Splitting techniques and Betti numbers of secant powers

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July 22, 2015

Abstract

Using ideal-splitting techniques, we prove a recursive formula relating the Betti numbers of the secant powers of the edge ideal of a graph H to those of the join of H with a finite independent set. We apply this result in conjunction with other splitting techniques to compute these Betti numbers for wheels, complete graphs and complete multipartite graphs, recovering and extending some known results about edge ideals.

1 Introduction

Let R be a polynomial ring in finitely many variables over a base field $\mathbb{K}$. One approach to studying modules over R is by constructing free resolutions and studying properties of these. If M is a finitely generated graded R-module, Hilbert's Syzygy Theorem implies that there exists a free resolution with only finitely many terms.

Furthermore, one can show that among these free resolutions, there is one which is minimal (in a sense which will be made precise later), and thereby defines a collection of integers, the *Betti numbers* of M. Of particular interest is the case when the module in question is an ideal of R. Even more specifically, if G is a simple graph with vertices $v_1, \ldots, v_n$, its *edge ideal* $I(G)$ is the ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials $x_i x_j$ such that $v_i v_j$ is an edge of G. The edge ideal was first defined by Villareal [17] and has attracted considerable interest as an algebraic object which encodes combinatorial information. In recent years, much attention has been devoted to studying the Betti numbers of edge ideals; see for example [4], [6], [7], [8], [10]. Betti numbers are also of interest in algebraic geometry ([14], [15]), as the edge ideal defines a (not necessarily irreducible) variety in n-dimensional projective space over $\mathbb{K}$.

A more general problem is that of computing the Betti numbers of the *secant powers* of the edge ideal. The actual definition of the secant powers of an ideal is somewhat delicate, but the idea is not hard to grasp. The first secant power is the ideal itself, and if V is the variety in n-dimensional projective space over $\mathbb{K}$ defined by the ideal, then its rth secant power is the ideal which defines the rth secant variety of V. The Betti numbers of secant powers of the edge ideal have also been studied in the literature (see [1], [12], and especially [14], [15]) but not nearly as extensively as those of the edge ideal. For convenience of reference, we will use the phrase "Betti numbers of G" throughout this article as shorthand for "Betti numbers of the secant powers of the edge ideal of G."

In his Ph.D. thesis [10], Sean Jacques studied and computed the Betti numbers of the edge ideals corresponding to various classes of graphs, including cycles, paths, forests, complete graphs, and complete bipartite graphs. His main tool is a formula of Hochster [9] which expresses the Betti numbers of a Stanley-Reisner ring over a simplicial complex in terms of the (simplicial) homology of the complex. Using this formula, Jacques was able to give exact computations of all the Betti numbers of complete graphs and complete bipartite graphs. His techniques have been applied in several works since (for example, [4]) and have proven to be quite fruitful.

An alternative approach to computing Betti numbers of edge ideals was initiated by Huy Tài Hà and Adam Van Tuyl in [7] and [8]; see in particular Theorems 3.6 and 4.6 of [7]. This technique, called *ideal splitting*, goes back to Eliahou and Kervaire [3] in the ungraded case and Fatabbi [5] in the graded case. The idea is to decompose the (monomial) ideal under consideration into simpler pieces, and make use of a formula relating the Betti numbers of the pieces to the Betti numbers of the original ideal. The advantage of this approach is that it obviates the need to compute simplicial homology groups and allows, at least in some cases, for the calculation of Betti numbers by induction.

The present article is written in the spirit of [7], but the notion of ideal splitting is applied in a different way, and in a different setting. Using a combinatorial description of higher secant ideals due to Sturmfels and Sullivant [16], we derive a recursive formula (Theorem 4.4) which allows us to relate the Betti numbers of join of a graph with a finite independent set to the Betti numbers of the graph itself. Since complete graphs and complete bipartite graphs can both be constructed by iterating this type of join operation, one can use this formula to compute the Betti numbers of *all the secant powers* of their edge ideals. In the process, we recover Jacques's calculations (for the edge ideal itself) by purely combinatorial means, without recourse to Hochster's formula. We emphasize that all our results are independent of the choice of base field $\mathbb{K}$.

Part of this work was done at the Summer Undergraduate Mathematical Sciences Research Institute held at Miami University during June and July 2013; it was later augmented and generalized by the first author. The authors also used the software `Macaulay2` to aid in the computation of examples. The authors thank the National Security Agency, the National Science Foundation, and Miami University for support during this time. They also thank Hamid Rahmati and Jessica Sidman for helpful discussions and correspondence.

2 Preliminaries

We now provide some background on minimal free resolutions; more detail may be found in any standard book on the subject, for example [2].

Throughout this article, we fix a base field $\mathbb{K}$. Let $x_1, \ldots, x_t$ be independent indeterminates and $R = \mathbb{K}[x_1, \ldots, x_t]$. Then R is an $\mathbb{N}$-graded ring in the natural way: $R = \bigoplus_e R_e$, where R_e is the $\mathbb{K}$-vector space spanned by the monomials in $x_1, \ldots, x_t$ of total degree e. Note also that R has a unique maximal ideal $\mathfrak{m}$ consisting of all elements of positive degree. For any integer d, we denote by $R(d)$ the graded ring whose degree e part is R_{d+e}. An ideal $I \subseteq R$ is called a *monomial ideal* if it is generated by monomials.

