the natural quotient map, define $S_{ij} = \pi_j(f(A_i))$; informally, this is the set of residue classes modulo p_j that appear in the j th coordinate of labels on vertices of A_i . Thus we have an embedding $h : G \hookrightarrow \text{Cay}(r) \cong \prod_{j=1}^s M(p_j, p_j^{e_j-1})$. For $j = 1, \dots, s$, fix an enumeration $P_{1j}, \dots, P_{p_j j}$ of the partite sets of $M(p_j, p_j^{e_j-1})$. For each $i = 1, \dots, t$ and $j = 1, \dots, s$, let $L_{ij} \subseteq \{1, \dots, p_j\}$ be the set of integers k for which there exists $x \in A_i$ such that $h(x) = (m_1, \dots, m_s)$ with $m_j \in P_{kj}$. Letting $\ell_{ij} = |L_{ij}|$, we may regard A_i as an induced subgraph of $\prod_{j=1}^s M(\ell_{ij}, p_j^{e_j-1})$. Observe that $|S_{ij}| = \ell_{ij}$. Now choose $j_i, 1 \leq j_i \leq s$, such that $\ell_{ij_i} \leq \ell_{ij}$ for all $j = 1, \dots, s$. By Lemma 5.2, $|A_i| \leq \alpha(\prod_{j=1}^s M(\ell_{ij}, p_j^{e_j-1})) = p_{j_i}^{e_{j_i}-1} \prod_{j \neq j_i} \ell_{ij} p_j^{e_j-1}$. Finally, for each $j = 1, \dots, s$, let $\beta_j : \mathbb{Z}_{p_j^{e_j}} \rightarrow \mathbb{Z}_{p_j}$ denote the quotient map, and for each $i = 1, \dots, t$, fix any $b_{j_i} \in S_{ij_i}$. We construct a coherent labeling $g : V(G) \rightarrow \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_s^{e_s}}$ by assigning vertices of A_i (in any order) labels of the form (x_1, \dots, x_s) where $\beta_j(x_j)$ is in $\beta_j(S_{ij})$ for $j \neq j_i$ and $\beta_{j_i}(x_{j_i}) = b_{j_i}$. This is possible because (by construction) there are $p_{j_i}^{e_{j_i}-1} \prod_{j \neq j_i} |S_{ij}| p_j^{e_j-1} = p_{j_i}^{e_{j_i}-1} \prod_{j \neq j_i} \ell_{ij} p_j^{e_j-1}$ such labels available, and $|A_i|$ does not exceed this quantity. \square

We now have the following multipartite analogue of Theorem 4.6:

Theorem 5.4. *Let G be a complete t -partite graph, where $t \geq 2$. For sufficiently large N , $\text{rep}(G)$ is of one of the following forms: p^a , $p^a q^b$, $p^a q^b u$, where $a \geq 1$, $b \geq 1$, and $p < q < u$ are primes.*

Proof.

Let p be the smallest prime dividing $r = \text{rep}(G)$ and q the second smallest prime. If r is not one of the above forms, then it has at least three distinct prime factors. For

convenience of notation, set $p_1 = p$, $p_2 = q$, $e_1 = a$, $e_2 = b$, and let $p_3 < \dots < p_s$ be the other prime factors of $r = \prod_{j=1}^s p_j^{e_j}$, where $s \geq 3$. Let $r' = r/\text{rad } r$. Now fix a representative labeling of G modulo r ; by Proposition 5.3, we may assume that for each i , $1 \leq i \leq t$, there exists some j_i , $1 \leq j_i \leq s$ such that the labels assigned to all vertices of A_i are congruent to a_{j_i} modulo p_{j_i} . We will call the j_i th coordinate the i -critical coordinate.

The proof proceeds in the same spirit as that of Lemma 3.8. The idea is to show that if r is not of any the above forms, then the total number of labels is higher than the upper bound imposed by Proposition 5.1. In the following, L_i will denote the set of (congruence classes of) labels actually used on the vertices of A_i ; L'_i will denote the set of labels which differ from some label in L_i in the i -critical coordinate but agree everywhere else. The set L''_i will consist of labels which agree with some label in L_i in all coordinates except the k th coordinate, where $k = 2$ if the first coordinate is i -critical and $k = 1$ otherwise. Since there are at least three coordinates, these three sets of labels are pairwise disjoint.