Now suppose I is an ideal of R. Because R/I is a finitely generated as an R-module, Hilbert's Syzygy Theorem [2, Corollary 19.7] implies that it has a finite resolution by free modules; that is, there exists an integer $n \leq t + 1$, finitely generated free

R-modules $F_0, \ldots, F_n$, and R-module homomorphisms $\phi_i : F_i \to F_{i-1}$, $i = 1, \ldots, n$ and $\phi_0 : F_0 \to R/I$ such that

$$0 \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \ldots \to F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} R/I \to 0$$

is an exact sequence.

It can be shown [2, Theorem 20.2] that R/I has a *minimal* free resolution of the above form, meaning that $\phi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$, for $i = 1, \ldots, n$. Furthermore, any two minimal free resolutions of I are isomorphic (as chain complexes), so the F_i are uniquely determined (as R-modules) up to isomorphism. Thus, each free module F_i may be written $\bigoplus_j R(-j)^{b_{i,j}(I)}$ in such a way so as to ensure that each of the maps $\phi_1, \ldots, \phi_n$ is a homomorphism of *graded* R-modules. Note that since F_i is finitely generated as an R-module, $b_{i,j}(I) = 0$ for all but finitely many j. The numbers $b_{i,j}(I)$ are called the (graded) *Betti numbers* of I. It is clear that for any R and nonzero ideal $I \subseteq R$, $b_{0,0}(I) = 1$ and $b_{0,j}(I) = 0$ for $j \neq 0$.

Since exactness is preserved under flat base change, we immediately have:

Proposition 2.1. *If R' is a flat graded R-algebra, then for any ideal I, $b_{i,j}(I \otimes_R R') = b_{i,j}(I)$.*

We will be most interested in the case $R = \mathbb{K}[x_1, \ldots, x_m]$, $R' = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$, where $x_1, \ldots, x_n, y_1, \ldots, y_m$ are independent indeterminates. In this situation, $I \otimes_R R'$ is simply the extension of the ideal $I \subseteq R$ to the larger ring R'.

It is also worth recording a standard result which follows directly from the construction of the Koszul complex:

Proposition 2.2. *[2, Corollary 19.3] For $i \geq 0$, $b_{i,i}(\mathfrak{m}) = \binom{t}{i}$.*

We will also be studying the secant powers of various monomial ideals in R. Since the definition itself is rather complicated and formulated in greater generality than we will need, we omit it here and instead refer the interested reader to [13] or [16] for details. The points we will need may be summarized as follows. There is an operation $*$ on ideals of R called the *join*, which is both associative and commutative; if I is a ideal of R, we define its *secant powers* by $I^{\{0\}} = \mathfrak{m}$, $I^{\{1\}} = I$, and for $r > 1$, $I^{\{r\}} = I * I^{\{r-1\}}$. Moreover, if I is a monomial ideal, then there is a convenient method for computing the generators of its secant powers in terms of its own generators; see [13, Proposition 3.1] for details. The "secant" terminology comes from algebraic geometry: if one considers I as defining a variety V in n-dimensional projective space over $\mathbb{K}$, then $I^{\{r\}}$ defines the r-fold secant variety of V.

3 Edge Ideals and Splitting

In this section, we define the edge ideal of a graph and recall a result which allows for a simple combinatorial description of a minimal generating set for each of its secant powers. Throughout this article, all graphs are assumed to be simple, with a finite vertex set. Given a subset S of vertices in a graph G, we denote by G_S the subgraph of G induced by S, i.e. the graph whose vertex set is S and whose edge set consists of those edges of G, both of whose endpoints lie in S. We denote by K_m the complete graph on m vertices and by $\overline{G}$ the complement of a graph G. If G and H are graphs with disjoint vertex sets, the *join* of G and H, denoted $G \vee H$, is the graph whose vertex set is $V(G) \cup V(H)$, and whose edge set is $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. (This join operation on graphs is not related to the join of ideals defined in Section 2.) Intuitively, one may think of the join of two graphs as constructed by taking disjoint copies of each and adding all possible edges with one endpoint in each of the two graphs. The join operation on graphs is easily seen to be associative. Finally, we denote by $\chi(G)$ the chromatic number of G; this is the smallest positive integer k such that there exists an assignment of an integer from $\{1, \ldots, k\}$ to each vertex of G in such a way that no two adjacent vertices are labeled with the same integer. For further details on graph theory, we refer the reader to [18] or any other standard textbook on the subject.

Let G be graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. Let $x_1, \ldots, x_n$ be independent indeterminates, and let $I(G)$ be the ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$ generated by all monomials $x_i x_j$ such that $v_i v_j$ is an edge of G; we call $I(G)$ the *edge ideal* of G. If $S = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, we denote by M_S the monomial $x_{i_1} \cdots x_{i_m} \in \mathbb{K}[x_1, \ldots, x_n]$. We also define

$$\mathcal{C}_r(G) = \{S \subseteq V(G) : \chi(G_S) = r + 1 \text{ and } \chi(G_T) \leq r \text{ for all proper } T \subseteq S\}.$$

Sturmfels and Sullivant have given a convenient combinatorial description of the secant ideals $I(G)^{\{r\}}$.

Theorem 3.1. *[16, Theorem 3.2]*
The ideal $I(G)^{\{r\}}$ is generated by $\{M_S : S \subseteq V(G)$ and $\chi(G_S) \geq r + 1\}$. A minimal generating set for $I(G)^{\{r\}}$ is given by $\mathcal{S}_r(G) = \{M_S : S \in \mathcal{C}_r(G)\}$.

The following elementary fact about monomial ideals is well-known:

Proposition 3.2. *Suppose I and J are monomial ideals in a polynomial ring R over a field, generated (respectively) by monomial sets A and B. Then $I \cap J$ is also a monomial ideal in R and is generated by $\{lcm(a, b) : a \in A, b \in B\}$.*

We now define the notion of a *splittable* ideal, due to Eliahou and Kervaire.

Definition 3.3. *[3]*
A monomial ideal I in a polynomial ring R (over a field) is called splittable *if there exist ideals J and K of R and minimal generating sets $\mathcal{G}(I)$, $\mathcal{G}(J)$, $\mathcal{G}(K)$ for I, J, and K (respectively), and a generating set $\mathcal{G}(J \cap K)$ for $J \cap K$, such that:*

1. *$I = J + K$*
2. *$\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$*
3. *There are functions $\phi : \mathcal{G}(J \cap K) \to \mathcal{G}(J)$ and $\psi : \mathcal{G}(J \cap K) \to \mathcal{G}(K)$ such that:*
   (a) *For all $u \in \mathcal{G}(J \cap K)$, $u = lcm(\phi(u), \psi(u))$.*
   (b) *For every subset $C \subseteq \mathcal{G}(J \cap K)$, both $lcm(\phi(C))$ and $lcm(\psi(C))$ strictly divide $lcm(C)$.*

In this situation, we say that $I = J + K$ is a splitting *of I and refer to the pair (ϕ, ψ) as a* splitting function*.*

Remark.
In the original formulation of this definition, the generating set for $J \cap K$ was also required to be minimal. However, since every generating set contains a minimal generating set, the two formulations are in fact equivalent.

The following result of Fatabbi relates splittability to the computation of the Betti numbers of the ideal in question.

Theorem 3.4. *[5, Proposition 3.2] Suppose I is a splittable monomial ideal in a polynomial ring over a field, with splitting $I = J + K$. Then*

$$b_{i,j}(I) = b_{i,j}(J) + b_{i,j}(K) + b_{i-1,j}(J \cap K)$$

for all integers $i \geq 1$ and j, provided we interpret $b_{0,j}(J \cap K)$ as 0.

4 Main Result

The goal of this section is to develop a formula relating the Betti numbers of the join of a graph H with an edgeless graph to those of H itself.

Let $v_1, \ldots, v_n$ be an ordering of the vertices in a graph H. Now let $w_1, \ldots, w_m$ be new vertices, and define, for $1 \leq \ell \leq m$, H_ℓ as the join of H with the edgeless graph on $W = \{w_1, \ldots, w_\ell\}$. If we set $H_0 = H$, then we may view each H_ℓ, $0 \leq \ell \leq m$, as isomorphic to $H \vee \overline{K_\ell}$. Now define $R = R_0 = \mathbb{K}[x_1, \ldots, x_n]$ and $R_\ell = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_\ell]$ for $1 \leq \ell \leq m$.

Lemma 4.1. *Suppose $1 \leq \ell \leq m$. Then the elements of $\mathcal{C}_r(H_\ell)$ are of two types:*

(i). Subsets $S \subseteq V(H)$ such that $S \in \mathcal{C}_r(H)$

(ii). Subsets of the form $S' \cup \{w\}$, where $S' \in \mathcal{C}_{r-1}(H)$ and $w \in W$.

Proof.
For convenience, set $H' = H_\ell$, and suppose $S \in \mathcal{C}_r(H')$. If $S \subseteq V(H)$, then clearly $S \in \mathcal{C}_r(H)$, so suppose S is not contained in $V(H)$. We claim that S contains exactly one of $w_1, \ldots, w_\ell$. Suppose to the contrary that w_i and w_j are both in S, where $1 \leq i < j \leq \ell$, and let $T = S - \{w_j\}$. Let $f' : T \to \{1, \ldots, t\}$ be a proper coloring of T. Since w_i is adjacent to all vertices of $T \cap V(H)$, we have $f'(w_i) \neq f'(v)$ for all $v \in T$. Extend f' to a function $f : S \to \{1, \ldots, t\}$ by setting $f(w_j) = f'(w_i)$. Since w_j is not adjacent to any vertex of $S \cap W$ but is adjacent to all vertices in $T \cap V(H)$, f is a proper t-coloring of S. This shows that $\chi(H'_S) \leq \chi(H'_T)$. Since obviously $\chi(H'_T) \leq \chi(H'_S)$, it follows that $\chi(H'_T) = \chi(H'_S)$, contradicting the hypothesis $S \in \mathcal{C}_r(H')$.