For each $j = 1, \dots, s$, let $\beta_j : \mathbb{Z}_{p_j^{e_j}} \rightarrow \mathbb{Z}_{p_j}$ denote the quotient map. Now fix i , $1 \leq i \leq t$. By construction, the labels on vertices of A_i are of the form $(y_1, \dots, y_s) \in R = \mathbb{Z}_{p_1^{e_1}} \times \dots \times \mathbb{Z}_{p_s^{e_s}}$, where for each $j = 1, \dots, s$, $\pi_j(y_j) \in L_{ij} \subseteq \mathbb{Z}_{p_j}$; in particular, $L_{ij_i} = \{b_{j_i}\}$. Let

$$L_i = \{(y_1, \dots, y_s) \in R : \pi_j(y_j) \in L_{ij} \text{ for all } j = 1, \dots, s\}$$

and

$$L'_i = \{(y_1, \dots, y_s) \in R : \pi_j(y_j) \in L_{ij} \text{ for all } j \neq j_i \text{ and } \pi_{j_i}(y_{j_i}) \neq b_{j_i}\}.$$

By construction, $L'_i \cap L_i = \emptyset$ and $|L'_i| = (p_{j_i} - 1)|L_i| \geq (p - 1)|L_i|$. Now, if $j_i = 1$, let $j'_i = 2$; otherwise let $j'_i = 1$, and define

$$L''_i = \{(y_1, \dots, y_s) \in R : \pi_j(y_j) \in L_{ij} \text{ for all } j \neq j'_i \text{ and } \pi_{j'_i}(y_{j'_i}) \notin L_{ij'_i}\}.$$

Once again, L''_i is disjoint from both L_i and L'_i ; moreover,

$$|L''_i| = \frac{p_{j'_i} - |L_{ij'_i}|}{|L_{ij'_i}|} \cdot |L_i| \geq \frac{1}{p_{j'_i} - 1} |L_i| \geq \frac{1}{q - 1} |L_i|.$$

Setting $\tilde{L}_i = L_i \cup L'_i \cup L''_i$, we have

$$|\tilde{L}_i| = |L_i| + |L'_i| + |L''_i| \geq |L_i| + (p - 1)|L_i| + \frac{1}{q - 1} |L_i| \geq \left(p + \frac{1}{q - 1}\right) |A_i|.$$

It is not hard to see that the sets \tilde{L}_i , $i = 1, \dots, t$ are pairwise disjoint, so letting $\tilde{L} = \cup_{i=1}^t \tilde{L}_i$, we have:

$$r \geq |\tilde{L}| \geq \sum_{i=1}^t \left(p + \frac{1}{q-1} \right) |A_i| = \left(p + \frac{1}{q-1} \right) N$$

Now our hypothesis tells us that r is divisible by at least two (not necessarily distinct) primes larger than q , so $q < r^{1/3}$. Moreover, by Proposition 5.1, $r \leq \ell p'$, where ℓ is the smallest prime $\geq t$ and p' is the smallest prime $\geq N$; by Bertrand's postulate, $r < 4tN$. By Proposition 5.1, $\ell < p < \ell^2 < 4t^2$.

Hence, $\frac{1}{q-1} > \frac{1}{q} > \frac{1}{r^{1/3}} > \frac{1}{(4t)^{1/3}N^{1/3}}$ and so

$$r \geq \left(p + \frac{1}{q-1} \right) N \geq pN + \frac{1}{(4t)^{1/3}} N^{2/3} = p \left(N + \frac{1}{p(4t)^{1/3}} N^{2/3} \right) \geq \ell \left(N + \frac{1}{4^{4/3}t^{7/3}} N^{2/3} \right)$$

However, t is fixed, so for sufficiently large N , Theorem 2.1 implies that there exists a prime between N and $N + \frac{1}{4^{4/3}t^{7/3}} \cdot N^{2/3}$. Hence $r > \ell p'$, which establishes a contradiction. \square

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