$\square$

We refer to members of $\mathcal{C}_r(H_\ell)$ as either of type (i) or type (ii), according to their classification in Lemma 4.1.

Define $A_{0,0} = I(H)^{\{r\}}$, and for $1 \leq k \leq \ell \leq m$, let $A_{k,\ell}$ be the ideal of R_ℓ generated by all M_S such that $S \in \mathcal{C}_r(H_\ell)$ is of type (ii) and $W \cap S = \{w_\ell\}$. Also define $B_{0,0} = 0$ and for $1 \leq k \leq \ell \leq m$, $B_{k,\ell} = \sum_{j=1}^{k} A_{j,\ell}$. Note further that if $0 \leq k \leq \ell \leq \ell' \leq m$, then by construction $A_{k,\ell'} = A_{k,\ell} \otimes_{R_\ell} R_{\ell'}$ and $B_{k,\ell'} = B_{k,\ell} \otimes_{R_\ell} R_{\ell'}$, so Proposition 2.1 implies:

$$b_{i,j}(A_{k,\ell}) = b_{i,j}(A_{k,\ell'}) \text{ and } b_{i,j}(B_{k,\ell}) = b_{i,j}(B_{k,\ell'}). \tag{1}$$

Lemma 4.2. *For $1 \leq k \leq \ell \leq m$, there are isomorphisms*

$$A_{k,\ell} \cong [I(H)^{\{r-1\}} \otimes_R R_\ell](-1) \text{ and } A_{k,\ell} \cap I(H)^{\{r\}} \cong [I(H)^{\{r\}} \otimes_R R_\ell](-1)$$

of graded R'-modules, and thus

$$b_{i,j}(A_{k,\ell}) = b_{i,j-1}(I(H)^{\{r-1\}}), \quad b_{i,j}(A_{k,\ell} \cap I(H)^{\{r\}}) = b_{i,j-1}(I(H)^{\{r\}}).$$

Proof.
By Lemma 4.1, $A_{k,\ell}$ is generated by monomials of the form $y_k M_{S'}$, where $S' \in \mathcal{C}_{r-1}(H)$. Thus, $A_{k,\ell} = y_k \left(I(H)^{\{r-1\}} \otimes_R R'\right)$, which is isomorphic (as a graded R'-module) to $[I(H)^{\{r-1\}} \otimes_R R'](-1)$. By Proposition 2.1, $b_{i,j}(A_{k,\ell}) = b_{i,j-1}(I(H)^{\{r-1\}})$, as predicted by the formula. Likewise, using Proposition 3.2, we see that $A_{k,\ell} \cap I(H)^{\{r\}}$ is generated by monomials of the form $y_k M_{S'}$, where $S' \in \mathcal{C}_r(H)$. Arguing as above, we have $A_{k,\ell} \cap I(H)^{\{r\}} \cong I(H)^{\{r\}}(-1)$, whence the result. $\square$

Lemma 4.3. *Let $r \geq 1$ and $1 \leq k \leq \ell \leq m$. Then there are splittings:*

$$B_{k,\ell} = B_{k-1,\ell} + A_{k,\ell}, \quad B_{k,\ell} \cap I(H)^{\{r\}} = B_{k-1,\ell} \cap I(H)^{\{r\}} + A_{k,\ell} \cap I(H)^{\{r\}}.$$

Thus,

$$b_{i,j}(B_{k,\ell}) = b_{i,j}(B_{k-1,\ell}) + b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(B_{k-1,\ell}),$$

$$b_{i,j}(B_{k,\ell} \cap I(H)^{\{r\}}) = b_{i,j}(B_{k-1,\ell} \cap I(H)^{\{r\}}) + b_{i,j-1}(I(H)^{\{r\}}) + b_{i-1,j-1}(B_{k-1,\ell} \cap I(H)^{\{r\}}).$$

Proof.
We will prove the first formula, the second being similar, *mutatis mutandis*. By Lemma 4.1, a set of minimal generators for $A_{k,\ell}$ is given by $y_k M_{S'}$, where $S' \in \mathcal{C}_{r-1}(H)$. By Proposition 3.2, a generating set for $B_{k-1,\ell} \cap A_{k,\ell}$ is given by the set of monomials $y_k M_{S'}$, where $S' \in \mathcal{C}_r(H_{k-1})$. Now let $\mu(S') = \max\{t : v_t \in S'\}$ and choose $T(S') \subseteq S' - \{v_{\mu(S')}\}$ such that $T(S') \in \mathcal{C}_{r-1}(H_{k-1})$. Observe also that $B_{k-1,\ell} \cap A_{k,\ell} \cong B_{k-1,\ell}(-1)$.

We claim that the correspondence $y_k M_{S'} \mapsto (M_{S'}, y_k M_{T(S')})$ defines a splitting function. The first and second conditions of Definition 3.3 are clearly satisfied. For the last condition, let $C = \{y_k M_{S'_d} : d \in D\}$ (where D is some set indexing the monomials) be a subset of the generating set for $B_{k-1,\ell} \cap A_{k,\ell}$ described above. Then the first coordinate of the image of any element of C under the above function does not involve the variable y_k. Furthermore, the second coordinate does not involve the variable x_M, where $M = \max_{d \in D} \mu(S'_d)$. This shows that $B_{k,\ell} = B_{k-1,\ell} + A_{k,\ell}$ defines a splitting. The remaining formulas follow from Theorem 3.4. $\square$

We now come to our main result.

Theorem 4.4. *Let H be graph and r, m positive integers. Then for all j,*

$$b_{1,j}(I(H_m)^{\{r\}}) = b_{1,j}(I(H)^{\{r\}}) + m b_{1,j-1}(I(H)^{\{r-1\}})$$

and for $i \geq 2$,

$$b_{i,j}(I(H_m)^{\{r\}}) = b_{i.j}(I(H_{m-1})^{\{r\}}) + b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1.j-1}(I(H_{m-1})^{\{r\}}).$$

Proof.
Let $1 \leq \ell \leq m$. The generators of $I(H_\ell)^{\{r\}}$ are described by Lemma 4.1: $J = I(H)^{\{r\}}$ is the ideal of R' generated by the monomials M_S where S is of type (i), and $K = B_{\ell,\ell}$ is the ideal generated by M_S for S of type (ii). We claim that

$$I(H_\ell)^{\{r\}} = I(H)^{\{r\}} + B_{\ell,\ell} \tag{2}$$

is in fact a splitting.

It is clear from the above description of the generators of $I(H_\ell)^{\{r\}}$ that condition (2) of Definition 3.3 is satisfied, so it remains to construct a splitting function. By Proposition 3.2 and Lemma 4.1, a generator $M_S \in \mathcal{G}(J \cap K)$ is a monomial of the form $y_j M_{S'}$, where $1 \leq j \leq \ell$ and $S' \in \mathcal{C}_r(H)$. Let $\mu(S') = \max\{i : v_i \in S'\}$; then choose $T(S') \subseteq S' - \{v_{\mu(S')}\}$ such that $T(S') \in \mathcal{C}_{r-1}(H)$. We claim that $y_j M_{S'} \mapsto (M_{S'}, y_j M_{T(S')})$ defines a splitting function.

As before, the first and second conditions of Definition 3.3 are clearly satisfied. With notation as above, let $C = \{y_{j_d} M_{S'_d} : d \in D, 1 \leq j_a \leq \ell\}$ be a subset of the generators of $J \cap K$. Now the monomial $\mathrm{lcm}(C)$ involves some indeterminate from among $y_1, \ldots, y_\ell$; however, the first coordinate of its image under the proposed function does not involve any of the y_j. Furthermore, the second coordinate does not involve x_N, where $N = \max_{d \in D} \max\{i : v_i \in T(S'_d)\}$, whereas $\mathrm{lcm}(C)$ does. Thus, (2) is a splitting, as claimed.

Applying Theorem 3.4 to (2) implies $b_{1,j}(I(H_m)^{\{r\}}) = b_{1,j}(I(H)^{\{r\}}) + b_{1,j}(B_{m,m})$. By Lemma 4.3 and (1), $b_{1,j}(B_{m,m}) = b_{1,j}(B_{m-1,m-1}) + b_{1,j-1}(I(H)^{\{r-1\}})$. Applying this successively yields $b_{1,j}(B_{m,m}) = b_{1,j}(B_{0,0}) + m b_{1,j-1}(I(H)^{\{r-1\}}) = b_{1,j}(I(H)^{\{r\}}) + m b_{1,j-1}(I(H)^{\{r-1\}})$, which establishes the first formula.

Now suppose $i \geq 2$. Applying Theorem 3.4 to (2) with $\ell = m$ yields

$$b_{i,j}(I(H_m)^{\{r\}}) = b_{i,j}(I(H)^{\{r\}}) + b_{i,j}(B_{m,m}) + b_{i-1,j}(B_{m,m} \cap I(H)^{\{r\}}),$$

which by Lemma 4.3 may be rewritten

$$b_{i,j}(I(H_m)^{\{r\}}) = b_{i,j}(I(H)^{\{r\}}) + b_{i,j}(B_{m-1,m}) + b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(B_{m-1,m})$$

$$+ b_{i-1,j}(B_{m-1,m} \cap I(H)^{\{r\}}) + b_{i-1,j-1}(I(H)^{\{r\}}) + b_{i-2,j-1}(B_{m-1,m} \cap I(H)^{\{r\}}).$$

Applying (1), this becomes

$$b_{i,j}(I(H_m)^{\{r\}}) = b_{i,j}(I(H)^{\{r\}}) + b_{i,j}(B_{m-1,m-1}) + b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(B_{m-1,m-1})$$

$$+b_{i-1,j}(B_{m-1,m-1} \cap I(H)^{\{r\}}) + b_{i-1,j-1}(I(H)^{\{r\}}) + b_{i-2,j-1}(B_{m-1,m-1} \cap I(H)^{\{r\}}). \quad (3)$$

However, Theorem 3.4 applied to (2) with $\ell = m-1$ yields

$$b_{i,j}(I(H_{m-1})^{\{r\}}) = b_{i,j}(I(H)^{\{r\}}) + b_{i,j}(B_{m-1,m-1}) + b_{i-1,j}(B_{m-1,m-1} \cap I(H)^{\{r\}}). \quad (4)$$

Subtracting (4) from (3), we obtain

$$\begin{aligned} & b_{i,j}(I(H_m)^{\{r\}}) - b_{i,j}(I(H_{m-1})^{\{r\}}) \\ &= b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(B_{m-1,m-1}) + b_{i-1,j-1}(I(H)^{\{r\}}) + b_{i-2,j-1}(B_{m-1,m-1} \cap I(H)^{\{r\}}). \\ &= b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(B_{m-1,m-1} + I(H)^{\{r\}}) \\ &= b_{i,j-1}(I(H)^{\{r-1\}}) + b_{i-1,j-1}(I(H_{m-1})^{\{r\}}). \end{aligned}$$

□

5 Applications

In this section, we apply Theorem 4.4 to calculate the Betti numbers for some common classes of graphs. To illustrate the key ideas, we begin with the relatively simple case of wheels, and then proceed to the case of complete graphs. Both of these calculations only use the case $m = 1$ of Theorem 4.4 and yield fairly elegant formulas for the Betti numbers. We conclude with the case of complete multipartite graphs, which is technically more complicated. Note that from the discussion of Section 2, we always have $b_{0,0} = 1$ and $b_{0,j} = 0$ for $j \neq 0$; hence we will focus on $b_{i,j}$ when $i \geq 1$.

1. **Wheels**. For an integer $n \geq 3$, the *n-cycle*, denoted C_n, is the graph on vertices $v_1, \ldots, v_n$ whose edges are $v_n v_1$ and $v_i v_{i+1}$, $1 \leq i \leq n-1$. The *n-wheel*, denoted W_n, is the join of C_n with $\overline{K_1}$. To compute the Betti numbers of W_n using Theorem 4.4, we will need the Betti numbers of C_n. For the edge ideal, these were calculated by Jacques [10, Theorem 7.6.28]: when $j < n$ and $2i \geq j$,

$$b_{i,j}(I(C_n)) = \frac{n}{n-2(j-i)} \binom{j-i}{2i-j} \binom{n-2(j-i)}{j-i}.$$

Moreover, if $n = 3m+1$ or $n = 3m+2$, then $b_{2m+1,n}(I(C_n)) = 1$, and if $n = 3m$, then $b_{2m,n}(I(C_n)) = 2$; all other Betti numbers of $I(C_n)$ are 0. Now if n is even, then $\chi(C_n) = 2$, so $I(C_n)^{\{r\}} = 0$ for $r \geq 2$. If n is odd, then $\chi(C_n) = 3$, so by Theorem 3.1, $I(C_n)^{\{2\}}$ is generated by the single monomial $x_1 \cdots x_n$. As such, we have $I(C_n)^{\{2\}} \cong R(-n)$; hence its only nonzero Betti number is $b_{1,n}(I(C_n)^{\{2\}}) = 1$. Clearly $I(C_n)^{\{r\}} = 0$ for $r \geq 3$.

We now turn to the computation of the Betti numbers of W_n, $n \geq 3$. In the interest of making the presentation more readable, we will express the Betti numbers of W_n in terms of those of C_n and other directly computable quantities. We begin with the edge ideal of W_n.

By Theorem 4.4 and Proposition 2.2,

$$b_{1,j}(I(W_n)) = b_{1,j}(I(C_n)) + b_{1,j-1}(I(C_n)^{\{0\}}) = \begin{cases} 2n & \text{if } j = 2 \\ 0 & \text{if } j \neq 2 \end{cases}$$

If $i \geq 2$, we have $b_{i,j}(I(W_n)) = b_{i,j}(I(C_n)) + b_{i,j-1}(I(C_n)^{\{0\}}) + b_{i-1,j-1}(I(C_n))$, so

$$b_{i.j}(I(W_n)) = \begin{cases} b_{i,i+1}(I(C_n)) + b_{i-1,i}(I(C_n)) + \binom{n}{i} & \text{if } j = i+1 \\ b_{i,j}(I(C_n)) + b_{i-1,j-1}(I(C_n)) & \text{if } j \neq i+1 \end{cases}$$

Turning our attention to the second secant ideal of W_n, we have:

$$b_{1,j}(I(W_n)^{\{2\}}) = b_{1,j}(I(C_n)^{\{2\}}) + b_{1,j-1}(I(C_n))$$

and for $i \geq 2$,

$$b_{i,j}(I(W_n)^{\{2\}}) = b_{i,j}(I(C_n)^{\{2\}}) + b_{i,j-1}(I(C_n)) + b_{i-1,j-1}(I(C_n)^{\{2\}}).$$

Thus, when n is even, $b_{i,j}(I(W_n)^{\{2\}}) = b_{i,j-1}(I(C_n))$ for all $i \geq 1$. When n is odd, we have $b_{i,j}(I(W_n)^{\{2\}}) = b_{i,j-1}(I(C_n)) + \varepsilon_{i,j}$, where $\varepsilon_{1,n} = \varepsilon_{2,n+1} = 1$ and $\varepsilon_{i,j} = 0$ otherwise.

When n is even, $I(W_n)^{\{r\}} = 0$ when $r \geq 3$. Finally, when n is odd, the only subgraph of W_n of chromatic number 4 is W_n itself, so $b_{1,n+1}(I(W_n)^{\{3\}}) = 1$ is the only nonzero Betti number of $I(W_n)^{\{3\}}$, and of course $I(W_n)^{\{r\}} = 0$ when $r \geq 4$.

2. **Complete Graphs**. Since $K_n = K_{n-1} \vee \overline{K_1}$, Theorem 4.4 provides a means of calculating its Betti numbers recursively. In fact, there is an elegant formula in closed form which recovers and extends Jacques's computation [10, Theorem 5.1.1] in the case of the edge ideal.

Theorem 5.1. *Suppose n, i are positive integers and r is a nonnegative integer. Then* $b_{i,i+r}(I(K_n)^{\{r\}}) = \binom{i+r-1}{r}\binom{n}{i+r}$. *If $j \neq i+r$, then* $b_{i,j}(I(K_n)^{\{r\}}) = 0$.

Proof.
We prove both assertions by induction on n. If $n = 1$, then $R = \mathbb{K}[x_1]$, so when $r = 0$ and $i = 1$, we have $I(K_1)^{\{0\}} = \mathfrak{m} = (x_1)$. Also, we have $b_{1,1}(I(K_1)^{\{0\}}) = 1$ and $b_{i,j}(I(K_1)^{\{0\}}) = 0$ for $j \neq i$, which agrees with the expression on the right of the asserted equality. When $r \geq 1$ or $i \geq 2$, we have $I(K_1)^{\{r\}} = 0$. Since $i + r \geq 2$, we also have $\binom{1}{i+r} = 0$.

Now suppose (by induction) that the formulas hold for $n-1$. If $i \geq 2$, then by Theorem 4.4:

$$b_{i,j}(I(K_n)^{\{r\}}) = b_{i,j}(I(K_{n-1})^{\{r\}}) + b_{i-1,j}(I(K_{n-1})^{\{r-1\}}) + b_{i-1,j-1}(I(K_{n-1})^{\{r\}})$$

If $j \neq i+r$, all terms on the right vanish by the induction hypothesis. If $j = i+r$, the induction hypothesis, in conjunction with the well-known combinatorial identity $\binom{m+1}{k+1} = \binom{m}{k+1} + \binom{m}{k}$ implies

$$b_{i,i+r}(I(K_n)^{\{r\}}) = \binom{i+r-1}{r}\binom{n-1}{i+r} + \binom{i+r-2}{r-1}\binom{n-1}{i+r-1} + \binom{i+r-2}{r}\binom{n-1}{i+r-1}$$

$$= \binom{i+r-1}{r}\binom{n-1}{i+r} + \left[\binom{i+r-2}{r-1} + \binom{i+r-2}{r}\right]\binom{n-1}{i+r-1}$$

$$= \binom{i+r-1}{r}\binom{n-1}{i+r} + \binom{i+r-1}{r}\binom{n-1}{i+r-1} = \binom{i+r-1}{r}\left[\binom{n-1}{i+r} + \binom{n-1}{i+r-1}\right]$$

$$= \binom{i+r-1}{r}\binom{n}{i+r}.$$

Finally, in the case $i = 1$, we have $b_{1,j}(I(K_n)^{\{r\}}) = b_{1,j}(I(K_{n-1})^{\{r\}}) + b_{1,j-1}(I(K_{n-1})^{\{r-1\}})$. If $j \neq 1 + r$, then both terms on the right vanish by induction. If $j = 1 + r$, the induction hypothesis implies

$$b_{1,1+r}(I(K_n)^{\{r\}}) = b_{1,1+r}(I(K_{n-1})^{\{r\}}) + b_{1,r}(I(K_{n-1})^{\{r-1\}}) = \binom{n-1}{r+1} + \binom{n-1}{r} = \binom{n}{r+1}.$$

This completes the inductive step.

$\square$

3. **Complete Multipartite Graphs**. If $m \geq 2$ and $n_1, \ldots, n_m$ are positive integers, the complete multipartite graph $K_{n_1,\ldots,n_m}$ may be defined as the m-fold join $\overline{K_{n_1}} \vee \cdots \vee \overline{K_{n_m}}$. It is easily seen that $\chi(K_{n_1,\ldots,n_m}) = m$. Jacques has computed the Betti numbers of the edge ideal of a complete bipartite graph; since its higher secant powers all vanish, there is nothing more to be done in this case.

Theorem 5.2. *(Jacques, [10, Theorem 5.2.4])*

$$b_{i,j}(I(K_{n_1,n_2})) = \begin{cases} \sum_{k,\ell \geq 1 : k+\ell=i+1} \binom{n_1}{k}\binom{n_2}{\ell} & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1 \end{cases}$$

If $m \geq 3$, we may realize $K_{n_1,\ldots,n_m}$ as $K_{n_1,\ldots,n_{m-1}} \vee \overline{K_{n_m}}$ and use Theorem 4.4 to perform a recursive computation, ultimately expressing everything in terms of the quantities appearing in Theorem 5.2. Unfortunately, there does not seem to be a nice formula in closed form. Nevertheless, it is quite easy to establish the following:

Proposition 5.3. *Let $m \geq 2$. If $j \neq i + r$, then $b_{i,j}(I(K_{n_1,\ldots,n_m})^{\{r\}}) = 0$.*

Proof.
We proceed by induction on m. The base case ($m = 2$) is Theorem 5.2. Suppose now that the result is known for all positive values $k \leq m-1$. If $i \neq 2$, then using Theorem 4.4, we have:

$$b_{i,j}(I(K_{n_1,\ldots,n_{m-1},1})^{\{r\}}) = b_{i,j}(I(K_{n_1,\ldots,n_{m-1}})^{\{r\}}) + b_{i,j-1}(I(K_{n_1,\ldots,n_{m-1}})^{\{r-1\}}) + b_{i-1,j-1}(I(K_{n_1,\ldots,n_{m-1}})^{\{r\}}).$$

If $j \neq i+r$, then all three terms on the right vanish by induction, and the result holds when $n_m = 1$. Now suppose the result holds when $n_m = k \geq 1$. Then

$$b_{i,j}(I(K_{n_1,\ldots,n_{m-1},k+1})^{\{r\}}) = b_{i,j}(I(K_{n_1,\ldots,n_{m-1},k})^{\{r\}}) + b_{i,j-1}(I(K_{n_1,\ldots,n_{m-1}})^{\{r-1\}}) + b_{i-1,j-1}(I(K_{n_1,\ldots,n_{m-1},k})^{\{r\}}$$

Again, all terms on the right vanish showing that the result holds for $n_m = k+1$. The argument for $i = 1$ is similar. $\square$

We conclude this discussion by giving a clean computation of the simplest nontrivial example in this family – the Betti numbers of the second secant power of the edge ideal of a complete tripartite graph – using a different type of edge-splitting argument. In preparation for the calculation, we introduce a counting function. For $i \geq 1$, $m \geq 2$ and $t \leq m$, define

$$P(i,t;n_1,\dots,n_m) = \sum_{\substack{1\le j_1<\dots<j_t\le m\\ \alpha_1+\dots+\alpha_t=i+1,\ \alpha_k>0}} \binom{n_{j_1}}{\alpha_1}\cdots\binom{n_{j_t}}{\alpha_t}$$

If we consider m bins with respective capacities $n_1,\dots,n_m$, the function defined above counts the number of ways of distributing $i+1$ balls among exactly t of these bins.
The Betti numbers of the edge ideal of the complete multipartite graph were also computed by Jacques:

Theorem 5.4. *[10, Theorem 5.3.8]*
Suppose $i,m \ge 1$. Then

$$b_{i,i+1}(I(B_{n_1,\dots,n_m})) = \sum_{t=2}^{m}(t-1)P(i,t;n_1,\dots,n_m)$$

We now have the tools necessary for our calculation:

Proposition 5.5. *Suppose $i \ge 1$. Then*

$$\begin{aligned} b_{i,i+2}(I(B_{n_1,n_2,n_3})^{(2)}) &= P(i+1,2;n_1,n_2,n_3) + 2P(i+1,3;n_1,n_2.n_3)\\ &\quad -P(i+1,2;n_1,n_3) - P(i+1,2;n_2,n_1+n_3)\end{aligned}$$

Proof.
For convenience, let $I = I(B_{n_1,n_2,n_3}) \subseteq \mathbb{K}[x_1,\dots,x_{n_1},y_1,\dots,y_{n_2},z_1,\dots,z_{n_3}]$, J the ideal generated by the various products x_iz_k, $1\le i\le n_1$, $1\le k\le n_3$, and K the ideal generated by the products x_iy_j and y_jz_k, $1\le i\le n_1$, $1\le j\le n_2$, $1\le k\le n_3$. By Proposition 3.2, $J\cap K$ is generated by the products $x_iy_jz_k$, where i, j, and k are as above. By Theorem 3.1, we see that in fact $I^{\{2\}} = J\cap K$. Furthermore, the map $x_iy_jz_k \mapsto (x_iz_k, x_iy_j)$ is splitting function, and thus witnesses that $I = J+K$ is a splitting.
By Theorem 3.4, we have:

$$b_{i,j}(I^{\{2\}}) = b_{i,j}(J\cap K) = b_{i+1,j}(I) - b_{i+1,j}(J) - b_{i+1,j}(K).$$

Now $I = I(B_{n_1,n_2,n_3})$, $J = I(B_{n_1,n_3})$, and $K = I(B_{n_2,n_1+n_3})$, so by Theorem 5.4, we have:

$$b_{i,i+2}(I^{\{2\}}) = P(i+1,2;n_1,n_2,n_3)+2P(i+1,3;n_1,n_2.n_3)-P(i+1,2;n_1,n_3)-P(i+1,2;n_2,n_1+n_3).$$

□

The key insight here was to identify the secant ideal as the intersection of two ideals which (along with their sum) are better understood, and to apply the ideal splitting formula in reverse. Unfortunately, this technique does not seem to extend to a more general setting.

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